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Qualitative analysis of solutions of obstacle elliptic inclusion problem with fractional Laplacian

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Abstract. In this paper, we study an elliptic obstacle problem with a generalized fractional Laplacian and a multivalued operator which is described by a generalized gradient. Under quite general assumptions on the data, we employ a surjectivity theorem for multivalued mappings generated by the sum of a maximal monotone multivalued operator and a bounded multivalued pseudomonotone mapping to prove that the set of weak solutions to the problem is nonempty, bounded and closed. Then, we introduce a sequence of penalized problems without obstacle constraints. Finally, we prove that the Kuratowski upper limit of the sets of solutions to penalized problems is nonempty and is contained in the set of solutions to original elliptic obstacle problem, i.e., $\emptyset \neq w\text{-}\limsup_{n\to\infty}\mathcal{S}_n = s\text{-}\limsup_{n\to\infty}\mathcal{S}_n \subset \mathcal{S}$.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $s \in (0,1)$ be such that $N > 2s$ and $\Omega^{\mathsf{C}} := \mathbb{R}^N\backslash\Omega$. We consider the following elliptic inclusion problem involving a generalized fractional Laplace operator, a multivalued term and obstacle effect

$$\begin{cases} \mathcal{L}_K u(x) + \partial j(x,u(x)) \ni f(x) & \text{in } \Omega \\ u(x) \leq \Phi(x) & \text{in } \Omega \\ u(x) = 0 & \text{in } \Omega^{\mathsf{C}}, \end{cases} \tag{1.1}$$

where the operator $\mathcal{L}_K$ stands for the generalized nonlocal fractional Laplace operator defined as follows

$$\mathcal{L}_K u(x) := -\int_{\mathbb{R}^N} \big(u(x+y) + u(x-y) - 2u(x)\big)K(y)\,\mathrm{d}y \quad \text{for a.e. } x \in \mathbb{R}^N,$$

and the term $\partial j(x,u(x))$ denotes the Clarke's generalized gradient of the locally Lipschitz function $j\colon\Omega\times\mathbb{R}\to\mathbb{R}$ with respect to the last variable. Through the paper, we assume that the kernel function K satisfies the following condition:
$\underline{H(K)}$: $K\colon\mathbb{R}^N\backslash\{0\}\to(0,+\infty)$ is such that

(i) the function $x\mapsto\min\{|x|^2,1\}K(x)$ belongs to $L^1(\mathbb{R}^N)$.
(ii) for all $x\in\mathbb{R}^N\backslash\{0\}$, there exists a constant $m_K>0$ such that

$$K(x) \geq m_K|x|^{-(N+2s)}.$$

(iii) for each $x\in\mathbb{R}^N\backslash\{0\}$, we have $K(x)=K(-x)$.

The weak solutions of problem (1.1) are understood as follows.

Definition 1.1. We say that $u \in X_0$ is a weak solution of problem (1.1) if $u(x) \le \Phi(x)$ for a.e. $x \in \Omega$ and the inequality holds

$$\int_{\mathbb{R}^N} (v(x) - u(x))\mathcal{L}_K(u)(x)\,\mathrm{d}x + \int_{\Omega} j^0(x, u(x); v(x) - u(x))\,\mathrm{d}x \ge \int_{\Omega} f(x)(v(x) - u(x))\,\mathrm{d}x$$

for all $v \in X_0$ with $v(x) \le \Phi(x)$ for a.e. $x \in \Omega$.

Particularly, if the kernel function K is specialized to the following formulation

$$K(x) := |x|^{-(N+2s)} \text{ for all } x \in \mathbb{R}^N \backslash \{0\},$$

and for some $s \in (0,1)$ such that $2s < N$, i.e., the generalized fractional nonlocal Laplace operator $\mathcal{L}_K$ becomes the classical fractional Laplace operator $(-\Delta)^s$,

$$(-\Delta)^s u(x) := -\int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}}\,\mathrm{d}y \text{ for a.e. } x \in \mathbb{R}^N,$$

then our problem (1.1) reduces to the following one

$$\begin{cases} (-\Delta)^s u(x) + \partial j(x, u(x)) \ni f(x) & \text{in } \Omega \\ u(x) \le \Phi(x) & \text{in } \Omega \\ u(x) = 0 & \text{in } \Omega^{\mathsf{C}}, \end{cases} \tag{1.2}$$

which was considered by Migórski et al. [35].

Problem (1.1) combines several interesting phenomena like a generalized fractional Laplace operator, a multivalued mapping provided by the Clarke generalized subdifferential and an obstacle inequality. However, in the present paper, we first apply the surjectivity theorem for multivalued mappings due to Le [29] to prove that the set of weak solutions to problem (1.1) is nonempty, bounded and closed. Then, by using penalty method, we consider a sequence of penalized problems without obstacle constraints corresponding to problem (1.1) (see problem (4.1)). Furthermore, we explore a significant convergence theorem that the Kuratowski upper limit of the sets of solutions to penalized problems is nonempty and is contained in the set of solutions to original inequality problem, i.e., $\emptyset \ne w\text{-}\limsup_{n\to\infty} \mathcal{S}_n = s\text{-}\limsup_{n\to\infty} \mathcal{S}_n \subset \mathcal{S}$.

The fractional calculus, as a natural generalization of the classical integer-order calculus, has been of great interest recently. Since fractional-order derivatives hold nice properties, for instance, nonlocal properties and memory effects, they have been widely applied to describe many phenomena, for example, in electrodynamics, biotechnology, aerodynamics, distributed propeller design and control of dynamical systems. Here, we refer to Liu et al. [32], Han et al. [26], Wu et al. [47], Zeng and Migórski [50], Wang et al. [46], Li et al. [30], Wang et al. [45], Zeng et al. [49], Zhang et al. [51], Migórski and Zeng [36,37].

Partial differential equations involving fractional Laplace operators have recently attracted a lot of attention, because fractional Laplace operators can describe accurately many complex systems in our real life, for example, anomalous diffusion phenomenon, dynamical networks behaviors and geophysical flows. For the problems with a fractional Laplace operator, we refer to Liu and Tan [33], Liu et al. [31], Migórski et al. [35,38], Autuori and Pucci [1], Chen et al. [9], Choi et al. [12], Stinga and Torrea [44], Mosconi et al. [39], Caffarelli et al. [6], Chen et al. [8]. On the other hand, for the problems dealing with multivalued terms modeled by Clarke's subdifferential we refer to the papers of Averna et al. [2], Denkowski et al. [13–16], Filippakis et al. [18,19], Gasiński [20,21], Gasiński et al. [22], Gasiński and Papageorgiou [24,25], Kalita and Kowalski [27], Papageorgiou et al. [41,42], Zeng et al. [48]. Finally, for the problems dealing with obstacle problems we refer to the papers of Caffarelli et al. [4], Caffarelli et al. [5], Choe [10], Choe and Lewis [11], Feehan and Pop [17], Oberman [40].

The paper is organized as follows. In Sect. 2, we recall some definitions of function spaces and important results in the sequel, in particular the surjectivity results of Le [29] and nonsmooth analysis. In Sect. 3, we establish a critical theorem which reveals that the set of weak solutions to problem (1.1) is nonempty, bounded and closed. In Sect. 4, we introduce a sequence of penalized problems without obstacle constraints by using penalty technique. Then, we prove the main convergence result that the Kuratowski upper limit of the sets of solutions to penalized problems is nonempty and is contained in the set of solutions to original elliptic obstacle problem.

2. Preliminaries

For a bounded domain $\Omega \subseteq \mathbb{R}^N$ and $1 \leq r \leq \infty$, in what follows, by $L^r(\Omega)$ and $L^r(\Omega; \mathbb{R}^N)$ we denote the usual Lebesgue spaces endowed with the norms denoted by $\|\cdot\|_r$. For $r > 1$, we denote by $r' = \frac{r}{r-1}$ its conjugate, the inner product in $\mathbb{R}^N$ is denoted by $\cdot$ and the norm of $\mathbb{R}^N$ is given by $|\cdot|$. Moreover, $\mathbb{R}_+ = [0, +\infty)$ and the Lebesgue measure in $\mathbb{R}^N$ is denoted by $|\cdot|_N$.

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary and $s \in (0,1)$ be such that $N > 2s$. In what follows, we adopt the symbols $\mathcal{S} := (\mathbb{R}^N \backslash \Omega) \times (\mathbb{R}^N \backslash \Omega)$, $\mathcal{P} := \mathbb{R}^{2N} \backslash \mathcal{S}$, and $2_s^* := \frac{2N}{N-2s}$ to denote the fractional critical exponent. Also, we denote by $u|_\Omega$ the function u restricted to the domain Ω. Consider the function space

$$X := \left\{ u \colon \mathbb{R}^N \to \mathbb{R} \mid u|_\Omega \in L^2(\Omega) \text{ and } (u(x) - u(y))^2 K(x-y) \in L^2(\mathcal{P}) \right\}.$$

It is obvious, see [43], that X is a normed linear space endowed with the norm

$$\|u\|_X := \|u\|_2 + \left(\int_{\mathcal{P}} |u(x) - u(y)|^2 K(x-y) \, \mathrm{d}y \, \mathrm{d}x \right)^{\frac{1}{2}}$$

for all $u \in X$. Since the boundary condition for problem (1.1) is the generalized Dirichlet boundary, so, we also introduce a subspace of X, given by

$$X_0 := \left\{ u \in X \mid u = 0 \ \text{ for a.e. } x \in \Omega^{\complement} \right\}.$$

Besides, we collect some important properties for the function space X_0 as follows.

Lemma 2.1. *Let $s \in (0,1)$ and Ω be a bounded, open subset of $\mathbb{R}^N$ with Lipschitz boundary and $N > 2s$. Then, we have*

(i) *X_0 is a Hilbert space with the inner product*

$$\langle u, v \rangle_{X_0} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} [u(x) - u(y)][v(x) - v(y)] K(x-y) \, \mathrm{d}x \, \mathrm{d}y$$

for all u, $v \in X_0$.

(ii) *If $p \in [1, 2_s^*]$, then there exists a positive constant $c(p)$ such that*

$$\|u\|_p \leq c(p) \|u\|_{X_0} \text{ for all } \ u \in X_0.$$

(iii) *The embedding from X_0 to $L^p(\mathbb{R}^N)$ is compact if $p \in [1, 2_s^*)$.*

Remark 2.2. Let X_0^* be the dual space of X_0. Note that $X_0 \subset L^2(\Omega) \subset X_0^*$ and $2 < 2_s^*$, so from Lemma 2.1, we can see that the embedding from X_0 to $L^2(\Omega)$ is compact.

Let E be a Banach space with its topological dual E^*. A function $J \colon E \to \mathbb{R}$ is said to be locally Lipschitz at $u \in E$ if there exist a neighborhood $N(u)$ of u and a constant $L_u > 0$ such that

$$|J(w) - J(v)| \leq L_u \|w - v\|_E \ \text{ for all } \ w, v \in N(u).$$

Definition 2.3. Let $J: E \to \mathbb{R}$ be a locally Lipschitz function and let $u, v \in E$. The generalized directional derivative $J^0(u; v)$ of J at the point u in the direction v is defined by

$$J^0(u; v) := \limsup_{w \to u,\, t \downarrow 0} \frac{J(w + tv) - J(w)}{t}.$$

The generalized gradient $\partial J: E \to 2^{E^*}$ of $J: E \to \mathbb{R}$ is defined by

$$\partial J(u) := \left\{ \xi \in E^* \mid J^0(u; v) \geq \langle \xi, v \rangle_{E^* \times E} \text{ for all } v \in E \right\} \text{ for all } u \in E.$$

The next proposition collects some basic results (see, e.g., Migórski et al. [34, Proposition 3.23]).

Proposition 2.4. *Let $J: E \to \mathbb{R}$ be locally Lipschitz of rank $L_u > 0$ at $u \in E$. Then, we have*

(a) *the function $v \mapsto J^0(u; v)$ is positively homogeneous, subadditive, and satisfies*

$$|J^0(u; v)| \leq L_u \|v\|_E \text{ for all } v \in E.$$

(b) *$(u, v) \mapsto J^0(u; v)$ is upper semicontinuous.*
(c) *for each $u \in E$, $\partial J(u)$ is a nonempty, convex and weak* compact subset of E^* with $\|\xi\|_{E^*} \leq L_u$ for all $\xi \in \partial J(u)$.*
(d) *$J^0(u; v) = \max\{\langle \xi, v \rangle_{E^* \times E} \mid \xi \in \partial J(u)\}$ for all $v \in E$.*
(e) *the multivalued function $E \ni u \mapsto \partial J(u) \subset E^*$ is upper semicontinuous from E into w^*-E^*.*

Besides, we recall the notions of pseudomonotonicity for multivalued operators (see, e.g., Gasiński and Papageorgiou [23, Definition 1.4.8]).

Definition 2.5. Let E be a real reflexive Banach space. The operator $A: E \to 2^{E^*}$ is called pseudomonotone if the following conditions hold:

(i) the set $A(u)$ is nonempty, bounded, closed and convex for all $u \in E$.
(ii) A is upper semicontinuous from each finite-dimensional subspace of E to the weak topology on E^*.
(iii) if $\{u_n\} \subset E$ with $u_n \rightharpoonup u$ in E and if $u_n^* \in A(u_n)$ is such that

$$\limsup_{n \to \infty} \langle u_n^*, u_n - u \rangle_{E^* \times E} \leq 0,$$

then to each element $v \in E$, exists $u^*(v) \in A(u)$ with

$$\langle u^*(v), u - v \rangle_{E^* \times E} \leq \liminf_{n \to \infty} \langle u_n^*, u_n - v \rangle_{E^* \times E}.$$

Throughout the paper, the symbols "$\rightharpoonup$" and "$\to$" stand for the weak and the strong convergence, respectively.

Definition 2.6. Let (Y, τ) be a Hausdorff topological space and $\{A_n\} \subset 2^Y$ for $n \geq 1$. We define

$$\tau\text{-}\liminf_{n \to \infty} A_n := \left\{ x \in Y \mid x = \tau\text{-}\lim_{n \to \infty} x_n,\, x_n \in A_n \text{ for all } n \geq 1 \right\},$$

and

$$\tau\text{-}\limsup_{n \to \infty} A_n := \left\{ x \in Y \mid x = \tau\text{-}\lim_{k \to \infty} x_{n_k},\, x_{n_k} \in A_{n_k},\, n_1 < n_2 < \ldots < n_k < \ldots \right\}.$$

The set $\tau\text{-}\liminf_{n \to \infty} A_n$ is called the τ-Kuratowski lower limit of the sets A_n, and $\tau\text{-}\limsup_{n \to \infty} A_n$ is called the τ-Kuratowski upper limit of the sets A_n. Further, if $A = \tau\text{-}\liminf_{n \to \infty} A_n = \tau\text{-}\limsup_{n \to \infty} A_n$, then A is called τ-Kuratowski limit of the sets A_n.

Finally, we will state the surjectivity theorem for multivalued mappings which are defined as the sum of a maximal monotone multivalued operator and a bounded multivalued pseudomonotone mapping. This theorem was proved in Le [29, Theorem 2.2]. We use the notation $B_R(0) = \{u \in E : \|u\|_E < R\}$.

Theorem 2.7. *Let E be a real reflexive Banach space, let $F: D(F) \subset E \to 2^{E^*}$ be a maximal monotone operator, let $G: D(G) = E \to 2^{E^*}$ be a bounded multivalued pseudomonotone operator, and let $L \in E^*$. Assume that there exist $u_0 \in E$ and $R \geq \|u_0\|_E$ such that $D(F) \cap B_R(0) \neq \varnothing$ and*

$$\langle \xi + \eta - L, u - u_0 \rangle_{E^* \times E} > 0$$

for all $u \in D(F)$ with $\|u\|_E = R$, all $\xi \in F(u)$ and all $\eta \in G(u)$. Then, there exists $u \in D(F) \cap D(G)$ such that

$$F(u) + G(u) \ni L.$$

3. Existence result

This section is devoted to explore the existence, boundedness and closedness of the set of weak solutions to problem (1.1). Our proof is based on a surjectivity theorem for multivalued mappings generated by the sum of a maximal monotone multivalued operator and a bounded multivalued pseudomonotone mapping.

To end this, we now impose the following assumptions for the data of problem (1.1).

$\underline{H(j)}$: $j: \Omega \times \mathbb{R} \to \mathbb{R}$ is such that

(i) for each $r \in \mathbb{R}$, the function $x \mapsto j(x, r)$ is measurable on Ω with $j(\cdot, 0)$ belonging to $L^1(\Omega)$;
(ii) for a.e. $x \in \Omega$, the function $r \mapsto j(x, r)$ is locally Lipschitz;
(iii) there exist $c_j > 0$, $1 \leq p < 2_s^*$ and $\alpha \in L_+^{\frac{p}{p-1}}(\Omega)$ such that
 - if $1 \leq p \leq 2$, then
$$|\xi| \leq \alpha(x) + c_j |r| \ \text{ for all } \xi \in \partial j(x, r)$$
for all $r \in \mathbb{R}$ and a.e. $x \in \Omega$.
 - if $2 < p < 2_s^*$,
$$|\xi| \leq \alpha(x) + c_j |r|^{p-1} \ \text{ for all } \xi \in \partial j(x, r)$$
for all $r \in \mathbb{R}$ and a.e. $x \in \Omega$.

(iv) there are $\beta_j \in L_+^1(\Omega)$ and $\eta_j > 0$ satisfying
$$-\xi r \leq \beta_j(x) + \eta_j |r|^2$$
for all $\xi \in \partial j(x, r)$, $r \in \mathbb{R}$, and a.e. $x \in \Omega$.

Remark 3.1. It is not difficult to see that condition $H(j)$(iv) is equivalently to the following inequality

$$j^0(x, r; -r) \leq \beta_j(x) + \eta_j |r|^2$$

for all $r \in \mathbb{R}$, and a.e. $x \in \Omega$. In fact, this condition has been used by Bai et al. [3] to explore the existence of solutions to a class of generalized mixed variational–hemivariational inequalities.

$\underline{\mathrm{H}(f)}$: $f \in L^{p'}(\Omega)$.

Consider the function $J: L^p(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega j(x, u(x))\,\mathrm{d}x \ \text{ for all } u \in L^p(\Omega). \tag{3.1}$$

On account of hypotheses H(j) and the definition of J (see (3.1)), the next lemma is a direct consequence of Theorem 3.47 of Migórski et al. [34].

Lemma 3.2. *Under the assumptions* H(j), *we have*

(i) *$J: L^p(\Omega) \to \mathbb{R}$ is locally Lipschitz continuous.*

(ii) *the inequality is true*

$$J^0(u;v) \leq \int_\Omega j^0(x,u(x);v(x))\,\mathrm{d}x$$

for all $u,v \in L^p(\Omega)$.

(iii) *for each* $u \in L^p(\Omega)$, *there hold*

$$\partial J(u) \subset \int_\Omega \partial j(x,u(x))\,\mathrm{d}x,$$
$$\|\xi\|_{p'} \leq d_J + c_j\|u\|_p^{p-1} \quad \text{for all } \xi \in \partial J(u),$$

with some $d_J > 0$.

Let C be a subset of X_0 defined by

$$C := \left\{u \in X_0 \;\middle|\; u(x) \leqslant \Phi(x) \text{ for a.e. } x \in \Omega\right\}, \tag{3.2}$$

where

$$\Phi\colon \Omega \to [0,+\infty] \text{ is a function.} \tag{3.3}$$

Remark 3.3. It is obvious that the set C is a nonempty, closed and convex subset of X_0 and $0 \in C$ due to the assumption (3.3).

The main results in the section are concerned with the following theorem.

Theorem 3.4. *Assume that* H(K), H(j), H(f) *and* (3.3) *hold. If, in addition,* $1 \leq p < 2_s^*$ *with* $\eta_j c(2)^2 < 1$, *then the set of weak solutions to problem* (1.1), *denoted by* $\mathcal{S}$, *is nonempty, bounded and closed in* X_0.

Proof. Let $I_C\colon X_0 \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be the indicator function of the set C, i.e.,

$$I_C(u) = \begin{cases} 0 & \text{if } u \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

It follows from Lemma 3.2 that $u \in X_0$ is a solution to the following problem

$$\langle Au, v-u\rangle_{X_0} + J^0(u;v-u) \geq \langle f, v-u\rangle_{X_0} \tag{3.4}$$

for all $v \in C$, and then, u is a weak solution to problem (1.1) as well, where C is the set given in (3.2) and operator $A\colon X_0 \to X_0^*$ is defined by

$$\langle Au, v\rangle_{X_0} := \int_{\mathbb{R}^N} v(x)\mathcal{L}_K u(x)\,\mathrm{d}x \quad \text{for all } u,v \in X_0.$$

Based on this critical conclusion, we next shall show that problem (3.4) has at least one solution in X_0. We start with the following claims.

Claim 1. $A\colon X_0 \to X_0^*$ *is a continuous, bounded and strongly monotone operator.*

For any $u, v \in X_0$, it yields

$$
\begin{aligned}
\langle Au, v\rangle_{X_0} &= \int_{\mathbb{R}^N} v(x)\mathcal{L}_K u(x)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^{2N}} v(x)\left[u(x+y)+u(x-y)-2u(x)\right]K(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^{2N}} v(x)\left[u(x+y)-u(x)\right]K(y)\,\mathrm{d}y\,\mathrm{d}x - \int_{\mathbb{R}^{2N}} v(x)\left[u(x-y)-u(x)\right]K(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^{2N}} v(x)\left[u(y)-u(x)\right]K(x-y)\,\mathrm{d}y\,\mathrm{d}x - \int_{\mathbb{R}^{2N}} v(x)\left[u(y)-u(x)\right]K(y-x)\,\mathrm{d}y\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^{2N}} v(x)\left[u(y)-u(x)\right]K(x-y)\,\mathrm{d}y\,\mathrm{d}x - \int_{\mathbb{R}^{2N}} v(y)\left[u(x)-u(y)\right]K(x-y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\mathbb{R}^{2N}} \left[v(x)-v(y)\right]\left[u(x)-u(y)\right]K(x-y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \langle u, v\rangle_{X_0}.
\end{aligned}
$$

Hence, we conclude that A is linear and bounded, more precisely,

$$\|Au\|_{X_0^*} \le \|u\|_{X_0} \quad \text{for all } u \in X_0.$$

Therefore, A is linear and continuous. Besides, the following equalities

$$\langle Au - Av, u - v\rangle_{X_0} = \langle u - v, u - v\rangle_{X_0} = \|u - v\|_{X_0}^2 \quad \text{for all } u, v \in X_0$$

indicate that A is strongly monotone with constant $m_A = 1$.

Claim 2. *$A + \partial J : X_0 \to 2^{X_0^*}$ is a bounded pseudomonotone multivalued operator such that for each $u \in X_0$, the set $A(u) + \partial J(u)$ is closed and convex in X_0^*.*

Employing Proposition 2.4 and Lemma 3.2 finds that the set $A(u) + \partial J(u)$ is closed and convex in X_0^* for each $u \in X_0$. Additionally, the boundedness of A, Lemma 3.2(iii) and the fact that the embedding from X_0 into $L^p(\Omega)$ is compact indicate that $X_0 \ni u \mapsto A(u) + \partial J(u) \subset X_0^*$ is a bounded map.

Next, we are going to illustrate that the map $X_0 \ni u \mapsto A(u) + \partial J(u) \subset X_0^*$ is upper semicontinuous from X_0 to X_0^* with weak topology. Invoking Proposition 3.8 of Migórski et al. [34], it is sufficient to verify that for any weakly closed subset D in X_0^*, the set $(A + \partial J)^-(D)$ is closed in X_0. Let $\{u_n\} \subset (A + \partial J)^-(D)$ be a sequence such that

$$u_n \to u \quad \text{in } X_0 \text{ as } n \to \infty, \text{ for some } u \in X_0. \tag{3.5}$$

Hence, for each $n \in \mathbb{N}$, there exists $\xi_n \in \partial J(u_n)$ satisfying

$$u_n^* = Au_n + \xi_n \in (A(u_n) + \partial J(u_n)) \cap D.$$

However, the continuity of A reveals that $A(u_n) \to A(u)$ in X_0^*, as $n \to \infty$. Besides, using Lemma 3.2(iii) and the convergence (3.5), it finds that the sequence $\{\xi_n\}$ is bounded in X_0^*, so, without any loss of generality, we may suppose that $\xi_n \rightharpoonup \xi$ in X_0^*, as $n \to \infty$, with some $\xi \in X_0^*$. Keeping in mind that ∂J is upper semicontinuous from X_0 to w-X_0^* and has bounded, convex, closed values (see Proposition 2.4(d)), therefore, it has a closed graph in $X_0 \times \text{w} - X_0^*$ (see cf. Kamenskii et al. [28, Theorem 1.1.4]). But, owing to the weak closedness of D, we obtain that $A(u) + \xi \in D$ and $\xi \in \partial J(u)$, which provides that $u \in (A + \partial J)^-(D)$. Consequently, $A + \partial J$ is upper semicontinuous from X_0 to X_0^* with weak topology.

Then, we will show that $A + \partial J$ is pseudomonotone. Let $\{u_n\}$ and $\{u_n^*\}$ be sequences such that

$$u_n \rightharpoonup u \quad \text{in } X_0, \tag{3.6}$$

$$u_n^* \in A(u_n) + \partial J(u_n) \quad \text{with} \quad \limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle_{X_0} \le 0. \tag{3.7}$$

It is enough to demonstrate that for each $v \in X_0$, we are able to find $u^*(v) \in A(u) + \partial J(u)$ satisfying

$$\liminf_{n\to\infty} \langle u_n^*, u_n - v\rangle_{X_0} \geq \langle u^*(v), u - v\rangle_{X_0}. \tag{3.8}$$

By virtue of (3.7), there exists a sequence $\{\xi_n\} \subset X_0^*$ such that for each $n \in \mathbb{N}$, $\xi_n \in \partial J(u_n)$ and

$$u_n^* = A(u_n) + \xi_n.$$

The latter combined with the inequality in (3.7) implies

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_{X_0} + \liminf_{n\to\infty} \langle \xi_n, u_n - u\rangle_{X_0} \leq 0. \tag{3.9}$$

But, using (3.6) and the compactness of the embedding of X_0 into $L^p(\Omega)$ yields that

$$u_n \to u \quad \text{in } L^p(\Omega) \quad \text{as } n \to \infty.$$

Additionally, utilizing Theorem 2.2 of Chang [7] finds

$$\partial(J|_{X_0})(u) \subset \partial(J|_{L^p(\Omega)})(u) \quad \text{for all } u \in X_0;$$

this ensures

$$\langle \xi_n, u_n - u\rangle_{X_0} = \langle \xi_n, u_n - u\rangle_{L^p(\Omega)}. \tag{3.10}$$

Moreover, Lemma 3.2(iii) and the boundedness of the sequence $\{u_n\}$ in X_0 guarantee that the sequence $\{\xi_n\}$ is contained in $L^{p'}(\Omega)$ as well. Then, passing to the limit in (3.10) as $n \to \infty$ to obtain

$$\lim_{n\to\infty} \langle \xi_n, u_n - u\rangle_{X_0} = \lim_{n\to\infty} \langle \xi_n, u_n - u\rangle_{L^p(\Omega)} = 0.$$

Inserting the above equality into (3.9) yields

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_{X_0} = \limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_{X_0} + \liminf_{n\to\infty} \langle \xi_n, u_n - u\rangle_{X_0} \leq 0.$$

However, the monotonicity of A deduces

$$\begin{aligned} 0 &\geq \limsup_{n\to\infty} \langle Au_n - Au + Au, u_n - u\rangle_{X_0} \\ &\geq \liminf_{n\to\infty} \langle Au, u_n - u\rangle_{X_0} + \limsup_{n\to\infty} \langle Au_n - Au, u_n - u\rangle_{X_0} \\ &\geq \limsup_{n\to\infty} \|u_n - u\|_{X_0}^2. \end{aligned}$$

This means that $u_n \to u$ in X_0, as $n \to \infty$. Besides, the reflexivity of X_0^* and boundedness of $\{\xi_n\} \subset X_0^*$ allow us to summarize that

$$\xi_n \rightharpoonup \xi \text{ in } X_0^* \text{ for some } \xi \in X_0^*.$$

As before we did, it is not difficult to see that $\xi \in \partial J(u)$ (see, e.g., Kamenskii et al. [28, Theorem 1.1.4]). Therefore, one has

$$\liminf_{n\to\infty} \langle u_n^*, u_n - v\rangle_{X_0} = \liminf_{n\to\infty} \langle A(u_n) + \xi_n, u_n - v\rangle_{X_0} = \langle A(u) + \xi, u - v\rangle_{X_0},$$

and it is clear that (3.8) holds with $u^* = A(u) + \xi \in A(u) + \partial J(u)$. Therefore, we conclude that $A + \partial J$ is pseudomonotone. This proves Claim 2.

Claim 3. There exists a constant $R > 0$ such that

$$\langle Au + \xi + \eta - f, u\rangle_{X_0} > 0 \tag{3.11}$$

for all $u \in C$ with $\|u\|_{X_0} = R$, all $\xi \in \partial J(u)$ and all $\eta \in \partial_C I_C(u)$, where the notation $\partial_C I_C$ stands for the subdifferential of I_C in the sense of convex analysis.

Let $u \in X_0$, $\xi \in \partial J(u)$ and $\eta \in \partial_C I_C(u)$ be arbitrary. Recall that $0 \in C$ and $f \in L^{p'}(\Omega) \subset X_0^*$, we have

$$\begin{aligned}
&\langle Au + \xi + \eta - f, u\rangle_{X_0} \\
&\quad \ge \|u\|_{X_0}^2 + \langle \xi, u\rangle_{L^p(\Omega)} + I_C(u) - I_C(0) - \|f\|_{X_0^*}\|u\|_{X_0} \\
&\quad \ge \|u\|_{X_0}^2 - \int_\Omega \xi(x)[-u(x)]\,\mathrm{d}x + I_C(u) - \|f\|_{X_0^*}\|u\|_{X_0} \\
&\quad \ge \|u\|_{X_0}^2 - \int_\Omega \beta_j(x) + \eta_j|u(x)|^2\,\mathrm{d}x + I_C(u) - \|f\|_{X_0^*}\|u\|_{X_0} \\
&\quad \ge \|u\|_{X_0}^2 - \|\beta_j\|_{L^1(\Omega)} - \eta_j\|u\|_{L^2(\Omega)}^2 + I_C(u) - \|f\|_{X_0^*}\|u\|_{X_0} \\
&\quad \ge \|u\|_{X_0}^2 - \|\beta_j\|_{L^1(\Omega)} - \eta_j c(2)^2\|u\|_{X_0}^2 + I_C(u) - \|f\|_{X_0^*}\|u\|_{X_0},
\end{aligned} \tag{3.12}$$

where we have used Lemma 3.2(iii). Keeping in mind that $I_C\colon X_0 \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function, so we now apply Proposition 1.3.1 in Gasiński and Papageorgiou [23], to find $a_C, b_C \ge 0$ such that

$$I_C(v) \ge -a_C\|v\|_{X_0} - b_C \quad \text{for all } v \in X_0. \tag{3.13}$$

Therefore, from (3.12) and (3.13), we have

$$\begin{aligned}
&\langle Au + \xi + \eta - f, u\rangle_{X_0} \\
&\quad \ge (1 - \eta_j c(2)^2)\|u\|_{X_0}^2 - \|\beta_j\|_{L^1(\Omega)} - a_C\|u\|_{X_0} - b_C - \|f\|_{X_0^*}\|u\|_{X_0}.
\end{aligned} \tag{3.14}$$

Since $1 - \eta_j c(2)^2 > 0$, so, we are able to find constant $R_0 > 0$ large enough such that

$$(1 - \eta_j c(2)^2)R_0^2 - \|\beta_j\|_{L^1(\Omega)} - a_C R_0 - b_C - \|f\|_{X_0^*}R_0 > 0.$$

Therefore, for each $R \ge R_0$ fixed, the desired inequality (3.11) holds.

Recall that $I_C\colon X_0 \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function, so, $\partial_C I_C\colon X_0 \to 2^{X_0^*}$ is maximal monotone. The latter together with Theorem 2.7 implies that there exists $u \in X_0$ resolving the inclusion problem:

Find $u \in X_0$ such that

$$Au + \partial J(u) + \partial_C I_C(u) \ni f.$$

Obviously, u solves problem (3.4) too; therefore, the set of weak solutions to problem (1.1) is nonempty, i.e., $\mathcal{S} \neq \emptyset$.

Next, we shall prove that the set $\mathcal{S}$ is closed in X_0. Let $\{u_n\} \subset \mathcal{S}$ be a sequence such that

$$u_n \to u \quad \text{in } X_0 \tag{3.15}$$

for some $u \in X_0$. For each $n \in \mathbb{N}$, we have

$$\langle A(u_n), v - u_n\rangle_{X_0} + \int_\Omega j^0(x, u_n(x); v(x) - u_n(x))\,\mathrm{d}x \ge \int_\Omega f(x)(v(x) - u_n(x))\,\mathrm{d}x$$

for all $v \in C$. Passing to the upper limit as $n \to \infty$ for the above inequality, it finds

$$\begin{aligned}
&\langle A(u), v-u\rangle_{X_0} + \int_\Omega j^0(x,u(x);v(x)-u(x))\,\mathrm{d}x \\
&\geq \langle A(u), v-u\rangle_{X_0} + \int_\Omega \limsup_{n\to\infty} j^0(x,u_n(x);v(x)-u_n(x))\,\mathrm{d}x \\
&\geq \limsup_{n\to\infty}\langle A(u_n), v-u_n\rangle_{X_0} + \limsup_{n\to\infty}\int_\Omega j^0(x,u_n(x);v(x)-u_n(x))\,\mathrm{d}x \\
&\geq \limsup_{n\to\infty}\left[\langle A(u_n), v-u_n\rangle_{X_0} + \int_\Omega j^0(x,u_n(x);v(x)-u_n(x))\,\mathrm{d}x\right] \\
&\geq \limsup_{n\to\infty}\int_\Omega f(x)(v(x)-u_n(x))\,\mathrm{d}x \\
&= \int_\Omega f(x)(v(x)-u(x))\,\mathrm{d}x
\end{aligned}$$

for all $v \in C$, where we have used Fatou Lemma (see, e.g., Migórski et al. [34, Theorem 1.64]), Lebesgue Dominated Convergence Theorem (see, e.g., Migórski et al. [34, Theorem 1.65]) and Proposition 2.4(b). Therefore, u solves problem (1.1); namely, the set $\mathcal{S}$ is closed.

Finally, we shall illustrate the set $\mathcal{S}$ is bounded. If the above were not true, then there would exist a sequence $\{u_n\} \subset \mathcal{S}$ such that

$$\|u_n\|_{X_0} \to \infty \quad \text{as } n \to \infty. \tag{3.16}$$

A simple calculating (see, for example, (3.14)) gives

$$\begin{aligned}
0 &\geq \langle Au_n - f, u_n\rangle_{X_0} - \int_\Omega j^0(x,u_n(x);-u_n(x))\,\mathrm{d}x \\
&\geq \|u_n\|_{X_0}^2 - \|\beta_j\|_{L^1(\Omega)} - \eta_j c(2)^2\|u_n\|_{X_0}^2 - \|f\|_{X_0^*}\|u_n\|_{X_0}.
\end{aligned}$$

Since $1 > \eta_j c(2)^2$, so, letting $n \to \infty$ for the above inequality, it finds a contradiction. Therefore, we conclude that $\mathcal{S}$ is bounded. □

Particularly, if K is specialized to $K(x) := |x|^{-(N+2s)}$ for all $x \in \mathbb{R}^N \backslash \{0\}$, then we have the following corollary, which extends the recent result, Migórski et al. [35, Theorem 1].

Corollary 3.5. *Assume that* H(j), H(f) *and* (3.3) *hold. If, in addition,* $1 \leq p < 2_s^*$ *with* $\eta_j c(2)^2 < 1$, *then the set of weak solutions to problem* (1.2) *is nonempty, bounded and closed in* X_0.

Remark 3.6. Recently, Migórski et al. [35] applied the Moreau–Yosida approximation method to show the solvability of (1.2). However, in the current paper, we use a different approach, which is a surjectivity theorem for multivalued mappings generated by the sum of a maximal monotone multivalued operator and a bounded multivalued pseudomonotone mapping, to prove the existence of weak solutions. In the meanwhile, we also provide the boundedness and closedness of the set of weak solution to the problem under consideration.

4. Convergence analysis

Let $\{\rho_n\}$ be a sequence with $\rho_n > 0$ for each $n \in \mathbb{N}$ such that $\rho_n \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, consider the following nonlocal elliptic inclusion problem with a penalty term:

$$\begin{cases} \mathcal{L}_K u(x) + \partial j(x, u(x)) + \dfrac{1}{\rho_n}(u(x) - \Phi(x))^+ \ni f(x) & \text{in } \Omega \\ u(x) = 0 & \text{in } \Omega^{\complement}, \end{cases} \tag{4.1}$$

where the superscript + stands for the positive part.

The weak solution to problem (4.1) is given as follows.

Definition 4.1. We say that $u \in X_0$ is a weak solution of problem (1.1) if the inequality holds

$$\int_{\mathbb{R}^N} v(x)\mathcal{L}_K(u)(x)\,\mathrm{d}x + \frac{1}{\rho_n}\int_{\Omega} (u(x) - \Phi(x))^+ v(x)\,\mathrm{d}x + \int_{\Omega} j^0(x, u(x); v(x))\,\mathrm{d}x \ge \int_{\Omega} f(x)v(x)\,\mathrm{d}x$$

for all $v \in X_0$.

In what follows, we denote by $\mathcal{S}_n$ the set of weak solutions to problem (4.1). As to details, in the section, we are interesting in the study of the essential relation between the sets $\mathcal{S}$ and $\mathcal{S}_n$. More precisely, the main results in the section are given the following theorem.

Theorem 4.2. *Assume that* $\mathrm{H}(K)$, $\mathrm{H}(j)$, $\mathrm{H}(f)$, (3.3) *and* $1 \le p < 2_s^*$ *with* $\eta_j c(2)^2 < 1$ *hold. If, in addition,* $\{\rho_n\}$ *is a sequence with* $\rho_n > 0$ *for each* $n \in \mathbb{N}$ *such that* $\rho_n \to 0$ *as* $n \to \infty$, *then the statements are true*

(i) *for each* $n \in \mathbb{N}$, *the set of weak solutions to problem* (4.1), $\mathcal{S}_n$, *is nonempty, bounded and closed.*

(ii) *it holds*

$$\emptyset \ne w\text{-}\limsup_{n\to\infty} \mathcal{S}_n = s\text{-}\limsup_{n\to\infty} \mathcal{S}_n \subset \mathcal{S}.$$

(iii) *for each* $u \in s\text{-}\limsup_{n\to\infty} \mathcal{S}_n$ *and any sequence* $\{\widetilde{u}_n\}$ *with*

$$\widetilde{u}_n \in \arg\min_{u_n \in \mathcal{S}_n} \|u_n - u\|_{X_0} \quad \text{for each } n \in \mathbb{N},$$

there exists a subsequence of $\{\widetilde{u}_n\}$ *converging strongly to* u *in* X_0, *where the set* $\arg\min_{u_n\in\mathcal{S}_n} \|u_n - u\|_{X_0}$ *is defined by*

$$\arg\min_{u_n \in \mathcal{S}_n} \|u_n - u\|_{X_0} := \left\{\widetilde{u} \in \mathcal{S}_n \mid \|u - \widetilde{u}\|_{X_0} \le \|u - v\|_{X_0} \text{ for all } v \in \mathcal{S}_n\right\}.$$

Proof. Ad (i). It can be proved directly by using the same argument as the proof of Theorem 3.4.
Ad (ii). Let us introduce a function $B\colon L^p(\Omega) \to L^{p'}(\Omega)$ given by

$$\langle Bu, v\rangle_{L^p(\Omega)} = \int_{\Omega} (u(x) - \Phi(x))^+ v(x)\,\mathrm{d}x \quad \text{for all } u, v \in L^p(\Omega). \tag{4.2}$$

First, we prove that the set $w\text{-}\limsup_{n\to\infty} \mathcal{S}_n$ is nonempty. Indeed, we have the following claim.

Claim 4. The set $\cup_{n\in\mathbb{N}}\mathcal{S}_n$ *is uniformly bounded in* X_0.

Arguing by contradiction, suppose that $\cup_{n\in\mathbb{N}}\mathcal{S}_n$ is unbounded. Without loss of generality, we may assume that there exists a sequence $\{u_n\} \subset X_0$ with $u_n \in \mathcal{S}_n$ for each $n \in \mathbb{N}$ such that

$$\|u_n\|_{X_0} \to \infty \quad \text{as } n \to \infty.$$

Hence, for each $n \in \mathbb{N}$, it has

$$\begin{aligned}&\langle Au_n, v\rangle_{X_0} + \int_\Omega j^0(x, u_n(x); v(x))\,\mathrm{d}x + \frac{1}{\rho_n}\int_\Omega (u_n(x) - \Phi(x))^+ v(x)\,\mathrm{d}x\\ &\geq \int_\Omega f(x)v(x)\,\mathrm{d}x\end{aligned}$$

for all $v \in X_0$. Inserting $v = -u_n$ into the above inequality yields

$$\begin{aligned}&\langle Au_n, u_n\rangle_{X_0} - \int_\Omega j^0(x, u_n(x); -u_n(x))\,\mathrm{d}x - \int_\Omega f(x)u_n(x)\,\mathrm{d}x\\ &\leq -\frac{1}{\rho_n}\int_\Omega (u_n(x) - \Phi(x))^+ u_n(x)\,\mathrm{d}x.\end{aligned}$$

Due to $\Phi(x) \geq 0$ for all $x \in \Omega$, we can use the monotonicity of the function $s \mapsto s^+$ to get

$$\begin{aligned}&\langle Au_n, u_n\rangle_{X_0} - \int_\Omega j^0(x, u_n(x); -u_n(x))\,\mathrm{d}x - \int_\Omega f(x)u_n(x)\,\mathrm{d}x\\ &\leq -\frac{1}{\rho_n}\int_\Omega \left[(u_n(x) - \Phi(x))^+ - (0 - \Phi(x))^+\right] u_n(x)\,\mathrm{d}x\\ &\leq 0,\end{aligned}$$

that is,

$$\|u_n\|_{X_0}^2 - \|f\|_{X_0^*}\|u_n\|_{X_0} - \int_\Omega j^0(x, u_n(x); -u_n(x))\,\mathrm{d}x \leq 0.$$

However, hypothesis H(j)(iv) reveals

$$\begin{aligned}0 &\geq \|u_n\|_{X_0}^2 - \|f\|_{X_0^*}\|u_n\|_{X_0} - \int_\Omega j^0(x, u_n(x); -u_n(x))\,\mathrm{d}x\\ &\geq (1 - \eta_j c(2)^2)\|u_n\|_{X_0}^2 - \|\beta_j\|_{L^1(\Omega)} - \|f\|_{X_0^*}\|u_n\|_{X_0}.\end{aligned}$$

Since $\eta_j c(2)^2 < 1$, we are able to find $R > 0$ large enough such that

$$(1 - \eta_j c(2)^2)R^2 - \|\beta_j\|_{L^1(\Omega)} - \|f\|_{X_0^*}R > 0.$$

This points out that for $n \in \mathbb{N}$ large enough we have

$$0 \geq \|u_n\|_{X_0}^2 - \|f\|_{X_0^*}\|u_n\|_{X_0} - \int_\Omega j^0(x, u_n(x); -u_n(x))\,\mathrm{d}x > 0,$$

which generates a contradiction. So, Claim 4 is valid.

Let $\{u_n\} \subset X_0$ with $u_n \in \mathcal{S}_n$ for each $n \in \mathbb{N}$ be an arbitrary sequence. Claim 4 indicates that $\{u_n\}$ is bounded in X_0. Then, we now may assume that along a relabeled subsequence, it has

$$u_n \rightharpoonup u \ \text{ as } n \to \infty \tag{4.3}$$

for some $u \in X_0$. This guarantees that the set $w\text{-}\limsup_{n\to\infty} \mathcal{S}_n$ is nonempty.

Next, we are going to demonstrate that $w\text{-}\limsup_{n\to\infty} \mathcal{S}_n$ is a subset of $\mathcal{S}$. Let $u \in w\text{-}\limsup_{n\to\infty} \mathcal{S}_n$ be arbitrary. Without loss of generality, we may suppose that there exists a subsequence $\{u_n\} \subset X_0$ with $u_n \in \mathcal{S}_n$ for all $n \in \mathbb{N}$ satisfying (4.3). Our goal is to prove that $u \in \mathcal{S}$.

Claim 5. $u(x) \leq \Phi(x)$ *for a.e.* $x \in \Omega$.

For every $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\frac{1}{\rho_n} \int_\Omega (u_n(x) - \Phi(x))^+ v(x)\,\mathrm{d}x \\
&\le \langle Au_n, -v\rangle_{X_0} + \int_\Omega j^0(x, u_n(x); -v(x))\,\mathrm{d}x + \int_\Omega f(x)v(x)\,\mathrm{d}x
\end{aligned} \tag{4.4}$$

for all $v \in X_0$. It is easy to calculate that

$$\frac{1}{\rho_n} \int_\Omega (u_n(x) - \Phi(x))^+ v(x)\,\mathrm{d}x \le M_0 \|v\|_{X_0}$$

for some $M_0 > 0$, which is independent of n and v. Hence,

$$\int_\Omega (u_n(x) - \Phi(x))^+ v(x)\,\mathrm{d}x \le \rho_n M_0 \|v\|_{X_0}.$$

Passing to the limit as $n \to \infty$ for the above inequality and using the convergence (4.3), it concludes from Lebesgue Dominated Convergence Theorem and the compactness of the embedding from X_0 to $L^p(\Omega)$ that

$$\begin{aligned}
&\int_\Omega (u(x) - \Phi(x))^+ v(x)\,\mathrm{d}x \\
&= \int_\Omega \lim_{n\to\infty} (u_n(x) - \Phi(x))^+ v(x)\,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_\Omega (u_n(x) - \Phi(x))^+ v(x)\,\mathrm{d}x \\
&\le \lim_{n\to\infty} \rho_n M_0 \|v\|_{X_0} \\
&= 0
\end{aligned}$$

for all $v \in X_0$. Therefore, we have $(u(x) - \Phi(x))^+ = 0$ for a.e. $x \in \Omega$, that is, $u(x) \le \Phi(x)$ for a.e. $x \in \Omega$.

Claim 6. $u \in \mathcal{S}$.

For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\langle Au_n, u_n - v\rangle_{X_0} \\
&\le \frac{1}{\rho_n} \int_\Omega (u_n(x) - \Phi(x))^+ (v(x) - u_n(x))\,\mathrm{d}x + \int_\Omega j^0(x, u_n(x); v(x) - u_n(x))\,\mathrm{d}x \\
&\quad + \int_\Omega f(x)(u_n(x) - v(x))\,\mathrm{d}x
\end{aligned}$$

for all $v \in X_0$. The latter combined with the monotonicity of $s \mapsto s^+$ deduces

$$\begin{aligned}
&\langle Au_n, u_n - v\rangle_{X_0} \\
&\le \frac{1}{\rho_n} \int_\Omega (v(x) - \Phi(x))^+ (v(x) - u_n(x))\,\mathrm{d}x + \int_\Omega j^0(x, u_n(x); v(x) - u_n(x))\,\mathrm{d}x \\
&\quad + \int_\Omega f(x)(u_n(x) - v(x))\,\mathrm{d}x
\end{aligned}$$

for all $v \in X_0$; hence,

$$\begin{aligned}&\langle Au_n, u_n - v\rangle_{X_0}\\&\quad\le \int_\Omega j^0(x, u_n(x); v(x) - u_n(x))\,\mathrm{d}x + \int_\Omega f(x)(u_n(x) - v(x))\,\mathrm{d}x\end{aligned}\tag{4.5}$$

for all $v \in C$. Inserting $v = u$ into the above inequality and passing to the upper limit as $n \to \infty$ for the resulting inequality, it finds

$$\begin{aligned}&\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle_{X_0}\\&\quad\le \limsup_{n\to\infty}\left(\int_\Omega j^0(x, u_n(x); u(x) - u_n(x))\,\mathrm{d}x + \int_\Omega f(x)(u_n(x) - u(x))\,\mathrm{d}x\right)\\&\quad\le \limsup_{n\to\infty}\int_\Omega j^0(x, u_n(x); u(x) - u_n(x))\,\mathrm{d}x + \limsup_{n\to\infty}\int_\Omega f(x)(u_n(x) - u(x))\,\mathrm{d}x\\&\quad\le \int_\Omega \limsup_{n\to\infty} j^0(x, u_n(x); u(x) - u_n(x))\,\mathrm{d}x\\&\quad\le 0,\end{aligned}$$

where we have used the compactness of the embedding from X_0 to $L^p(\Omega)$, Fatou Lemma and Lebesgue Dominated Convergence Theorem. The latter combined with the strong monotonicity of A implies

$$\begin{aligned}&\limsup_{n\to\infty}\|u_n - u\|_{X_0}^2\\&\quad\le \limsup_{n\to\infty}\langle Au_n - Au, u_n - u\rangle_{X_0} + \liminf_{n\to\infty}\langle Au, u_n - u\rangle_{X_0}\\&\quad\le \limsup_{n\to\infty}\langle Au_n - Au + Au, u_n - u\rangle_{X_0}\\&\quad\le 0,\end{aligned}$$

so we conclude $u_n \to u$ as $n \to \infty$. Passing to the upper limit as $n \to \infty$ for inequality (4.5), we can employ Fatou Lemma and Lebesgue Dominated Convergence Theorem again to conclude that

$$\begin{aligned}&\langle Au, u - v\rangle_{X_0}\\&\quad\le \int_\Omega j^0(x, u(x); v(x) - u(x))\,\mathrm{d}x + \int_\Omega f(x)(u(x) - v(x))\,\mathrm{d}x\end{aligned}$$

for all $v \in C$. Therefore, one finds $u \in \mathcal{S}$. This means $\emptyset \neq w\text{-}\limsup_{n\to\infty}\mathcal{S}_n \subset \mathcal{S}$.

Claim 7. It holds $w\text{-}\limsup_{n\to\infty}\mathcal{S}_n = s\text{-}\limsup_{n\to\infty}\mathcal{S}_n$.

Since $s\text{-}\limsup_{n\to\infty}\mathcal{S}_n \subset w\text{-}\limsup_{n\to\infty}\mathcal{S}_n$, it is enough to verify the condition $w\text{-}\limsup_{n\to\infty}\mathcal{S}_n \subset s\text{-}\limsup_{n\to\infty}\mathcal{S}_n$. Let $u \in w\text{-}\limsup_{n\to\infty}\mathcal{S}_n$ be arbitrary. Without any loss of generality, there exists a sequence, still denoted by $\{u_n\}$ with $u_n \in \mathcal{S}_n$ such that $u_n \rightharpoonup u$ as $n \to \infty$. We claim that $u_n \to u$ as $n \to \infty$. For each $n \in \mathbb{N}$, the inequality (4.5) holds. Inserting $v = u$ into (4.5) and passing to the upper limit as $n \to \infty$ for the resulting inequality, it is easy to see

$$\limsup_{n\to\infty}\|u_n - u\|_{X_0}^2 \le 0.$$

Then, one has $u_n \to u$ as $n \to \infty$, namely, $u \in s\text{-}\limsup_{n\to\infty}\mathcal{S}_n$. Consequently, it is valid that $s\text{-}\limsup_{n\to\infty}\mathcal{S}_n = w\text{-}\limsup_{n\to\infty}\mathcal{S}_n$.

Ad (iii). Let $u \in s\text{-}\limsup_{n\to\infty}\mathcal{S}_n$ be arbitrary. Since $\mathcal{S}_n$ is nonempty, bounded and closed, so, the set $\arg\min_{u_n\in\mathcal{S}_n}\|u_n - u\|_{X_0}$ is nonempty. Let $\{\widetilde{u}_n\}$ be any sequence such that

$$\widetilde{u}_n \in \arg\min_{u_n\in\mathcal{S}_n}\|u_n - u\|_{X_0} \quad \text{for each } n \in \mathbb{N}.$$

It follows from Claim 4 that the sequence $\{\widetilde{u}_n\}$ is bounded. So, we may assume, by passing to a subsequence, not relabeled, that

$$\widetilde{u}_n \rightharpoonup \widetilde{u} \ \text{ as } n \to \infty$$

for some $\widetilde{u} \in X_0$, whereas using the same argument as the proof of Claim 5, it finds $\widetilde{u} \in C$. Then, for each $n \in \mathbb{N}$, (4.5) is available. Employing the same process with the proof of Claim 6, it concludes that $\widetilde{u}$ is a solution to problem (1.1) and $\widetilde{u}_n \to u$ as $n \to \infty$. Consequently, the desired conclusion is true. □

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