Introduction to stress and strain analysis

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1

The uniform state of stress

Consider a continuous 3-dimensional body subjected to an arbitrary system of forces. The state of stress at a point O in the body can be studied by considering an infinitesimal parallelepiped erected at this point. It is assumed that the resultant forces acting on any two parallel faces are the same, *i.e.*, a uniform state or field of stress exists.

The isolated element is shown in Fig. 1.1 referred to a Cartesian coordinate system. The double subscript is interpreted as follows: The first subscript indicates the direction of a normal to the plane or face on which the stress component acts; the second subscript relates to the direction of the stress itself. Note that $\sigma_x \equiv \sigma_{xx}$. Thus, τ_{xy} is the shear stress on the x-face in the y-direction.

The following sign convention is used: If face (F) and direction (D) are both positive, τ_{FD} is positive; if F and D are both negative, τ_{FD} is positive; if F and D are of opposite signs, τ_{FD} is negative.

By taking moments of the forces due to the stresses about each axis, we can show that:

$$\begin{aligned} \tau_{xy} &= \tau_{yx} \ ; \\ \tau_{xz} &= \tau_{zx} \ ; \\ \tau_{yz} &= \tau_{zy} \ . \end{aligned} \tag{1.1}$$

In future therefore, no distinction will be made between τ_{xy} and τ_{yx}, τ_{xz} and τ_{zx},

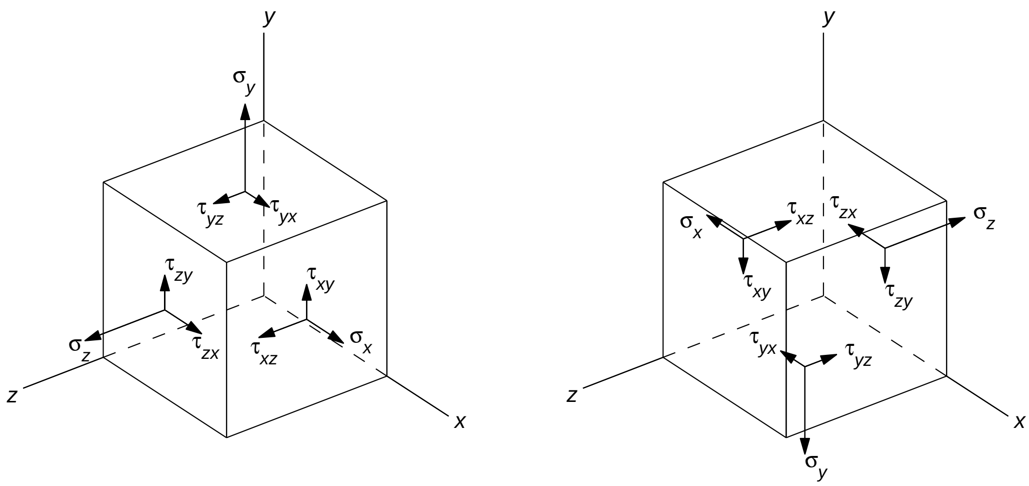

Figure 1.1: Stresses acting on the positive *(left)* and negative *(right)* faces of an infinitesimal body.

or τ_{yz} and τ_{zy}. This means that only 6 Cartesian components are necessary for the complete specification of the state of stress at any point in the body. These terms can be conveniently assembled into the so-called **stress tensor**:

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}. \tag{1.2}$$

2

Stress on an inclined plane

It is required to find the state of stress on a plane inclined to the axes previously setup, represented by the face ABC of the tetrahedron in Fig. 2.1, assuming that the stresses on faces OBC, OCA, and OAB are known. The position of the plane can be specified by the length and orientation of the normal OD drawn from the origin O to the plane ABC such that the angles ODA, ODB, and ODC are all right angles. The length of OD is equal to r, and its position is given by the angles AOD, BOD, and COD. The cosines of these angles are known as **direction cosines** and are denoted by:

$$\begin{aligned} \cos\ \mathrm{AOD} &= l\ ; \\ \cos\ \mathrm{BOD} &= m\ ; \\ \cos\ \mathrm{COD} &= n\ . \end{aligned} \tag{2.1}$$

It can be proven on geometrical grounds, that

$$l^2 + m^2 + n^2 = 1, \tag{2.2}$$

so that only two of the direction cosines are independent.

The areas of the perpendicular planes OBC, OCA, and OAB may now be

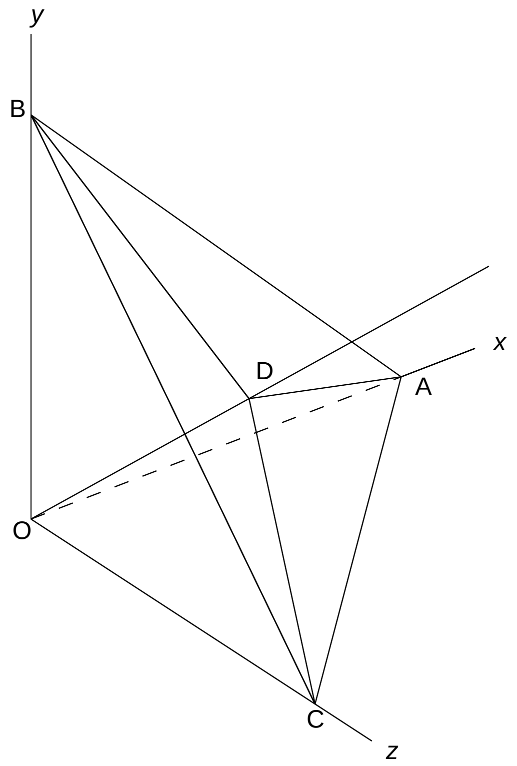

Figure 2.1: Plane inclined to the axes, represented by the face ABC of a tetrahedron.

found in terms of these direction cosines:

$$\begin{aligned} \mathrm{OA} &= \frac{r}{l}\ ; \\ \mathrm{OB} &= \frac{r}{m}\ ; \\ \mathrm{OC} &= \frac{r}{n}\ . \end{aligned} \tag{2.3}$$

Let the area of face ABC be A, and that of OCA be A_y. The volume of the tetrahedron can be written as $\frac{1}{3}Ar = \frac{1}{3}A_y\mathrm{OB} = \frac{1}{3}A_y\frac{r}{m}$, from which it follows that $A_y = A\,m$. Hence,

$$\frac{A_x}{l} = \frac{A_y}{m} = \frac{A_z}{n} = A. \tag{2.4}$$

These are the relationships between the areas of the four faces of the tetrahedron.

With the stress on the inclined plane represented by its three Cartesian components s_x, s_y, and s_z, the general state of stress on the tetrahedron is shown in Fig. 2.2. The equilibrium of the element can be examined by resolving the forces

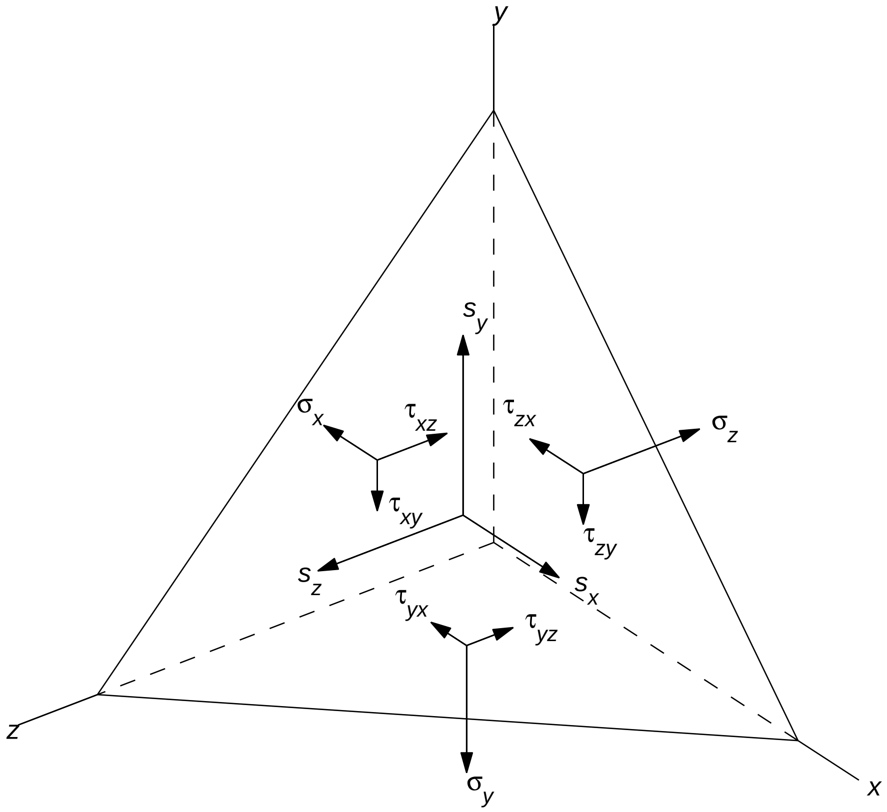

Figure 2.2: State of stress on a tetrahedron.

acting on it in the directions of the three axes. For example, in the x-direction,

$$s_x A - \sigma_x A_x - \tau_{yx} A_y - \tau_{zx} A_z = 0. \tag{2.5}$$

Hence,

$$s_x = \sigma_x l + \tau_{yx} m + \tau_{zx} n. \tag{2.6}$$

Similar expressions exist for s_y and s_z. These three equations can be expressed in matrix form:

$$\begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix}. \tag{2.7}$$

The **resultant stress** on the inclined plane is given by the resultant forces acting on ABC divided by the area of ABC. Therefore,

$$s = \sqrt{s_x^2 + s_y^2 + s_z^2}. \tag{2.8}$$

To find the **normal stress** on the plane, the forces parallel to the normal have to be resolved, noting that the area ABC is common to all forces acting on this face:

$$\begin{aligned} N = s_x l + s_y m + s_z n = & \begin{bmatrix} l & m & n \end{bmatrix} \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} = \\ = & \begin{bmatrix} l & m & n \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix}. \end{aligned} \tag{2.9}$$

The square of the resultant stress is equal to the sum of the squares of the normal stress and the **shear stress** on the plane, so that

$$T = \sqrt{s^2 - N^2}. \tag{2.10}$$

The direction cosines of this resultant shear stress may also be found. Let l_1, m_1, and n_1 be the direction cosines of T with respect to XYZ. Then, in the x-direction,

$$s_x A = N\ A\ l + \tau\ A\ l_1. \tag{2.11}$$

Therefore,

$$\begin{aligned} l_1 &= \frac{s_x - N\, l}{\tau}; \\ m_1 &= \frac{s_y - N\, m}{\tau}; \\ n_1 &= \frac{s_z - N\, n}{\tau}. \end{aligned} \tag{2.12}$$

3

Transformation of stresses for rotation of axes

By extending the theory developed in the previous Chapter, it is possible to transform the tensor of a given state of stress known for one set of axes to the tensor of the same state in a second set of axes. For example, if the stress tensor in the (x, y, z) coordinate system is $[\sigma]$, it is possible to determine the stress tensor $[\sigma']$ in another coordinate system (x', y', z').

The relative rotation of the second system to the first is defined by 9 direction cosines, where for example l_1 is the angle between the x'-axis and the x-axis, m_1 is the angle between the x'-axis and the y-axis, and l_2 is the angle between the y'-axis and the x-axis.

The full set of cosines is:

$$[L] = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}. \tag{3.1}$$

Consider a second Cartesian system (x', y', z') and the set used in the previous Chapter. Let the x'-axis be coincident with the normal of the plane that was examined: $\sigma_{x'} = N$. The normal stress on the plane follows from (2.9):

$$\sigma_{x'} = N = \begin{bmatrix} l_1 & m_1 & n_1 \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{bmatrix} l_1 \\ m_1 \\ n_1 \end{bmatrix}. \qquad (3.2)$$

By using the same method for the other axes, the other components of the stress tensor $[\sigma']$ can be determined:

$$[\sigma'] = [L][\sigma][L]^{\mathsf{T}}. \qquad (3.3)$$

Using this equation, a tensorial state of stress in one coordinate system can be easily converted into another system.

4
Principal stresses

A principal plane is defined as one on which the shear stress is zero. The normal stress on this plane is known as the principal stress and is denoted by p. If the direction cosines of the plane are l, m, and n, then resolving stresses on the plane in the coordinate directions gives:

$$\begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} = \begin{bmatrix} p\,l \\ p\,m \\ p\,n \end{bmatrix} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix}. \tag{4.1}$$

Rearranging this with (2.7) gives:

$$\begin{bmatrix} \sigma_x - p & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - p & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - p \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0. \tag{4.2}$$

This matrix represents three linear equations in l, m, and n, which will have non-trivial solutions if and only if

$$\det |[\sigma] - [p]| = 0. \tag{4.3}$$

Or, after expansion of the determinant:

$$\begin{aligned} p^3 \quad & -(\sigma_x + \sigma_y + \sigma_z)p^2 \\ & +(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)p \\ & -(\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2) = 0. \end{aligned} \tag{4.4}$$

This is known as the **stress cubic**. Its roots, which are always real, are the values of the three principal stresses, p_1, p_2, and p_3, which exist on three perpendicular planes. Note that the equation is independent of the direction cosines and therefore the coordinate system. If a different set of axes had been used to describe the element at point O, different values of applied stress σ_x, τ_{xy}, ... would have resulted. However, the principal stresses would remain unaltered and the same stress cubic would have been derived. This means that the coefficients are the same, regardless of the original choice of axes, *i.e.*, they are invariant. By convention, the coefficients of the stress cubic are referred to as first, second, and third **stress invariants**:

$$\begin{aligned} I_1 &= \sigma_x + \sigma_y + \sigma_z \ ; \\ I_2 &= \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 \ ; \\ I_3 &= \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2 \ . \end{aligned} \tag{4.5}$$

Thus, the stress cubic can be rewritten in terms of p and I:

$$p^3 - I_1p^2 + I_2p - I_3 = 0. \tag{4.6}$$

and, as the stress invariants have the same values regardless of the axes chosen, they must also be the same when the axes of reference correspond to the axes perpendicular to the principal planes. Therefore,

$$\begin{aligned} I_1 &= p_1 + p_2 + p_3 \ ; \\ I_2 &= p_1p_2 + p_2p_3 + p_3p_1 \ ; \\ I_3 &= p_1p_2p_3 \ . \end{aligned} \tag{4.7}$$

By convention, it is assumed that $p_1 > p_2 > p_3$.

One further invariant dubbed I_2' is derived from I_1 and I_2:

$$I_2' = 2(I_1^2 - 3I_2). \tag{4.8}$$

From (4.5),

$$I_2' = (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2), \tag{4.9}$$

or in terms of principal stresses

$$\mathrm{I}_2' = (p_1 - p_2)^2 + (p_2 - p_3)^3 + (p_3 - p_1)^2. \tag{4.10}$$

The so-called **Von Mises** yield stress σ_{ys} is related to I_2' as follows:

$$\sigma_{\mathrm{ys}}^2 = \frac{\mathrm{I}_2'}{2}. \tag{4.11}$$

I_1 and I_2' are of particular interest to us because they are related to the octahedral normal and octahedral shear stress, respectively.

The directions of the principal stresses can be found by substituting each of the three values of p in turn into (4.2) and, using (2.2), solving for each set of l, m, and n.

5

Stationary values of shear stress

Shear planes contain maximum shear stresses corresponding to two of the principal stresses (*cf.* the 2-dimensional situation, where $\tau_{\text{max}} = \frac{1}{2}(p_1 - p_2)$, evident from Mohr's circle of stress). Therefore, there will be three of these planes and their corresponding stresses.

Having determined the positions of the principal planes and values of the principal stresses, then rotate the Cartesian coordinate system to correspond with the directions of the normals of the principal planes, as demonstrated in Fig. 5.1. The stress components on the plane are:

$$\begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} p_1\, l \\ p_2\, m \\ p_3\, n \end{bmatrix}. \tag{5.1}$$

The resultant stress is

$$s = \sqrt{p_1^2 l^2 + p_2^2 m^2 + p_3^2 n^2}, \tag{5.2}$$

and the normal stress is

$$N = p_1 l^2 + p_2 m^2 + p_3 n^2. \tag{5.3}$$

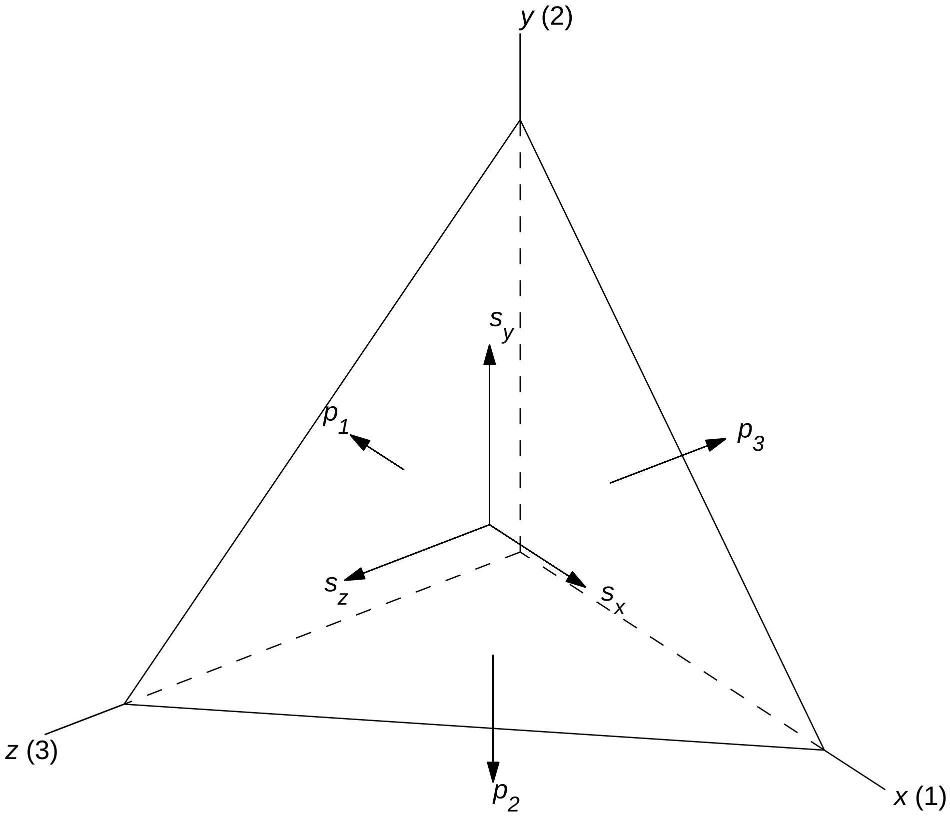

Figure 5.1: State of stress after rotation of the Cartesian coordinate system.

Therefore, the shear stress is given by

$$T^2 = S^2 - N^2 = (p_1^2 l^2 + p_2^2 m^2 + p_3^2 n^2) - (p_1 l^2 + p_2 m^2 + p_3 n^2)^2. \qquad (5.4)$$

Using (2.2), one direction cosine can be eliminated:

$$\begin{aligned} T^2 = \; & l^2(p_1^2 - p_3^2) + m^2(p_2^2 - p_3^2) + p_3^2 \\ & - \left\{ l^2(p_1 - p_3) + m^2(p_2 - p_3) + p_3 \right\}^2 . \end{aligned} \qquad (5.5)$$

The values of l and m, and therefore n, which maximize or minimize T are found by differentiating (5.5) with respect to l and m and equating to zero.

$$T\frac{\partial T}{\partial l} = l(p_1 - p_3)\left\{(p_1 + p_3) - 2\left[l^2(p_1 - p_3) + m^2(p_2 - p_3) + p_3\right]\right\} = 0;$$
$$T\frac{\partial T}{\partial m} = m(p_2 - p_3)\left\{(p_2 + p_3) - 2\left[l^2(p_1 - p_3) + m^2(p_2 - p_3) + p_3\right]\right\} = 0. \tag{5.6}$$

Clearly, these equations vanish if $l = m = 0$ and $n = 1$, but this locates the plane on which p_3 acts, which by definition is a principal plane on which the shear stress $T = 0$. The other minima can similarly be found on the other principal planes. The maxima are found as follows. Assume $l = 0$. This satisfies the first equation in (5.6). The second will vanish if

$$(p_2 + p_3) - 2\left[m^2(p_2 - p_3) + p_3\right] = 0, \tag{5.7}$$

or, simplified, if

$$(p_2 - p_3)(1 - 2m^2) = 0. \tag{5.8}$$

This holds for

$$m = \pm\frac{1}{\sqrt{2}}. \tag{5.9}$$

Thus,

$$n = \pm\frac{1}{\sqrt{2}}. \tag{5.10}$$

Substituting back into (5.4), the shear is found to be

$$T = \frac{1}{2}(p_2 - p_3). \tag{5.11}$$

By a similar approach, the other shear planes and shear stresses can be found; the results are summarized in Table 5.1. The planes on which the maximum shear stresses act are illustrated in Fig. 5.2. Note for instance that the shear stress $\frac{1}{2}(p_1 - p_2)$ acts on a plane given by $l = \pm\frac{1}{\sqrt{2}}$, $m = \pm\frac{1}{\sqrt{2}}$, $n = 0$. The normal to this plane is at right angles to the z (3) axis and bisects the right angle between the x (1) and y (2) axes.

Corresponding to each shear stress, there is a normal stress (5.3). With $l = m = \frac{1}{\sqrt{2}}$ and $n = 0$, $N = \frac{1}{2}(p_1 + p_2)$. Therefore, on this principal plane of shear, there is a shear stress $T = \frac{1}{2}(p_1 - p_2)$ and a normal stress $N = \frac{1}{2}(p_1 + p_2)$.

Table 5.1: Minimal and maximal shear stresses and corresponding direction cosines.

	$T_{\min}$			$T_{\max}$		
l	0	0	± 1	0	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$
m	0	± 1	0	$\pm\frac{1}{\sqrt{2}}$	0	$\pm\frac{1}{\sqrt{2}}$
n	± 1	0	0	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	0
T	0	0	0	$\frac{1}{2}(p_2 - p_3)$	$\frac{1}{2}(p_1 - p_3)$	$\frac{1}{2}(p_1 - p_2)$

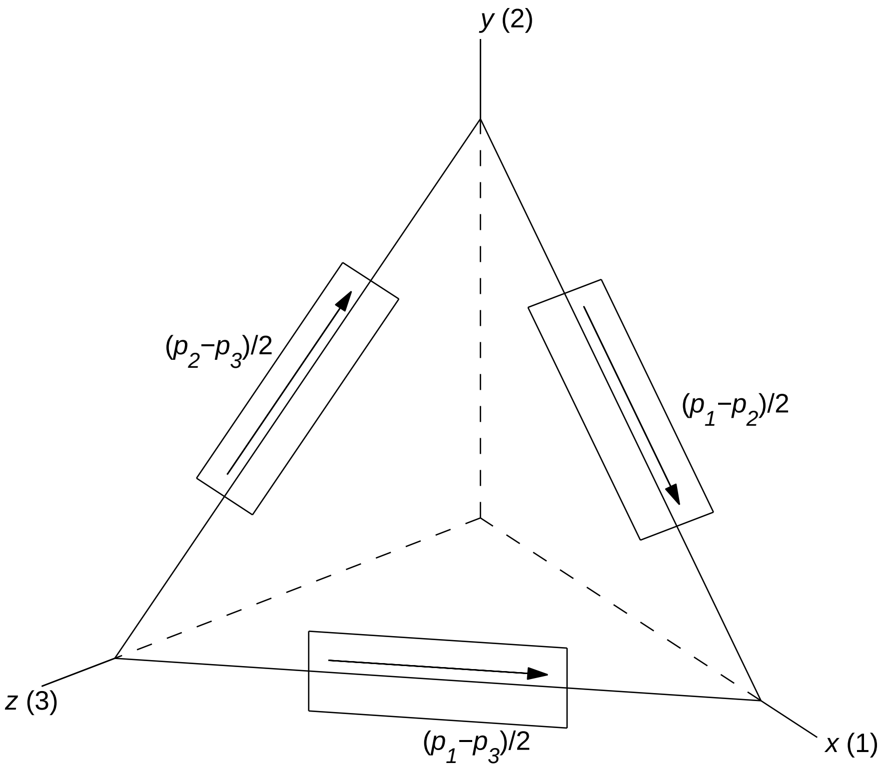

Figure 5.2: Planes of maximum shear stresses.

6
Octahedral stresses

It is possible to find a more meaningful indication of the state of stress, rather than the tensorial description, by considering the octahedral sections of the element. These are planes equally inclined to the principle stress axes. From consideration of Fig. 6.1 it is obvious that each of the eight planes will be subjected to the same value of direct stress and the same value of shear stress. This means that whereas it took six parameters to describe the state of stress in a set of rectangular sections, it takes only to describe the magnitudes (though not directions) of the stresses on octahedral sections. Note that octahedral normal and shear stresses correspond to two fundamental effects of uniform dilation and uniform shear.

The magnitudes of the octahedral stresses are easily obtained. If the direction cosines of the normal to the octahedral plane are l, m, and n, their values must be $l^2 = m^2 = n^2 = \frac{1}{3}$. Also,

$$\sigma_0 = \frac{1}{3}(p_1 + p_2 + p_3) = \frac{\mathrm{I}_1}{3} \tag{6.1}$$

and

$$\tau_0 = \frac{1}{3}\sqrt{(p_1 - p_2)^2 + (p_1 - p^3)^2 + (p_2 - p_3)^2} = \frac{1}{3}\sqrt{\mathrm{I}_2'}. \tag{6.2}$$

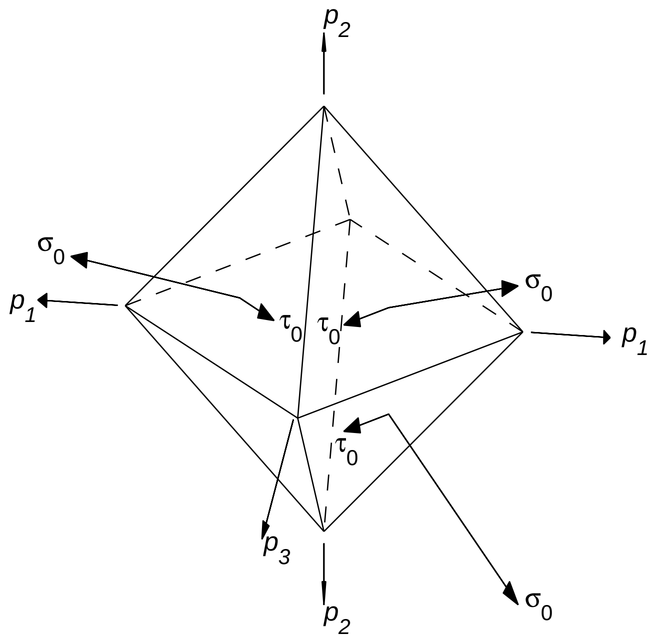

Figure 6.1: State of stress on octahedral sections of an element. σ_0 is the octahedral normal stress, τ_0 is the octahedral shear stress.

The octahedral stresses can obviously be quoted in terms of the components of any tensor of the state:

$$\sigma_0 = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z), \tag{6.3}$$

and

$$\tau_0 = \frac{1}{3}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2)}. \tag{6.4}$$

The octahedral shear stress is related to the Von Mises yield stress. Failure occurs when

$$\tau_0 = \frac{\sqrt{2}}{3}\sigma_{\text{ys}}. \tag{6.5}$$

7
Hydrostatic (dilational) and deviatoric stress tensors

It is sometimes necessary to take account of the directions of the octahedral shear stresses as well as their magnitudes. In this Chapter, it is shown that any general stress tensor can be split into two, where one part describes the effect of the direct stress on the octahedral section, whereas the other describes the effect of the octahedral shear stress.

The octahedral normal stress may be expressed in tensor form as

$$[\sigma_0] = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{bmatrix}. \tag{7.1}$$

This is called the hydrostatic of dilational stress tensor. It describes a state of stress without shearing present. The value of σ_0 is the average of the direct stresses on the leading diagonal of the general stress tensor $[\sigma]$. The remainder of the tensor, $[\sigma]-[\sigma_0]$, must describe the effect of the uniform octahedral shears and is called the deviatoric tensor of the state.

Note that the dilational tensor produces a change in volume without change in shape, while the deviatoric tensor produces a change in shape without change in volume.

8
Strains and displacements

To examine the relationship between strain and displacement, in the first instance only a 2-dimensional case will be considered. The displacements are assumed to be small, so that the strains are small compared to unity.

For the element ABCD in Fig. 8.1, stressing the element will result in a displacement of the element in addition to straining. Direct strains are responsible for increases in length of the sides of the element, while shear strains produce rotation of the lines, *i.e.*, change of shape. The increase in length of AB to A′B′ is $\mathrm{d}x \times$ (the variation of u with respect to x) $= \frac{\partial u}{\partial x}\,\mathrm{d}x$.

Therefore, the direct strain in the x-direction is

$$\varepsilon_x = \frac{(\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}x) - \mathrm{d}x}{\mathrm{d}x} = \frac{\partial u}{\partial x}. \tag{8.1}$$

Similarly, in the y-direction

$$\varepsilon_y = \frac{\partial u}{\partial y}. \tag{8.2}$$

Because of shear, AB and AD rotate through small angles θ and λ. For small strains ($\mathrm{d}x \gg \frac{\partial u}{\partial x}\,\mathrm{d}x$),

$$\theta \approx \tan\theta = \frac{\frac{\partial v}{\partial x}}{\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}x} = \frac{\partial v}{\partial x}, \tag{8.3}$$

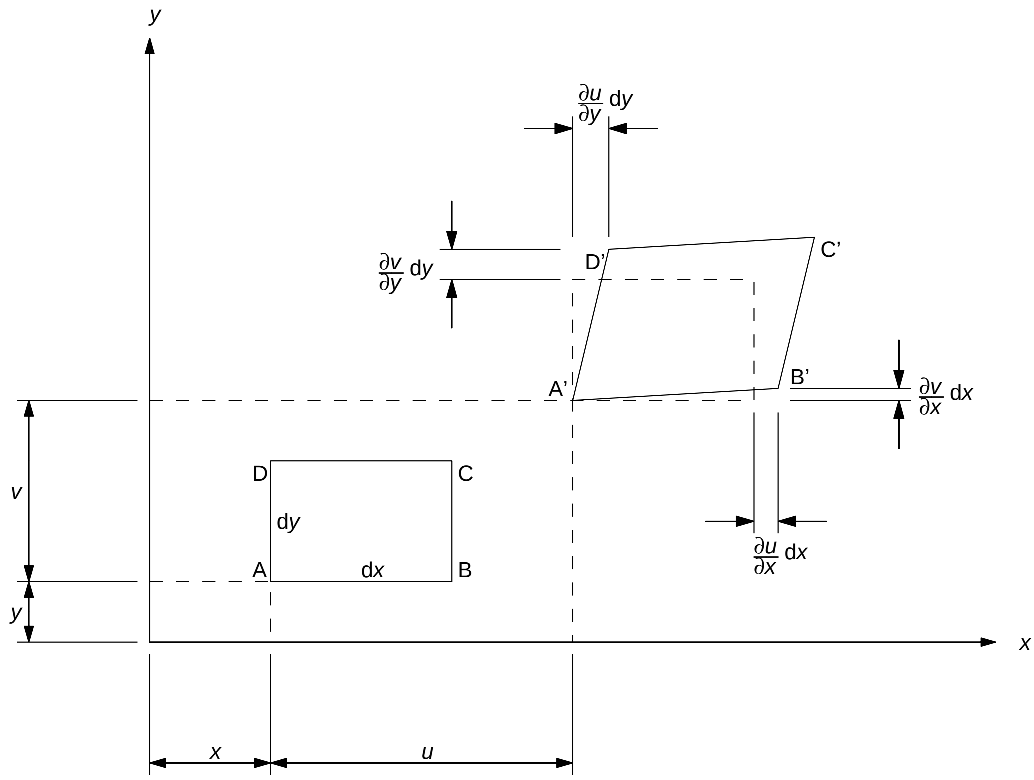

Figure 8.1: Stressing an element.

and

$$\lambda \approx \tan \lambda = \frac{\partial u}{\partial y}. \tag{8.4}$$

The shear strain is the change in angle between two lines originally at right angles:

$$\gamma_{xy} = \theta + \lambda = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}. \tag{8.5}$$

Clearly, $\gamma_{yx} = \gamma_{xy}$.

In the case of a 3-dimensional rectangular prism with sides dx, dy, and dz, a similar analysis will give

$$\varepsilon_x = \frac{\partial u}{\partial x}; \qquad \varepsilon_y = \frac{\partial v}{\partial y}; \qquad \varepsilon_z = \frac{\partial w}{\partial z};$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}; \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}; \quad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}. \tag{8.6}$$

Like stress, strain is a tensor quantity. It may be stored in a matrix

$$\begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{yx} & \frac{1}{2}\gamma_{zx} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{zy} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix}. \tag{8.7}$$

A strain tensor has all the properties of a stress tensor, and the same concepts derived in previous Chapters will apply to it:

1. Strain resolution
   The normal strain on a plane with direction cosines l, m, and n is given by

$$\varepsilon_N = \begin{bmatrix} l & m & n \end{bmatrix} \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{yx} & \frac{1}{2}\gamma_{zx} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{zy} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix}. \tag{8.8}$$

2. Transformation of axes
   The state of strain $[\varepsilon']$ can be found in any second coordinate system defined by the direction cosine matrix $[L]$ with respect to the first system by the equation

$$[\varepsilon'] = [L][\varepsilon][L]^{\mathrm{T}}. \tag{8.9}$$

3. Principal strains
   In any pattern of deformation there is one set of rectangular directions which suffer no relative rotation and are therefore free of shear. The linear strains of these lines are called principal strains. They are found by solving

$$\begin{bmatrix} \varepsilon_x - \varepsilon_p & \frac{1}{2}\gamma_{yx} & \frac{1}{2}\gamma_{zx} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y - \varepsilon_p & \frac{1}{2}\gamma_{zy} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z - \varepsilon_p \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0. \tag{8.10}$$

   In an isotropic material the principal axes of strain will coincide with the principal axes of stress.

4. Stationary values of shear strain
   The same technique can be applied to small strains to derive those planes where the maximum and minimum values of the shear strain exist: $\frac{1}{2}\gamma_{13} = \frac{1}{2}(\varepsilon_{p_1} - \varepsilon_{p_3})$.

5. Volumetric and octahedral stains
   The linear strain along each of the four octahedral axes which are equally inclined to the three principal axes is

$$\varepsilon_0 = \frac{1}{3}(\varepsilon_{p_1} + \varepsilon_{p_2} + \varepsilon_{p_3}) = \frac{1}{3}(\varepsilon_x + \varepsilon_y + \varepsilon_z), \tag{8.11}$$

   which is related to the volumetric strain $\Delta = \varepsilon_{p_1} + \varepsilon_{p_2} + \varepsilon_{p_3}$. This is proved by considering the volumetric strain of a prism with sides parallel to the principal axes. If the sides are a_1, a_2, a_3 and increase to $a_1 + \varepsilon_{p_1}$, $a_2 + \varepsilon_{p_2}$, $a_3 + \varepsilon_{p_3}$, the volumetric strain

$$\begin{aligned} \Delta &= \frac{\text{change in volume}}{\text{original volume}} \\ &= \frac{a_1 a_2 a_3 (1 + \varepsilon_{p_1})(1 + \varepsilon_{p_2})(1 + \varepsilon_{p_3}) - a_1 a_2 a_3}{a_1 a_2 a_3}. \end{aligned} \tag{8.12}$$

   Ignoring second order terms,

$$\Delta = \varepsilon_{p_1} + \varepsilon_{p_2} + \varepsilon_{p_3} = 3\varepsilon_0. \tag{8.13}$$

6. Hydrostatic and deviatoric strain tensors
   As for the stress tensor, the strain tensor can be split into its dilational and deviatoric parts. The following relationships apply, which will be discussed in the next Chapter:

$$\sigma_0[U] = \kappa \Delta [U] \tag{8.14}$$

   for the hydrostatic terms, and

$$[\sigma] - \sigma_0[U] = 2G\left([\varepsilon] - \frac{\Delta}{3}[U]\right) \tag{8.15}$$

   for the deviatoric terms. Here, κ is the Bulk modulus and G is the shear modulus.

9
Generalized Hooke's law

Hooke's law states that the strain produced in an elastic material is proportional to the applied stress. For a linear elastic material the principle of superposition applied, so that the deformations may be determined independently and added. Remember that a normal stress only produces a normal strain, while a shear stress only produces a shear strain. Therefore, the strain due to σ_x in the x-direction is $\frac{\sigma_x}{E}$ (extension), with the Poisson effect producing a strain of $-\frac{\nu\sigma_x}{E}$ in the y- and z-directions. Hence, for a triaxial stress, the total normal strains are given by

$$\begin{aligned}
\varepsilon_x &= \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} = \frac{1}{E}(\sigma_x - \nu(\sigma_y + \sigma_z))\,; \\
\varepsilon_y &= \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E} - \frac{\nu\sigma_z}{E} = \frac{1}{E}(\sigma_y - \nu(\sigma_z + \sigma_x))\,; \\
\varepsilon_z &= \frac{\sigma_z}{E} - \frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} = \frac{1}{E}(\sigma_z - \nu(\sigma_x + \sigma_y))\,,
\end{aligned} \tag{9.1}$$

where E is Young's modulus. Alternatively, in matrix form,

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix}. \tag{9.2}$$

By solving for the direct stresses we obtain:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix} = \frac{\nu E}{(1-2\nu)(1+\nu)} \begin{bmatrix} \frac{1-\nu}{\nu} & 1 & 1 \\ 1 & \frac{1-\nu}{\nu} & 1 \\ 1 & 1 & \frac{1-\nu}{\nu} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix}. \tag{9.3}$$

The shear strains are given by

$$\begin{bmatrix} \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \frac{1}{G} \begin{bmatrix} \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} \tag{9.4}$$

or

$$\begin{bmatrix} \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = G \begin{bmatrix} \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}. \tag{9.5}$$

The elastic constants, E, ν, and G, are related so that there are only two independent constants. To determine the relationship consider a 2-dimensional element in pure shear, *i.e.*, $\sigma_x = \sigma_y = \sigma_z = \tau_{xz} = \tau_{yz} = 0$ and $\tau_{xy} \neq 0$.

The principal stresses for this system, acting on planes at 45° to the xy-axes, are

$$\begin{aligned} p_1 &= \tau_{xy} \ ; \\ p_2 &= -\tau_{xy} \ ; \\ p_3 &= 0 \ . \end{aligned} \tag{9.6}$$

From Hooke's law,

$$\varepsilon_{p_1} = \frac{p_1}{E} - \frac{\nu p_2}{E} = \frac{\tau_{xy}}{E}(1+\nu). \tag{9.7}$$

If the strains are resolved into a direction at 45° to the xy-axes, $l = \cos\theta = m = \sin\theta = \frac{1}{\sqrt{2}}$. Combining this with (9.4) gives

$$\varepsilon_{p_1} = \frac{1}{2}\gamma_{xy} = \frac{\tau_{xy}}{2G}. \tag{9.8}$$

From (9.7) it now follows that

$$G = \frac{E}{2(1+\nu)}. \tag{9.9}$$

The Bulk modulus

Summing the three equations in (9.1) gives

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z). \tag{9.10}$$

The sum on the left is the invariant which determines the volumetric strain (8.13), while the right-hand term is the invariant which is equal to three times the hydrostatic stress (6.3). Therefore,

$$\Delta = \frac{3(1-2\nu)}{E}\sigma_0 = \frac{\sigma_0}{\kappa}, \tag{9.11}$$

where $\kappa = \frac{E}{3(1-2\nu)}$ is an elastic constant called the bulk modulus.

The relationship between the deviatoric tensors can easily be determined. From (9.3), by subtracting the second and third equation from twice the first, we obtain

$$\sigma_x - \sigma_0 = \frac{E}{1+\nu}\left(\varepsilon_x - \frac{\Delta}{3}\right) = 2G\left(\varepsilon_x - \frac{\Delta}{3}\right). \tag{9.12}$$

Thus, each direct stress of the deviatoric tensor, like each shear component, is related by $2G$ to the corresponding element of the deviatoric strain tensor.

Lamé's constant

It has been shown that the total stress tensor can be split into its hydrostatic and deviatoric parts. Each of these effects may then be described as a simple proportion of the corresponding strain effect. This is advantageous since we can often ignore the hydrostatic effect. In other cases, however, it may be more convenient to retain the total tensors intact. Here we show that the split relationship can be recombined to describe the total stress tensor as a linear function of the total strain tensor and the volumetric strain:

$$[\sigma] = 2G[\varepsilon] + \left(\kappa - \frac{2G}{3}\right)\Delta[U]. \tag{9.13}$$

Because κ and G are constants,

$$[\sigma] = 2G[\varepsilon] + \lambda\Delta[U], \tag{9.14}$$

where λ is Lamé's constant

$$\lambda = \kappa - \frac{2G}{3} = \frac{E\nu}{(1+\nu)(1-2\nu)}. \tag{9.15}$$

10

Equilibrium equations for three dimensions

Consider a small cuboid of finite size $\Delta x \times \Delta y \times \Delta z$, with stresses acting on each coordinate plane and their variations on opposite faces. Resolving the forces in the x-direction and equating to zero for equilibrium, ignoring body forces, yields

$$\begin{aligned} &\sigma_x \, \Delta z \, \Delta y - \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} \Delta x \right) \Delta z \, \Delta y \\ &+\tau_{yx} \, \Delta x \, \Delta z - \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \Delta y \right) \Delta x \, \Delta z \\ &+\tau_{zx} \, \Delta x \, \Delta y - \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \Delta z \right) \Delta x \, \Delta y \quad = 0, \end{aligned} \tag{10.1}$$

which after simplifying becomes

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0. \tag{10.2}$$

Resolving in the other directions similarly gives

$$[\sigma]\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}. \tag{10.3}$$

These equations ensure that equilibrium of the material is maintained.

In two dimensions they can be simplified to

$$\begin{aligned}\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} &= 0\ ;\\ \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} &= 0\ .\end{aligned} \tag{10.4}$$

11
Strain compatibility equations

The fundamental equations (8.6) relate the six components of strain in a 3-dimensional system to only three components of displacement. The strains therefore cannot be independent of one another.

Consider a 2-dimensional case. Since

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{11.1}$$

and

$$\varepsilon_y = \frac{\partial v}{\partial y}, \tag{11.2}$$

it follows that

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2} \tag{11.3}$$

and

$$\frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \partial y}. \tag{11.4}$$

Also,

$$\frac{\partial \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y}. \tag{11.5}$$

Therefore,

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}. \tag{11.6}$$

This is the condition of compatibility for 2-dimensional problems. The 3-dimensional equations of compatibility are derived in a similar manner:

$$\begin{aligned} \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} &= \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}; \\ \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} &= \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}; \\ \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} &= \frac{\partial^2 \gamma_{xz}}{\partial x \partial z} \end{aligned} \tag{11.7}$$

and

$$\begin{aligned} 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} &= \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right); \\ 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right); \\ 2\frac{\partial^2 \varepsilon_x}{\partial x \partial y} &= \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right). \end{aligned} \tag{11.8}$$

12
Plane strain

Consider a long cylinder held between fixed, rigid end plates. It is subjected to an internal pressure, *i.e.*, a purely lateral load. The cylinder can only deform in the x and y directions. So, $w = 0$ along the length of the cylinder. From (8.6),

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0 \tag{12.1}$$

and

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \ ; \\ \varepsilon_y &= \frac{\partial v}{\partial y} \ ; \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \ . \end{aligned} \tag{12.2}$$

This is a state of plain strain, where each point remains within its transverse plane following application of the load. Since $\varepsilon_z = 0$,

$$\varepsilon_z = \frac{1}{E}\left(\sigma_x - \nu(\sigma_x + \sigma_y)\right) = 0, \tag{12.3}$$

so that

$$\varepsilon_x = \frac{1-\nu^2}{E}\left(\sigma_x - \frac{\nu}{1-\nu}\sigma_y\right);$$
$$\varepsilon_y = \frac{1-\nu^2}{E}\left(\sigma_y - \frac{\nu}{1-\nu}\sigma_x\right); \qquad (12.4)$$
$$\gamma_{xy} = \frac{1}{G}\tau_{xy}.$$

The compatibility equation must also be satisfied by this stressing regime, for two dimensions that is:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}. \qquad (12.5)$$

Differentiating (12.4) and using (10.4) with (12.5) gives

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0. \qquad (12.6)$$

This is the equation of compatibility in terms of stress. Therefore, we now have three equations defining three unknowns:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0\ ;$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0\ ; \qquad (12.7)$$
$$\nabla^2(\sigma_x + \sigma_y) = 0\ .$$

13
Plane stress

Consider a thin plate whose loading is evenly distributed over the thickness parallel to the plane of the plate. On the faces of the plate, $\sigma_z = \tau_{zx} = \tau_{zy} = 0$. Since the plate is thin, it can be assumed the same is true through its thickness. The stress–strain relationships are therefore

$$\begin{aligned} \varepsilon_x &= \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) ; \\ \varepsilon_y &= \frac{1}{E}\left(\sigma_y - \nu\sigma_z\right) ; \\ \varepsilon_z &= -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right) ; \\ \gamma_{xy} &= \frac{1}{G}\,\tau_{xy} . \end{aligned} \tag{13.1}$$

Again, using the same strain compatibility equation with the equations for equilibrium, it can be proven that

$$\nabla^2(\sigma_x + \sigma_y) = 0, \tag{13.2}$$

which is the same equation of compatibility as for the plane strain case.

14
Polar coordinates

Where axial symmetry exists, it is much more convenient to use polar coordinates. The polar coordinate system (r, θ) and the Cartesian coordinate system (x, y) are related by the expressions

$$\begin{aligned} x &= r\cos\theta; \\ y &= r\sin\theta; \\ r^2 &= x^2 + y^2; \\ \theta &= \tan^{-1}\tfrac{y}{x}. \end{aligned} \tag{14.1}$$

Strain components in polar coordinates

Consider a pure radial strain, where the internal face of the element with original length $\mathrm{d}r$ is displaced by a radial distance u and a tangential distance v. The strained length of the element is $\mathrm{d}r + \frac{\partial u}{\partial r}\mathrm{d}r$. Therefore, the radial strain

$$\varepsilon_r = \frac{\frac{\partial u}{\partial r}\mathrm{d}r}{\mathrm{d}r} = \frac{\partial u}{\partial r}. \tag{14.2}$$

The tangential strain arises from the radial displacement

$$\varepsilon_{\theta_r} = \frac{(r+u)\,\mathrm{d}\theta - r\,\mathrm{d}\theta}{r\,\mathrm{d}\theta} = \frac{u}{r} \tag{14.3}$$

and the tangential displacement

$$\varepsilon_{\theta_\theta} = \frac{\left(r + \frac{\partial v}{\partial \theta}\right) \mathrm{d}\theta - r \, \mathrm{d}\theta}{r \, \mathrm{d}\theta} = \frac{1}{r} \frac{\partial v}{\partial \theta}. \tag{14.4}$$

Thus,

$$\varepsilon_\theta = \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta}. \tag{14.5}$$

The shear strain also has two components arising from each of the u and v components. The total shear strain is

$$\gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{v}{r}. \tag{14.6}$$

So, in two dimensions, the strain components are

$$\begin{aligned} \varepsilon_r &= \frac{\partial u}{\partial r}; \\ \varepsilon_\theta &= \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta}; \\ \gamma_{r\theta} &= \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{v}{r}. \end{aligned} \tag{14.7}$$

Hooke's law

To write Hooke's law in polar coordinates, the x and y subscripts are simply replaced by r and θ. For example, in plane stress:

$$\begin{aligned} \varepsilon_r &= \frac{1}{E} \left(\sigma_r - \nu \sigma_\theta\right) ; \\ \varepsilon_\theta &= \frac{1}{E} \left(\sigma_\theta - \nu \sigma_r\right) ; \\ \gamma_{r\theta} &= \frac{1}{G} \, \tau_{r\theta} \; . \end{aligned} \tag{14.8}$$

Equilibrium equations

By considering equilibrium of the radial and circumferential forces acting on an element of unit thickness, we obtain

$$\begin{aligned} \frac{r\partial\sigma_r}{\partial r} + \frac{\partial\tau_{r\theta}}{\partial\theta} + \sigma_r - \sigma_\theta = 0 \ ; \\ \frac{\partial\sigma_\theta}{\partial\theta} + \frac{r\partial\tau_{r\theta}}{\partial r} + 2\tau_{r\theta} = 0 \ . \end{aligned} \tag{14.9}$$

Strain compatibility equation

The three compatibility equations (14.7) defining the strain components can be combined to give the equation of compatibility

$$\frac{\partial^2\varepsilon_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\varepsilon_r}{\partial\theta^2} + \frac{2}{r}\frac{\partial\varepsilon_\theta}{\partial r} - \frac{1}{r}\frac{\partial\varepsilon_r}{\partial r} = \frac{1}{r}\frac{\partial^2\gamma_{r\theta}}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\gamma_{r\theta}}{\partial\theta}. \tag{14.10}$$

Stress compatibility equation

Substituting (14.8) into (14.10) yields

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\sigma_r + \sigma_\theta) = 0. \tag{14.11}$$

For axially symmetrical problems, this may be further simplified to

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)(\sigma_r + \sigma_\theta) = 0. \tag{14.12}$$

15
Stress functions

The preceding theory has shown that the solution of 2-dimensional problems in elasticity requires the integration of the differential equations of equilibrium together with the compatibility equations and the boundary conditions for the problem.

In 1862, G.B. Airy proposed a stress function $\phi(x, y)$ that satisfied the above conditions, defined by:

$$\begin{aligned} \sigma_x &= \frac{\partial^2 \phi}{\partial y^2}; \\ \sigma_y &= \frac{\partial^2 \phi}{\partial x^2}; \\ \tau_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y}. \end{aligned} \tag{15.1}$$

When substituted into 13.2, we obtain

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2}\right) = 0 \tag{15.2}$$

or

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0, \tag{15.3}$$

which is equal to

$$\nabla^4\phi = 0. \tag{15.4}$$

This is a biharmonic differential equation representing the compatibility equation for stress. It is difficult to solve for any but the simplest of problems. The advantage of the Airy stress function, however, is that functions of x and y can be devised which represent particular stressing regimes. This is known as the semi-inverse method of analysis, while developing ϕ from known boundary conditions is the direct method.

Examples

1. Single powers of x and y are no use because they give $\sigma_x = \sigma_y = \tau_{xy} = 0$.

2. $\phi = Ax^2$ represents simple tension in the x-direction: $\sigma_x = 0$; $\sigma_y = 2A$; $\tau_{xy} = 0$.

3. $\phi = Axy$ represents pure shear parallel to the axes: $\sigma_x = \sigma_y = 0$; $\tau_{xy} = -A$.

4. $\phi = Ay^3$ represents pure bending: $\sigma_x = 6Ay$; $\sigma_y = 0$; $\tau_{xy} = 0$.

5. $\phi = Ax^4$ does not satisfy (15.4).

6. $\phi = A\left(xy^3 - \frac{3}{4}xyh^2\right)$ represents a thin, cantilever of thickness h, end loaded by force F: $\sigma_x = 6Axy$; $\sigma_y = 0$; $\tau_{xy} = 3A\left(\frac{h^2}{4} - y^2\right)$.
   The boundary conditions are satisfied: $\tau_{xy} = 0$ at $y = \pm\frac{h}{2}$ for all values of x; $\sigma_y = 0$ at $y = \pm\frac{h}{2}$ for all values of x; $\sigma_x = 0$ at $x = 0$ for all values of y.
   The magnitude of F can be found by different approaches, for example: $P = 2\int_0^{\frac{h}{2}} \tau_{xy} b \, \mathrm{d}y$ for beam width b, giving $F = \dfrac{Ah^3b}{2}$.

16

Stress functions in polar coordinates

The Airy stress function may also be used in polar coordinates:

$$\begin{aligned}\sigma_r &= \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r};\\ \sigma_\theta &= \frac{\partial^2\phi}{\partial r^2};\\ \tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right).\end{aligned} \tag{16.1}$$

The biharmonic equation then becomes

$$\nabla^4\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right) = 0. \tag{16.2}$$

Hole in a plate subjected to pure tension

Consider the stress function $\phi = \frac{1}{2}\sigma_0 y^2$, which defines a plate under tension in the x-direction. To analyse the stresses in a plate with a hole, it is more convenient to use polar coordinates, and therefore replace y by $r\sin\theta$, so that

$$\phi = \frac{\sigma_0}{2}r^2\sin^2\theta \tag{16.3}$$

or

$$\phi = \frac{\sigma_0}{4} r^2 (1 - \cos 2\theta). \tag{16.4}$$

Therefore, from (16.1), the stresses in the plate without a hole can be written as

$$\begin{aligned} \sigma_r &= \tfrac{1}{2}\sigma_0(1 + \cos 2\theta); \\ \sigma_\theta &= \tfrac{1}{2}\sigma_0(1 - \cos 2\theta); \\ \tau_{r\theta} &= -\tfrac{1}{2}\sigma_0 \sin 2\theta. \end{aligned} \tag{16.5}$$

Assume that for a plate with a hole with radius a the stress function will be of the same form as

$$\phi = f_1(r) + f_2(r)(1 - \cos 2\theta). \tag{16.6}$$

After substituting (16.6) into the biharmonic equation (16.2) and solving for f_1 and f_2, it is found that

$$\begin{aligned} f_1 &= \frac{\sigma_0}{4} r^2 - \frac{a^2 \sigma_0}{2} \ln r; \\ f_2 &= -\frac{\sigma_0}{4} r^2 - \frac{a^4}{r^4}\frac{\sigma_0}{4} + \frac{a^2\sigma_0}{2}, \end{aligned} \tag{16.7}$$

so that

$$\begin{aligned} \sigma_r &= \tfrac{1}{2}\sigma_0 \left[\left(1 - \frac{a^2}{r^2}\right) + \left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right) \cos 2\theta \right]; \\ \sigma_\theta &= \tfrac{1}{2}\sigma_0 \left[\left(1 + \frac{a^2}{r^2}\right) - \left(1 + \frac{3a^4}{r^4}\right) \cos 2\theta \right]; \\ \tau_{r\theta} &= -\tfrac{1}{2}\sigma_0 \left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right) \sin 2\theta. \end{aligned} \tag{16.8}$$

Note that σ_θ has a maximum along $\theta = \pm\frac{\pi}{2}$. At $r = a$, $(\sigma_\theta)_{\mathrm{max}} = 3\sigma_0$. But for a plate without a hole, using (16.5), $(\sigma_\theta)_{\mathrm{max}} = \sigma_0$. Therefore, the stress concentration factor of the hole

$$K = \frac{3\sigma_0}{\sigma_0} = 3. \tag{16.9}$$

The distribution of stresses across the central section of the plate is shown in Fig. 16.1. For $\theta = \pm\frac{\pi}{2}$, $\tau_{xy} = 0$. Note that the effect of the hole is only negligible beyond distances over nine radii, with the stress field returning to that of simple tension.

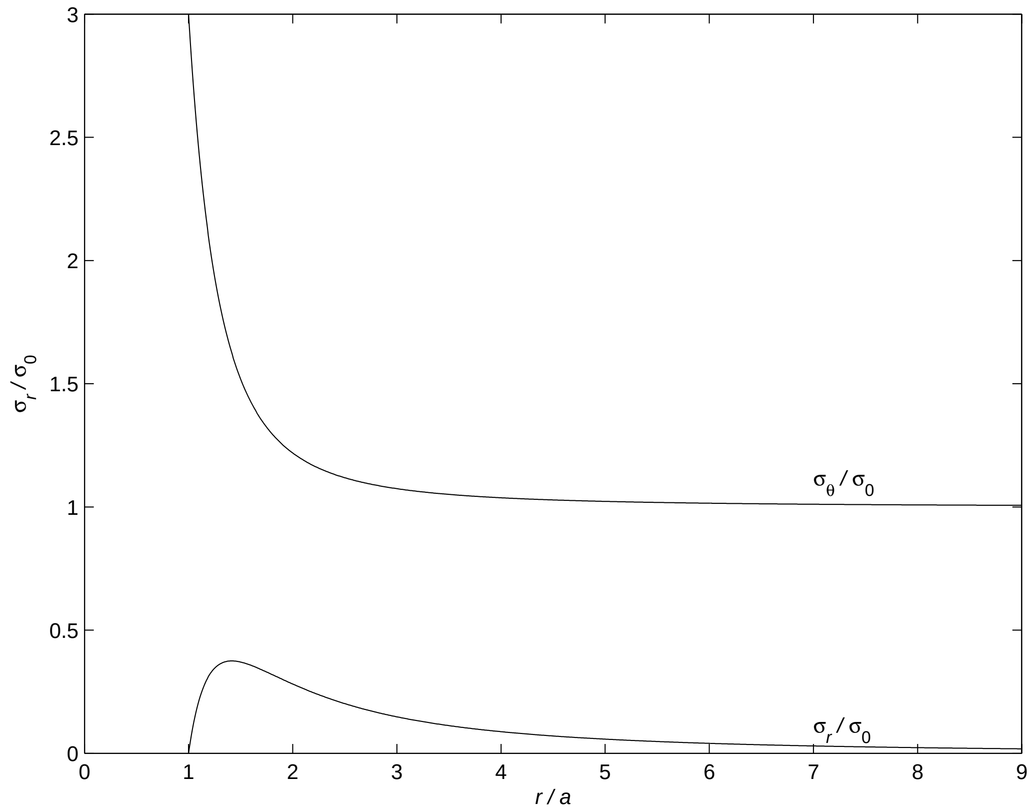

Figure 16.1: Variation of σ_r and σ_θ across the central cross-section of a plate containing a small hole of diameter 2a.

For an elliptical hole with width 2a in the x-direction and length 2b in the y-direction, at $\theta = \pm\frac{\pi}{2}$,

$$(\sigma_\theta)_{\max} = \sigma_0 \left(1 + \frac{2b}{a}\right). \tag{16.10}$$

Thus, if $b : a = 20 : 1$, $K = 41$. Therefore, a crack, which can be considered as a very long ellipse, has a very high stress concentration factor.

17
Torsion of non-circular prisms

For circular bars under torsion,

$$\frac{M_\mathrm{t}}{J} = \frac{\tau}{r} = \frac{G\theta}{l}, \tag{17.1}$$

where G is the shear modulus, J is the polar moment of inertia, *i.e.*, $\frac{1}{2}\pi a^4$ for a circular cross-section of radius a, l is the length of the bar, M_t is the applied torque, r is the radius at which the stress is required, θ is the angle of twist, and τ is the shear stress.

We may assume that all transverse sections remain plain after twisting and that the cross-sections are undistorted in their individual planes, *i.e.*, the shear strain is a linear function of r. Initially plane cross-sections experience out-of-plane deformation or **warping** (*cf.* Fig. 17.1).

General solution of the torsion problem

Consider a bar of arbitrary cross-section. The origin of the coordinate system is at the centre of twist, where the displacements, u and v, are zero. It is assumed that the warping deformation is independent of the axial position:

$$w = f(x, y), \tag{17.2}$$

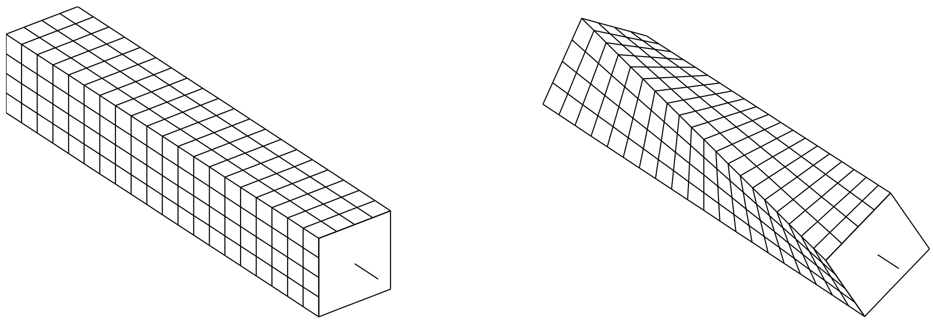

Figure 17.1: Rectangular bar *(left)* subjected to warping *(right)*.

that the projection of the xy-plane of any warped cross-section rotates as a rigid body, and that the twist per unit length θ_1 is constant. Consider Fig. 17.2 which shows any cross-section of the bar at distance z from the origin. Point P has moved to P' under a torsional load. Assuming that θ_1 is small and that no rotation occurs at the origin, then

$$\begin{aligned} u &= -r\theta_1 z \sin\alpha = -y\theta_1 z; \\ v &= r\theta_1 z \cos\alpha = x\theta_1 z. \end{aligned} \tag{17.3}$$

The strains in the bar are therefore

$$\begin{aligned} \varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} &= 0; \\ \gamma_{yz} &= \frac{\partial w}{\partial y} + x\theta_1; \\ \gamma_{zx} &= \frac{\partial w}{\partial x} - y\theta_1. \end{aligned} \tag{17.4}$$

Therefore, the stresses are

$$\begin{aligned} \sigma_x = \sigma_y = \sigma_z = \tau_{xy} &= 0; \\ \tau_{yz} &= G\left(\frac{\partial w}{\partial y} + x\theta_1\right); \\ \tau_{zx} &= G\left(\frac{\partial w}{\partial x} - y\theta_1\right). \end{aligned} \tag{17.5}$$

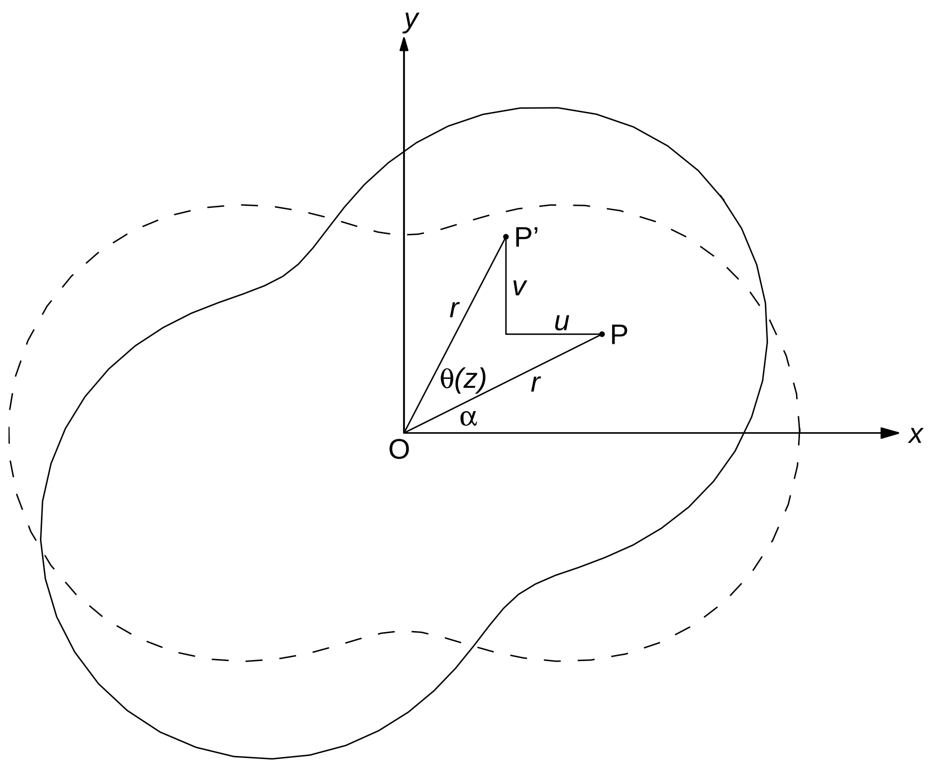

Figure 17.2: Cross-section of a warped bar. Point P has moved to P' under a torsional load.

Substituting these into the equilibrium equations (10.3) gives:

$$\begin{aligned}\frac{\partial \tau_{yz}}{\partial z} &= 0; \\ \frac{\partial \tau_{zx}}{\partial z} &= 0; \\ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} &= 0.\end{aligned} \tag{17.6}$$

Furthermore,

$$\frac{\partial \tau_{zx}}{\partial y} - \frac{\partial \tau_{yz}}{\partial x} = G\left(\frac{\partial^2 w}{\partial x \partial y} - \theta_1\right) - G\left(\frac{\partial^2 w}{\partial x \partial y} + \theta_1\right) = -2G\theta_1. \tag{17.7}$$

Equations (17.6) and (17.7) are usually solved by using a stress function $\phi(x, y)$, called the **Prandtl stress function**, such that

$$\begin{aligned} \tau_{yz} &= -\frac{\partial \phi}{\partial x}; \\ \tau_{zx} &= \quad \frac{\partial \phi}{\partial y}. \end{aligned} \tag{17.8}$$

Equation (17.7) therefore becomes

$$\nabla^2 \phi = -2G\theta_1 = \text{constant}. \tag{17.9}$$

Boundary conditions

1. The load-free sides of the bar
   ϕ may be considered as a surface defined by x and y, so that τ_{xy} is the slope of the ϕ-surface in the x-direction. At a free edge of any section, the normal shear stress, and therefore the slope of ϕ, must be equal to zero. Thus ϕ is a constant around the boundary. We may arbitrarily choose $\phi = 0$.

2. The loaded ends of the bar
   The total torque at the end surface with area A is

$$\begin{aligned} M_\text{t} &= -\int_A y\,\tau_{zx}\,\text{d}x\,\text{d}y + \int_A x\,\tau_{yz}\,\text{d}x\,\text{d}y \\ &= -\int_A y\left(\frac{\partial \phi}{\partial y}\right)\text{d}x\,\text{d}y + \int_A x\left(\frac{\partial \phi}{\partial x}\right)\text{d}x\,\text{d}y. \end{aligned} \tag{17.10}$$

   Let's concentrate on the first term of (17.10): $-\int_A y\left(\frac{\partial \phi}{\partial y}\right)\text{d}x\,\text{d}y$. For constant x, $\frac{\partial \phi}{\partial y}\,\text{d}y = \text{d}\phi$, so that integrating by parts from C to D (*cf.* Fig. 17.3) gives

$$-\iint_\text{C}^\text{D} y\,\text{d}\phi\,\text{d}x = -\int\left\{\left[y\phi\right]_\text{C}^\text{D} - \int_\text{C}^\text{D}\phi\,\text{d}y\right\}\text{d}x. \tag{17.11}$$

   Because ϕ is constant around the edge, $\phi_\text{C} = \phi_\text{D}$, and

$$-\iint_\text{C}^\text{D} y\,\text{d}\phi\,\text{d}x = \iint_\text{C}^\text{D}\phi\,\text{d}y\,\text{d}x = \int_A \phi\,\text{d}A. \tag{17.12}$$

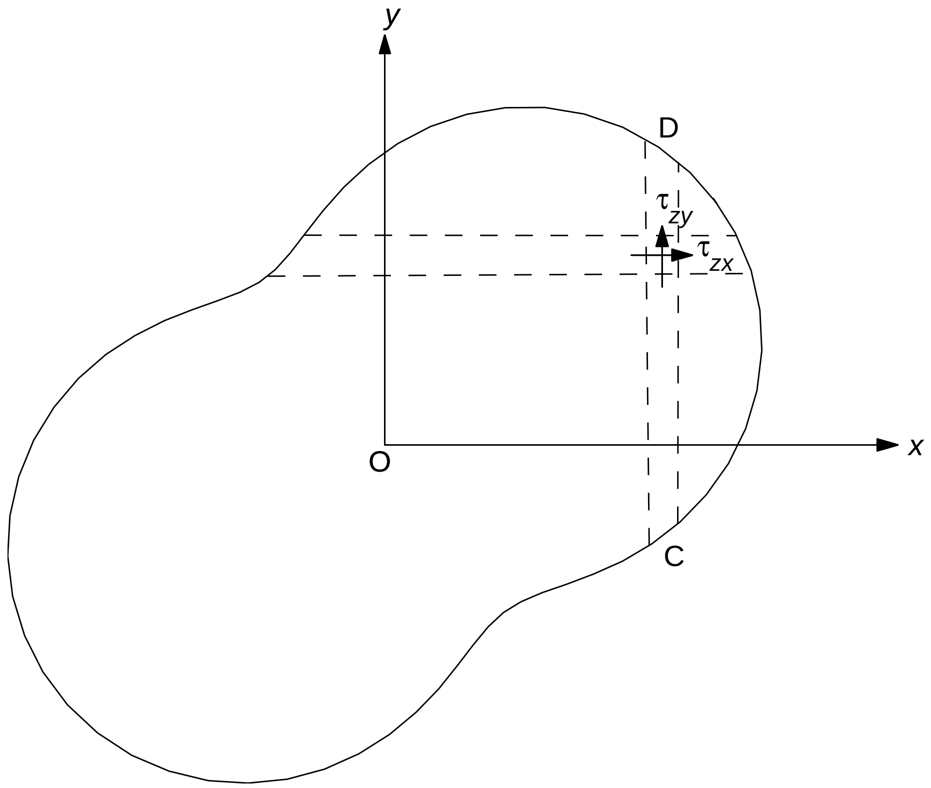

Figure 17.3: Cross-section of a warped bar with an element dx×dy.

A similar result is found for the second integral in (17.10). Therefore, the total torque

$$M_t = 2\int_A \phi \, dA. \tag{17.13}$$

Thus, the total torque equals twice the volume under the ϕ-surface if $\phi = 0$ at the perimeter.

18
Prandtl's membrane analogy

Consider an edge-supported membrane (soap film) spanning a hole cut in a plate. The shape of the hole is the same as that of the twisted bar to be studied. Let the membrane be blown-up by a small pressure p, so that all deflections are small. Let the tensile force per unit length in the membrane be S (*cf.* Fig. 18.1).

Considering the normal equilibrium of an element $\mathrm{d}x \times \mathrm{d}y$, we obtain

$$\begin{aligned} p\,\mathrm{d}x\,\mathrm{d}y = {} & S\,\mathrm{d}y\frac{\partial z}{\partial x} - S\,\mathrm{d}y\frac{\mathrm{d}}{\mathrm{d}x}\left(z + \frac{\partial z}{\partial x}\,\mathrm{d}x\right) \\ & + S\,\mathrm{d}x\frac{\partial z}{\partial y} - S\,\mathrm{d}x\frac{\mathrm{d}}{\mathrm{d}y}\left(z + \frac{\partial z}{\partial y}\,\mathrm{d}y\right), \end{aligned} \tag{18.1}$$

which leads to

$$\nabla^2 z = -\frac{p}{S}. \tag{18.2}$$

This may be compared to (17.9), taking into account a scaling factor:

$$\phi = \frac{2G\theta_1}{p/S}\,z, \tag{18.3}$$

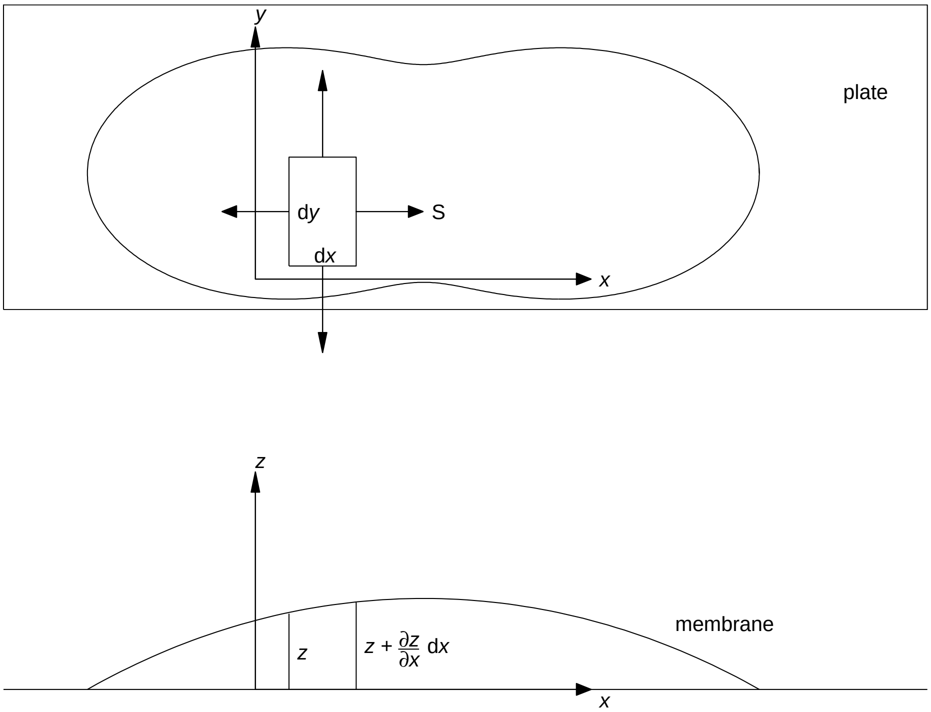

Figure 18.1: Edge-supported membrane spanning a hole in a plate.

so that

$$\begin{aligned}\tau_{xz} &= \frac{\partial\phi}{\partial y} = \frac{2G\theta_1}{p/S}\frac{\partial z}{\partial y};\\ \tau_{yz} &= -\frac{\partial\phi}{\partial x} = -\frac{2G\theta_1}{p/S}\frac{\partial z}{\partial x},\end{aligned} \tag{18.4}$$

and

$$M_t = 2\int_A \phi \, dA = \frac{2G\theta_1}{p/S}\int_A z \, dA. \tag{18.5}$$

19
Torsion of thin-walled members — open sections

Consider a rectangular section where the length l is much greater than the width $b = 2x$ (*cf.* Fig. 19.1).

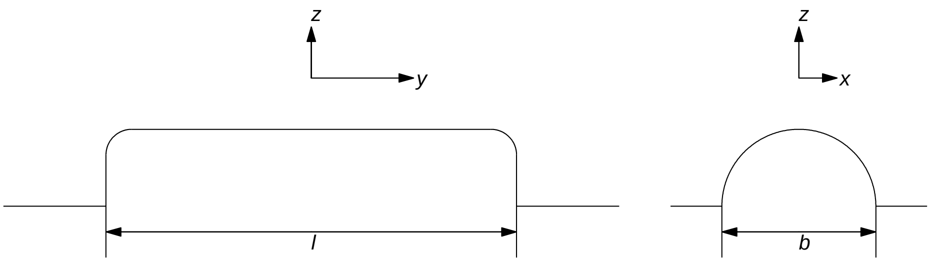

Figure 19.1: Rectangular section with length l and width b.

Assuming the membrane is squared off at the ends,

$$p\,l\,2x = 2\,S\,l\left(-\frac{\mathrm{d}z}{\mathrm{d}x}\right), \tag{19.1}$$

or

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{p}{S}\,x, \tag{19.2}$$

giving

$$z = -\frac{p}{S}\frac{x^2}{2} + \text{constant}. \tag{19.3}$$

At $x = \frac{b}{2}$, $z = 0$. Therefore,

$$\text{constant} = \frac{p}{S}\frac{b^2}{8}, \tag{19.4}$$

so that

$$z = \frac{p}{2S}\left(\frac{b^2}{4} - x^2\right). \tag{19.5}$$

Rephrasing (18.5),

$$M_\mathrm{t} = \frac{4G\theta_1}{p/s} \times \text{volume under the membrane.} \tag{19.6}$$

The volume under the membrane

$$\int_A z\ \mathrm{d}A = 2\int_0^{\frac{b}{2}} z\ l\,\mathrm{d}x = \frac{p\,l}{S}\int_0^{\frac{b}{2}}\left(\frac{b^2}{4} - x\right)\mathrm{d}x = \frac{p}{S}\frac{lb^3}{12}. \tag{19.7}$$

Therefore, the torque transmitted by a rectangular section

$$M_\mathrm{t} = \frac{G\theta_1 lb^3}{3}, \tag{19.8}$$

and the shear stress

$$\tau = \frac{2G\theta_1}{p/S}\frac{\mathrm{d}z}{\mathrm{d}x} = -2G\theta_1 x. \tag{19.9}$$

20

Torsion of thin-walled members — closed sections

The membrane analogy can also be applied to closed sections if care is taken defining the shape of the membrane.

Consider a hollow thin-walled member with wall thickness b. The membrane surface is approximated by a flat surface of thickness h, as shown in Fig. 20.1. The slope of the membrane $\frac{h}{b}$ is taken constant over the thin wall. If L is the perimeter length,

$$p\,A = \int_0^L \frac{h}{b} S \,\mathrm{d}L, \tag{20.1}$$

or

$$\frac{p}{S} = \frac{h}{A} \int_0^L \frac{1}{b} \,\mathrm{d}L. \tag{20.2}$$

Also,

$$\tau = \frac{\partial \phi}{\partial r} = \frac{2G\theta_1}{p/S} \frac{\mathrm{d}z}{\mathrm{d}r} = \frac{2G\theta_1}{p/S} \frac{h}{b}. \tag{20.3}$$

Therefore,

$$\tau = \frac{2G\theta_1}{\dfrac{h}{A} \displaystyle\int_0^L \frac{1}{b} \,\mathrm{d}L} \frac{h}{b} = \frac{2G\theta_1 A}{b \displaystyle\int_0^L \frac{1}{b} \,\mathrm{d}L}. \tag{20.4}$$

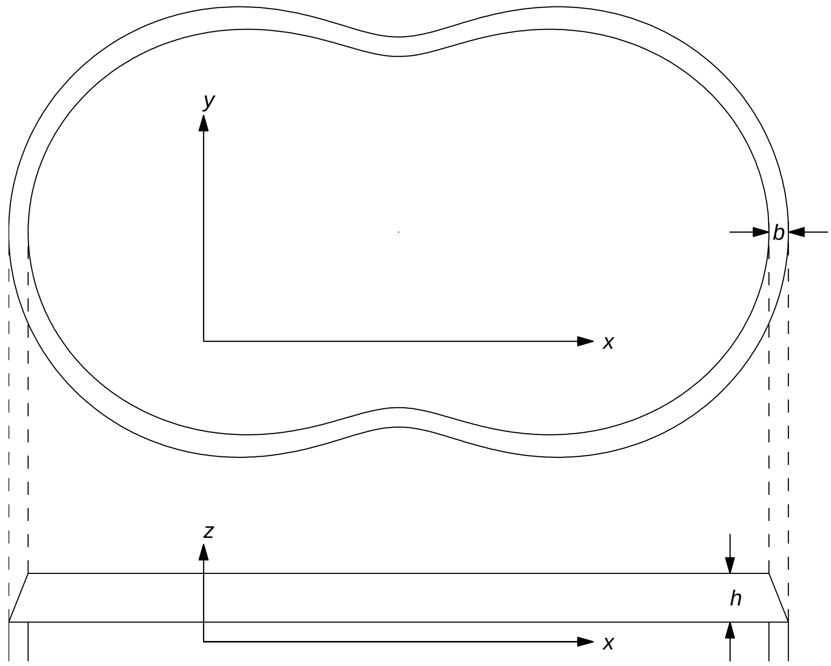

Figure 20.1: Closed section approximated by a flat surface and constant slopes over the thin wall.

For a constant the wall thickness,

$$\tau = \frac{2G\theta_1 A}{L}. \tag{20.5}$$

The torque carried by the section

$$M_\mathrm{t} = \frac{4G\theta_1}{p/S}\int_A z \, \mathrm{d}A = \frac{4G\theta_1}{p/S} hA = \frac{4G\theta_1 hA}{\dfrac{h}{A}\displaystyle\int_0^L \frac{1}{b}\,\mathrm{d}L} = \frac{4G\theta_1 A^2}{\displaystyle\int_0^L \frac{1}{b}\,\mathrm{d}L}. \tag{20.6}$$

For a constant the wall thickness, the total torque transmitted

$$M_\mathrm{t} = \frac{4G\theta_1 A^2 b}{L}, \tag{20.7}$$

so that

$$\tau = \frac{M_\mathrm{t}}{2Ab}. \tag{20.8}$$

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