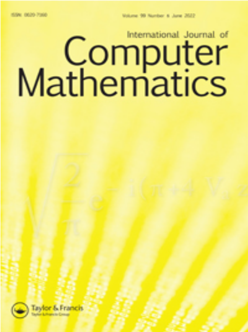

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ARTICLE

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ABSTRACT

In this paper, we study a dynamic contact problem with Clarke subdifferential boundary conditions. The material is assumed to be viscoplastic which has an implicit expression of the stress field in constitutive law. The weak form of the model is governed by an evolutionary hemivariational inequality coupled with an integral equation. We study a fully discrete approximation scheme of the problem and bound the errors. Under appropriate solution regularity assumptions, optimal-order error estimates can be derived. Finally, a numerical example is also included to support our theoretical analysis. Particularly, it gives numerical evidence on the theoretically predicted optimal convergence order.

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1. Introduction

Variational and hemivariational inequalities are important tools for studying nonlinear problems. While the family of variational inequalities is concerned with convex functionals, the theory of hemivariational inequalities was introduced in the 1980s [20] to deal with nonsmooth problems with a nonconvex structure. Hemivariational inequalities are shown to be useful in many subjects, especially in contact mechanics. We refer to the book [19] and references therein for the mathematical modelling and unique solvability result. The numerical analysis and simulations can be found in [12,13] for a static contact, [2,22] for a quasistatic contact, and [1] for a dynamic contact, among others. We also refer to the comprehensive book [11].

In applications, it is also important to consider the viscoplastic materials like rubbers, metals, rocks and so on. The constitutive law of such materials is in the form of

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}'(t)) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}'(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,\mathrm{d}s, \tag{1}$$

where $\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})$ denote the displacement field, the stress tensor and the linearized strain tensor, respectively. Operator $\mathcal{A}$ describes the purely viscous properties of the material. Operator $\mathcal{E}$ is an elasticity operator. Operator $\mathcal{G}$, which is always nonlinear, describes the viscoplastic behaviour. For simplicity, the dependence on the spatial variable $\boldsymbol{x}$ is not indicated. From (1.1), we can observe that the constitutive law is history-dependent and has an implicit expression of the stress field $\boldsymbol{\sigma}$. It means that we need to consider a coupled system which is a history-dependent hemivariational inequality combined with an integral equation. When deriving error estimates, we have to handle $\boldsymbol{\sigma}$, $\boldsymbol{u}$ and $\boldsymbol{u}'$ at the same time, rather than only $\boldsymbol{u}$ and $\boldsymbol{u}'$. As a result, analysis and numerical simulations of viscoplastic models are more complicated than those of viscoelastic models.

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In the past decades, various viscoplastic contact models in the framework of variational inequalities have been studied extensively and thoroughly, see, e.g. [9,10,18]. However, studies on viscoplastic contact in the framework of hemivariational inequalities are still in the development, and we list them in the following. Existence and uniqueness result is obtained for the quasistatic viscoplastic contact problem in [5], in [14] with an internal state term and a memory term, in [15] with an internal state term, a memory term and a damage term. The unique solvability for a dynamic contact problem is the topic of [17], and [16] with a damage term. Numerical analysis and simulations are only studied for quasistatic viscoplastic contact models, see [4] and [23] recently. This paper is firstly devoted to the numerical analysis and simulations of a dynamic viscoplastic contact problem in the framework of hemivariational inequalities.

The paper is organized as follows. In Section 2, we present necessary notation and basic results. In Section 3, we describe the contact problem and the corresponding weak formulation, and state the existence and uniqueness result. In Section 4, we consider a fully discrete approximation scheme and derive the optimal-order error estimates under solution regularity assumptions. Finally, in Section 5, we report a numerical example and illustrate the theoretically predicted convergence orders.

2. Preliminaries

In this section, we present some necessary preliminary material.

Let X be a Banach space. Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz function. From [7], the generalized directional derivative of φ at $x \in X$ in the direction $v \in X$ is defined by

$$\varphi^0(x; v) = \limsup_{y \to x, t \downarrow 0} \frac{\varphi(y + tv) - \varphi(y)}{t}.$$

The generalized gradient of φ at x is a subset of a dual space X^*, given by $\partial\varphi(x) = \{\zeta \in X^* \mid \varphi^0(x; v) \geq \langle \zeta, v \rangle_{X^* \times X}$ for all $v \in X\}$. Two useful properties are provided

$$\varphi^0(x; v) = \max\{\langle \zeta, v \rangle_{X^* \times X} \mid \zeta \in \partial\varphi(x)\}, \tag{2}$$

$$\varphi^0(x; v_1 + v_2) \leq \varphi^0(x; v_1) + \varphi^0(x; v_2). \tag{3}$$

Let d be a positive integer. Let $\mathbb{S}^d$ be the linear space of second-order symmetric tensors on $\mathbb{R}^d$. The inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\|_{\mathbb{R}^d} = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij} \tau_{ij}, \quad \|\boldsymbol{\tau}\|_{\mathbb{S}^d} = (\boldsymbol{\tau} : \boldsymbol{\tau})^{1/2} \quad \text{for all } \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

The convention of summation over repeated indices is used in this paper.

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain with Lipschitz continuous boundary Γ. We split the boundary into three mutually disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $\text{meas}(\Gamma_1) > 0$. The unit outward normal vector on Γ is denoted by $\boldsymbol{\nu} = (\nu_i) \in \mathbb{R}^d$. For a vector field $\boldsymbol{v}$, the normal and tangential components of $\boldsymbol{v}$ are denoted by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. For a tensor field $\boldsymbol{\sigma}$, the normal and tangential components are denoted by $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$.

We introduce two Hilbert spaces with their inner products

$$V = \{\boldsymbol{v} = (v_i) \in H^1(\Omega; \mathbb{R}^d) : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\}, \quad (\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \, \mathrm{d}x,$$

$$Q = \{\boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega; \mathbb{S}^d) : \tau_{ij} = \tau_{ji}\}, \quad (\boldsymbol{\tau}, \boldsymbol{\sigma})_Q = \int_\Omega \boldsymbol{\tau} : \boldsymbol{\sigma} \, \mathrm{d}x,$$

where $\boldsymbol{\varepsilon} : H^1(\Omega;\mathbb{R}^d) \to L^2(\Omega;\mathbb{S}^d)$ is the deformation operator defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \tfrac{1}{2}(u_{i,j} + u_{j,i}).$$

The index following comma indicates a partial derivative. We denote the associated norms in V and Q by $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. From assumption $\text{meas}(\Gamma_1) > 0$ and Korn's inequality, we have the completeness of the space $(V, \|\cdot\|_V)$. Let $H = L^2(\Omega;\mathbb{R}^d)$. The embedding of $(V, \|\cdot\|_V)$ into $(H, \|\cdot\|_H)$ is continuous and dense. The dual of V is denoted by V^*, and the duality pairing of V and V^* is denoted by $\langle\cdot,\cdot\rangle$. Then $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = (\boldsymbol{u}, \boldsymbol{v})_H \ \forall \boldsymbol{u} \in H, \boldsymbol{v} \in V$. By the Sobolev trace theorem, we have

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3;\mathbb{R}^d)} \le \|\gamma\|\|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V,$$

where $\|\gamma\|$ represents the norm of the trace operator $\gamma : V \to L^2(\Gamma_3;\mathbb{R}^d)$.

We also recall the following Green's formula:

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x + \int_\Omega \operatorname{Div}\boldsymbol{\sigma}\cdot\boldsymbol{v}\,\mathrm{d}x = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{v}\,\mathrm{d}\Gamma \quad \forall\, \boldsymbol{v} \in V,\ \boldsymbol{\sigma} \in Q,$$

with the divergence operator defined by $\operatorname{Div}\boldsymbol{\sigma} = (\sigma_{ij,j})$.

3. A viscoplastic contact model

In this section, we describe the dynamic viscoplastic contact and the corresponding weak formulation, and present the existence and uniqueness theorem.

Let a viscoplastic body occupy a bounded domain Ω described in Section 2. The body is clamped on Γ_1 and may come in contact with an obstacle on Γ_3. A volume force of density $\boldsymbol{f}_0$ acts on Ω and a surface traction of density $\boldsymbol{f}_2$ acts on Γ_2. Assume that the contact process is dynamic. We study the following classical formulation of the contact problem in the time interval $[0, T]$.

Problem P: *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}'(t)) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}'(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,\mathrm{d}s \quad \text{in } \Omega \times (0,T), \tag{4}$$

$$\boldsymbol{u}''(t) - \operatorname{Div}\boldsymbol{\sigma}(t) = \boldsymbol{f}_0(t) \quad \text{in } \Omega \times (0,T), \tag{5}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Gamma_1 \times (0,T), \tag{6}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on } \Gamma_2 \times (0,T), \tag{7}$$

$$-\sigma_\nu(t) \in \partial j(u'_\nu(t)), \quad \boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \quad \text{on } \Gamma_3 \times (0,T), \tag{8}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}'(0) = \boldsymbol{v}_0 \quad \text{in } \Omega. \tag{9}$$

■

Equation (4) represents the constitutive law of viscoplastic material introduced in Section 1. Equation (5) is the normalized equilibrium equation for the dynamic process. The general equilibrium equation is in the form of

$$\rho\boldsymbol{u}''(t) - \operatorname{Div}\boldsymbol{\sigma}(t) = \boldsymbol{f}_0(t),$$

where ρ denotes the density of mass. However, here we assume ρ is the constant 1 after the scaling of the equation for simplicity. We have the clamped boundary condition (6) and the traction boundary condition (7). Boundary condition (8) is used to model the frictionless contact with normal damped response. The initial conditions are given by (9). Further interpretations of the model can be found in [17].

We need the following assumptions on the data:

$$\begin{cases} H(\mathcal{A})\colon \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d \text{ satisfies} \\ \quad \text{(i) } \mathcal{A}(\cdot, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d; \\ \quad \text{(ii) } \mathcal{A}(\boldsymbol{x}, \cdot) \text{ is continuous on } \mathbb{S}^d \text{ for a.e. } \boldsymbol{x} \in \Omega; \\ \quad \text{(iii) } \|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon})\|_{\mathbb{S}^d} \le c_1(h(\boldsymbol{x}) + \|\boldsymbol{\varepsilon}\|_{\mathbb{S}^d}) \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega \text{ with } h \in L^2(\Omega), \\ \qquad h \ge 0 \text{ and } c_1 > 0; \\ \quad \text{(iv) } (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(x, \boldsymbol{\varepsilon}_2)) : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2_{\mathbb{S}^d} \text{ for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e.} \\ \qquad \boldsymbol{x} \in \Omega \text{ with } m_{\mathcal{A}} > 0; \\ \quad \text{(v) } \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) : \boldsymbol{\varepsilon} \ge \alpha \|\boldsymbol{\varepsilon}\|^2_{\mathbb{S}^d} \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega \text{ with } \alpha > 0. \end{cases}$$

$$\begin{cases} H(\mathcal{E})\colon \mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d \text{ satisfies} \\ \quad \text{(i) } \|\mathcal{E}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{E}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\|_{\mathbb{S}^d} \le L_{\mathcal{E}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d} \text{ for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega \text{ with} \\ \qquad L_{\mathcal{E}} > 0; \\ \quad \text{(ii) } \mathcal{E}(\cdot, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d; \\ \quad \text{(iii) } \mathcal{E}(\cdot, \mathbf{0}) \in L^2(\Omega, \mathbb{S}^d). \end{cases}$$

$$\begin{cases} H(\mathcal{G})\colon \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d \text{ satisfies} \\ \quad \text{(i) } \|\mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\|_{\mathbb{S}^d} \le L_{\mathcal{G}}(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{\mathbb{S}^d} + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d}) \text{ for all } \boldsymbol{\sigma}_1, \\ \qquad \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega \text{ with } L_{\mathcal{G}} > 0; \\ \quad \text{(ii) } \mathcal{G}(\cdot, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d; \\ \quad \text{(iii) } \mathcal{G}(\cdot, \mathbf{0}, \mathbf{0}) \in L^2(\Omega; \mathbb{S}^d). \end{cases}$$

$$\begin{cases} H(j)\colon j : \Gamma_3 \times \mathbb{R} \to \mathbb{R} \text{ satisfies} \\ \quad \text{(i) } j(\cdot, r) \text{ is measurable for all } r \in \mathbb{R} \text{ and } j(\cdot, 0) \in L^1(\Gamma_3); \\ \quad \text{(ii) } j(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz for a.e. } \boldsymbol{x} \in \Gamma_3; \\ \quad \text{(iii) } |\partial j(\boldsymbol{x}, r)| \le c_j(1 + |r|) \text{ for a.e. } \boldsymbol{x} \in \Gamma_3, \text{ all } r \in \mathbb{R} \text{ with } c_j > 0; \\ \quad \text{(iv) } j^0(\boldsymbol{x}, r; -r) \le d_j(1 + |r|) \text{ for a.e. } \boldsymbol{x} \in \Gamma_3, \text{ all } r \in \mathbb{R} \text{ with } d_j \ge 0; \\ \quad \text{(v) } (\eta_1 - \eta_2)(r_1 - r_2) \ge -m_j |r_1 - r_2|^2 \text{ for all } \eta_1 \in \partial j(\boldsymbol{x}, r_1), \eta_2 \in \partial j(\boldsymbol{x}, r_2), \\ \qquad r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3 \text{ with } m_j \ge 0; \\ \quad \text{(vi) either } j(\boldsymbol{x}, \cdot) \text{ or } -j(\boldsymbol{x}, \cdot) \text{ is regular for a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases}$$

$H(f)$: The densities of body forces and surface traction satisfy

$$\boldsymbol{f}_0 \in L^2(0, T; H), \quad \boldsymbol{f}_2 \in L^2(0, T; L^2(\Gamma_2; \mathbb{R}^d)).$$

$H(0)$: The initial data have the regularity $\boldsymbol{u}_0, \boldsymbol{v}_0 \in V$.

We note that the relaxed monotonicity condition of subdifferential $H(j)(v)$ is equivalent to the following $H(j)(v')$

$$j^0(\boldsymbol{x}, r_1; r_2 - r_1) + j^0(\boldsymbol{x}, r_2; r_1 - r_2) \le m_j |r_1 - r_2|^2, \quad \text{for all } r_1, r_2 \in \mathbb{R} \text{ and a.e. } \boldsymbol{x} \in \Gamma_3.$$

Further define the function $\boldsymbol{f} : (0, T) \to V^*$ by

$$\langle \boldsymbol{f}(t), \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v} \, \mathrm{d}\Gamma \quad \text{for all } \boldsymbol{v} \in V, \text{ a.e. } t \in (0, T). \tag{10}$$

Through a standard derivation, we obtain the following weak formulation of Problem P.

Problem P_V^0: *Find a displacement field $\boldsymbol{u} : [0, T] \to V$, a stress field $\boldsymbol{\sigma} : [0, T] \to Q$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\boldsymbol{u}'(0) = \boldsymbol{v}_0$ and*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}'(t)) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}'(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s))) \, \mathrm{d}s, \tag{11}$$

$$\langle \boldsymbol{u}''(t), \boldsymbol{w}\rangle + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + \int_{\Gamma_3} j^0(u'_\nu(t); w_\nu)\, \mathrm{d}\Gamma \geq \langle \boldsymbol{f}(t), \boldsymbol{w}\rangle \tag{12}$$

hold for all $\boldsymbol{w} \in V$ *and a.e.* $t \in (0, T)$. ■

The unique solvability of Problem P_V^0 is proved in [17].

Theorem 3.1: *Assume* $H(\mathcal{A})$, $H(\mathcal{E})$, $H(\mathcal{G})$, $H(j)$, $H(f)$, $H(0)$ *and*

$$m_{\mathcal{A}} > m_j \|\gamma\|^2. \tag{13}$$

Then Problem P_V^0 *has a unique solution with the following regularity:*

$$\boldsymbol{u} \in W^{1,2}(0, T; V) \cap C^1(0, T; H), \quad \boldsymbol{u}'' \in L^2(0, T; V^*), \tag{14}$$

$$\boldsymbol{\sigma} \in L^2(0, T; Q), \quad \mathrm{Div}\, \boldsymbol{\sigma} \in L^2(0, T; V^*). \tag{15}$$

Introduce the velocity variable $\boldsymbol{v} = \boldsymbol{u}'$, then $\boldsymbol{u}(t) = \boldsymbol{u}(0) + \int_0^t \boldsymbol{v}(s)\, \mathrm{d}s$. It follows from Theorem 3.1 that $\boldsymbol{v} \in L^2(0, T; V) \cap C(0, T; H)$, $\boldsymbol{v}' \in L^2(0, T; V^*)$. We consider the following equivalent problem in terms of the velocity.

Problem P_V: *Find a velocity field* $\boldsymbol{v} : [0, T] \to V$, *a stress field* $\boldsymbol{\sigma} : [0, T] \to Q$ *such that* $\boldsymbol{v}(0) = \boldsymbol{v}_0$ *and*

$$\boldsymbol{u}(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}(s)\, \mathrm{d}s, \tag{16}$$

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(t)) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, \mathrm{d}s, \tag{17}$$

$$\langle \boldsymbol{v}'(t), \boldsymbol{w}\rangle + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + \int_{\Gamma_3} j^0(v_\nu(t); w_\nu)\, \mathrm{d}\Gamma \geq \langle \boldsymbol{f}(t), \boldsymbol{w}\rangle \tag{18}$$

hold for all $\boldsymbol{w} \in V$ *and a.e.* $t \in (0, T)$. ■

4. A fully discrete scheme and error estimate

In this section, we construct a fully discrete approximation of Problem P_V and provide a result on error estimates.

We use a uniform partition of the time interval $[0, T]$. For a positive integer N, define the time step size by $k = T/N$. We have a uniform partition: $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ with the nodes $t_n = nk$, $n = 0, 1, \ldots, N$. For a continuous function $g = g(t)$, we write $g_n = g(t_n)$. Backward difference approximation is used for the time derivative

$$g'(t_n) \approx \frac{g(t_n) - g(t_{n-1})}{k}, \quad 1 \leq n \leq N.$$

We assume that Ω is a polygonal or polyhedral domain. Let $\mathcal{T}^h$ be a regular family of finite element triangulations of $\overline{\Omega}$ into triangles or tetrahedrons. Here, $h > 0$ is a spatial discretization parameter.

We use piecewise linear finite element space to approximate the velocity field:

$$V^h = \{\boldsymbol{v}^h \in C(\overline{\Omega})^d \mid \boldsymbol{v}^h|_T \in \mathbb{P}_1(T)^d\ \forall\, T \in \mathcal{T}^h,\ \boldsymbol{v}^h = 0 \text{ on } \Gamma_1\}, \tag{19}$$

and piecewise constant finite element space to approximate the stress field

$$Q^h = \{\boldsymbol{\tau}^h \in Q \mid \boldsymbol{\tau}^h|_T \in \mathbb{R}^{d\times d}\ \forall\, T \in \mathcal{T}^h\}. \tag{20}$$

Define the orthogonal projection $\mathcal{P}_{Q^h} : Q \to Q^h$ by

$$(\mathcal{P}_{Q^h}\boldsymbol{\tau}, \boldsymbol{\tau}^h)_Q = (\boldsymbol{\tau}, \boldsymbol{\tau}^h)_Q \quad \forall\, \boldsymbol{\tau} \in Q,\ \boldsymbol{\tau}^h \in Q^h. \tag{21}$$

It has nonexpansive property: $\|\mathcal{P}_{Q^h}\boldsymbol{\tau}\|_Q \le \|\boldsymbol{\tau}\|_Q$ for all $\boldsymbol{\tau} \in Q$ and the estimate

$$\|\boldsymbol{\tau} - \mathcal{P}_{Q^h}\boldsymbol{\tau}\|_Q \le ch|\boldsymbol{\tau}|_{H^1(\Omega;\mathbb{S}^d)} \quad \forall\, \boldsymbol{\tau} \in H^1(\Omega;\mathbb{S}^d). \tag{22}$$

With additional assumptions $\boldsymbol{f}_0 \in C(0,T;H)$, $\boldsymbol{f}_2 \in C(0,T;L^2(\Gamma_2;\mathbb{R}^d))$, we have $\boldsymbol{f} \in C(0,T;V^*)$ from definition (10). Choose $\boldsymbol{v}_0^h, \boldsymbol{u}_0^h \in V^h$ to approximate the initial values $\boldsymbol{v}_0$ and $\boldsymbol{u}_0$. We construct the following fully discrete approximation scheme for Problem P_V.

Problem P_V^{hk}: *Find* $\boldsymbol{v}^{hk} = \{\boldsymbol{v}_n^{hk}\}_{n=0}^N \subset V^h$ *and* $\boldsymbol{\sigma}^{hk} = \{\boldsymbol{\sigma}_n^{hk}\}_{n=0}^N \subset Q^h$ *such that* $\boldsymbol{v}_0^{hk} = \boldsymbol{v}_0^h$, $\boldsymbol{\sigma}_0^{hk} = \mathcal{P}_{Q^h}\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_0^h) + \mathcal{P}_{Q^h}\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h)$,

$$\boldsymbol{u}_n^{hk} = \boldsymbol{u}_0^h + k\sum_{j=1}^n \boldsymbol{v}_j^{hk}, \quad 0 \le n \le N \tag{23}$$

and

$$\boldsymbol{\sigma}_n^{hk} = \mathcal{P}_{Q^h}\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}) + \mathcal{P}_{Q^h}\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}) + k\sum_{j=0}^{n-1}\mathcal{P}_{Q^h}\mathcal{G}(\boldsymbol{\sigma}_j^{hk} - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_j^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk})), \tag{24}$$

$$\left(\frac{\boldsymbol{v}_n^{hk} - \boldsymbol{v}_{n-1}^{hk}}{k}, \boldsymbol{w}^h\right)_H + (\boldsymbol{\sigma}_n^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q + \int_{\Gamma_3} j^0(v_{n\nu}^{hk}; w_\nu^h)\,\mathrm{d}\Gamma \ge \left\langle \boldsymbol{f}_n, \boldsymbol{w}^h\right\rangle \tag{25}$$

hold for $n = 1, 2, \ldots, N$, *and for all* $\boldsymbol{w}^h \in V^h$. ■

Note that for Problem P_V^{hk}, with $\{\boldsymbol{v}_j^{hk}\}_{j\le n-1}$ known, $\boldsymbol{v}_n^{hk}$ is determined by

$$\frac{1}{k}(\boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_H + (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q + \int_{\Gamma_3} j^0(v_{n\nu}^{hk}; w_\nu^h)\,\mathrm{d}\Gamma \ge \left\langle \boldsymbol{f}_n, \boldsymbol{w}^h\right\rangle + \frac{1}{k}(\boldsymbol{v}_{n-1}^{hk}, \boldsymbol{w}^h)_H$$
$$- (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q - k\sum_{j=0}^{n-1}(\mathcal{G}(\boldsymbol{\sigma}_j^{hk} - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_j^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q \quad \forall\, \boldsymbol{w}^h \in V^h.$$

This is an elliptic hemivariational inequality and has a unique solution. Thus, combined with (24), the unique solvability of our Problem P_V^{hk} is implied.

We first explore the boundedness of the discrete solutions. Note that c, whose value may change in different inequalities, will denote a general positive constant independent of discretization parameters h and k.

Theorem 4.1: *Let $\{\boldsymbol{v}_n^{hk}\}_{n=0}^N$ and $\{\boldsymbol{\sigma}_n^{hk}\}_{n=0}^N$ be the unique solution of Problem P_V^{hk}. Then, there exists a constant $c > 0$ such that*

$$\max_{1\le i\le N} \|\boldsymbol{v}_n^{hk}\|_H^2 + \sum_{i=1}^N \|\boldsymbol{v}_i^{hk} - \boldsymbol{v}_{i-1}^{hk}\|_H^2 + k\sum_{i=1}^N (\|\boldsymbol{v}_i^{hk}\|_V^2 + \|\boldsymbol{u}_i^{hk}\|_V^2 + \|\boldsymbol{\sigma}_i^{hk}\|_Q^2) \le c. \tag{26}$$

Proof: From (23),

$$\|\boldsymbol{u}_n^{hk}\|_V \le \|\boldsymbol{u}_0^h\|_V + k\sum_{j=1}^n \|\boldsymbol{v}_j^{hk}\|_V, \tag{27}$$

then we have

$$\|\boldsymbol{u}_n^{hk}\|_V^2 \le c + ck\sum_{j=1}^n \|\boldsymbol{v}_j^{hk}\|_V^2. \tag{28}$$

Multiply this inequality by k, change n to i, and sum over $i = 1$ to n, we obtain

$$k\sum_{i=1}^n \|\boldsymbol{u}_i^{hk}\|_V^2 \le c + ck\sum_{i=1}^n k\sum_{j=1}^i \|\boldsymbol{v}_j^{hk}\|_V^2. \tag{29}$$

Combine (24) and (25), and take $\boldsymbol{w}^h = -\boldsymbol{v}_n^{hk}$, we have

$$\begin{aligned}
&\left(\frac{\boldsymbol{v}_n^{hk} - \boldsymbol{v}_{n-1}^{hk}}{k}, \boldsymbol{v}_n^{hk}\right)_H + (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}))_Q \\
&\quad\le \int_{\Gamma_3} j^0(v_{n\nu}^{hk}; -v_{n\nu}^{hk})\,\mathrm{d}\Gamma + \left\langle \boldsymbol{f}_n, \boldsymbol{v}_n^{hk}\right\rangle \\
&\qquad - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}) + k\sum_{j=0}^{n-1} \mathcal{G}(\boldsymbol{\sigma}_j^{hk} - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_j^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}))_Q.
\end{aligned} \tag{30}$$

From $H(\mathcal{A})$,

$$\begin{aligned}
(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}))_Q &= (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}) - \mathcal{A}\mathbf{0}, \boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}) - \mathbf{0})_Q + (\mathcal{A}\mathbf{0}, \boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}))_Q \\
&\ge m_{\mathcal{A}}\|\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})\|_Q^2 - \|\mathcal{A}\mathbf{0}\|_Q\|\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})\|_Q \\
&\ge m_{\mathcal{A}}\|\boldsymbol{v}_n^{hk}\|_V^2 - c\|\boldsymbol{v}_n^{hk}\|_V.
\end{aligned}$$

From $H(j)$ and (2),

$$\begin{aligned}
\int_{\Gamma_3} j^0(v_{n\nu}^{hk}; -v_{n\nu}^{hk})\,\mathrm{d}\Gamma &= \int_{\Gamma_3} j^0(v_{n\nu}^{hk}; -v_{n\nu}^{hk}) + j^0(0; v_{n\nu}^{hk})\,\mathrm{d}\Gamma - \int_{\Gamma_3} j^0(0; v_{n\nu}^{hk})\,\mathrm{d}\Gamma \\
&\le \int_{\Gamma_3} m_j|v_{n\nu}^{hk}|^2\,\mathrm{d}\Gamma + \int_{\Gamma_3} c_j|v_{n\nu}^{hk}|\,\mathrm{d}\Gamma \\
&\le m_j\|\boldsymbol{v}_n^{hk}\|_{L^2(\Gamma_3;\mathbb{R}^d)}^2 + c_j\sqrt{\mathrm{meas}(\Gamma_3)}\|\boldsymbol{v}_n^{hk}\|_{L^2(\Gamma_3;\mathbb{R}^d)} \\
&\le m_j\ \|\gamma\|^2\ \ \|\boldsymbol{v}_n^{hk}\|_V^2 + c\|\boldsymbol{v}_n^{hk}\|_V.
\end{aligned}$$

From $H(\mathcal{E})$,

$$-(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}))_Q \le \|\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})\|_Q\|\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})\|_Q$$

$$\leq (\|\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}) - \mathcal{E}\mathbf{0}\|_Q + \|\mathcal{E}\mathbf{0}\|_Q)\ \|\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})\|_Q$$
$$\leq c\ (\|\boldsymbol{u}_{n-1}^{hk}\|_V + 1)\ \|\boldsymbol{v}_n^{hk}\|_V.$$

Similarly, from $H(\mathcal{G})$ and H($\mathcal{A}$),

$$-\left(k\sum_{j=0}^{n-1}\mathcal{G}(\boldsymbol{\sigma}_j^{hk} - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_j^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})\right)_Q$$
$$\leq \left(k\sum_{j=0}^{n-1}\|\mathcal{G}(\boldsymbol{\sigma}_j^{hk} - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_j^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk}))\|_Q\right)\ \|\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})\|_Q$$
$$\leq k\sum_{j=0}^{n-1}(\|\boldsymbol{\sigma}_j^{hk}\|_Q + \|\boldsymbol{v}_j^{hk}\|_V + \|\boldsymbol{u}_j^{hk}\|_V + 1)\ \|\boldsymbol{v}_n^{hk}\|_V.$$

For any $\epsilon > 0$, by applying the modified Cauchy–Schwarz inequality and combining (29) and the above inequalities to (30), we have

$$\left(\frac{\boldsymbol{v}_n^{hk} - \boldsymbol{v}_{n-1}^{hk}}{k}, \boldsymbol{v}_n^{hk}\right)_H + m_{\mathcal{A}}\|\boldsymbol{v}_n^{hk}\|_V^2$$
$$\leq m_j\|\gamma\|^2\|\boldsymbol{v}_n^{hk}\|_V^2 + c\|\boldsymbol{v}_n^{hk}\|_V + c\ \|\boldsymbol{u}_{n-1}^{hk}\|_V\ \|\boldsymbol{v}_n^{hk}\|_V$$
$$+ ck\sum_{j=0}^{n-1}(\|\boldsymbol{\sigma}_j^{hk}\|_Q + \|\boldsymbol{v}_j^{hk}\|_V + \|\boldsymbol{u}_j^{hk}\|_V)\ \|\boldsymbol{v}_n^{hk}\|_V$$
$$\leq \epsilon\|\boldsymbol{v}_n^{hk}\|^2 + c + m_j\|\gamma\|^2\|\boldsymbol{v}_n^{hk}\|_V^2 + c\|\boldsymbol{u}_{n-1}^{hk}\|_V^2 + ck\sum_{j=0}^{n-1}(\|\boldsymbol{\sigma}_j^{hk}\|_Q^2 + \|\boldsymbol{v}_j^{hk}\|_V^2 + \|\boldsymbol{u}_j^{hk}\|_V^2).$$

Since

$$\left(\frac{\boldsymbol{v}_n^{hk} - \boldsymbol{v}_{n-1}^{hk}}{k}, \boldsymbol{v}_n^{hk}\right)_H = \frac{1}{2k}(\|\boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1}^{hk}\|_H^2 + \|\boldsymbol{v}_n^{hk} - \boldsymbol{v}_{n-1}^{hk}\|_H^2),$$

combined with (28), we have

$$\frac{1}{2k}(\|\boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1}^{hk}\|_H^2 + \|\boldsymbol{v}_n^{hk} - \boldsymbol{v}_{n-1}^{hk}\|_H^2) + (m_{\mathcal{A}} - m_j\|\gamma\|^2 - \epsilon)\|\boldsymbol{v}_n^{hk}\|_V^2$$
$$\leq c + ck\sum_{j=0}^{n-1}(\|\boldsymbol{\sigma}_j^{hk}\|_Q^2 + \|\boldsymbol{v}_j^{hk}\|_V^2 + \|\boldsymbol{u}_j^{hk}\|_V^2).$$

Let $\epsilon = \frac{1}{2}(m_{\mathcal{A}} - m_j\|\gamma\|^2)$. Multiply the above inequality by $2k$, change n to i, and sum over $i = 1$ to n, we have

$$\|\boldsymbol{v}_n^{hk}\|_H^2 + \sum_{i=1}^{n}\|\boldsymbol{v}_i^{hk} - \boldsymbol{v}_{i-1}^{hk}\|_H^2 + \bar{c}k\sum_{i=1}^{n}\|\boldsymbol{v}_i^{hk}\|_V^2$$
$$\leq c + ck\sum_{i=1}^{n}k\sum_{j=0}^{i-1}(\|\boldsymbol{\sigma}_j^{hk}\|_Q^2 + \|\boldsymbol{v}_j^{hk}\|_V^2 + \|\boldsymbol{u}_j^{hk}\|_V^2). \tag{31}$$

It directly leads to

$$k\sum_{i=1}^{n}\|\boldsymbol{v}_i^{hk}\|_V^2 \le c + ck\sum_{i=1}^{n}k\sum_{j=0}^{i-1}(\|\boldsymbol{\sigma}_j^{hk}\|_Q^2 + \|\boldsymbol{v}_j^{hk}\|_V^2 + \|\boldsymbol{u}_j^{hk}\|_V^2). \tag{32}$$

From (24) and (27),

$$\begin{aligned}\|\boldsymbol{\sigma}_n^{hk}\|_Q &\le \|\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})\|_Q + \|\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})\|_Q + k\sum_{j=0}^{n-1}\|\mathcal{G}(\boldsymbol{\sigma}_j^{hk} - \mathcal{A}\boldsymbol{\varepsilon}(v_j^{hk}), \boldsymbol{\varepsilon}(u_j^{hk}))\|_Q \\ &\le c\|\boldsymbol{v}_n^{hk}\|_V + c\|\boldsymbol{u}_{n-1}^{hk}\|_V + ck\sum_{j=0}^{n-1}(\|\boldsymbol{\sigma}_j^{hk}\|_Q + \|\boldsymbol{v}_j^{hk}\|_V + \|\boldsymbol{u}_j^{hk}\|_V) + c \\ &\le c\|\boldsymbol{v}_n^{hk}\|_V + ck\sum_{j=0}^{n-1}(\|\boldsymbol{\sigma}_j^{hk}\|_Q + \|\boldsymbol{v}_j^{hk}\|_V + \|\boldsymbol{u}_j^{hk}\|_V) + c.\end{aligned}$$

Take the squares of both sides

$$\|\boldsymbol{\sigma}_n^{hk}\|_Q^2 \le c\|\boldsymbol{v}_n^{hk}\|_V^2 + ck\sum_{j=0}^{n-1}(\|\boldsymbol{\sigma}_j^{hk}\|_Q^2 + \|\boldsymbol{v}_j^{hk}\|_V^2 + \|\boldsymbol{u}_j^{hk}\|_V^2) + c.$$

Multiply the above inequality by k, change n to i, and sum over $i = 1$ to n, we have

$$k\sum_{i=1}^{n}\|\boldsymbol{\sigma}_i^{hk}\|_Q^2 \le ck\sum_{i=1}^{n}\|\boldsymbol{v}_i^{hk}\|_V^2 + ck\sum_{i=1}^{n}k\sum_{j=0}^{i-1}(\|\boldsymbol{\sigma}_j^{hk}\|_Q^2 + \|\boldsymbol{v}_j^{hk}\|_V^2 + \|\boldsymbol{u}_j^{hk}\|_V^2) + c$$

Apply (32), we have

$$k\sum_{i=1}^{n}\|\boldsymbol{\sigma}_i^{hk}\|_Q^2 \le ck\sum_{i=1}^{n}k\sum_{j=0}^{i-1}(\|\boldsymbol{\sigma}_j^{hk}\|_Q^2 + \|\boldsymbol{v}_j^{hk}\|_V^2 + \|\boldsymbol{u}_j^{hk}\|_V^2) + c. \tag{33}$$

Combine (29), (32) and (33),

$$k\sum_{i=1}^{n}(\|\boldsymbol{v}_i^{hk}\|_V^2 + \|\boldsymbol{u}_i^{hk}\|_V^2 + \|\boldsymbol{\sigma}_i^{hk}\|_Q^2) \qquad \le ck\sum_{i=1}^{n}k\sum_{j=1}^{i}(\|\boldsymbol{\sigma}_j^{hk}\|_Q^2 + \|\boldsymbol{v}_j^{hk}\|_V^2 + \|\boldsymbol{u}_j^{hk}\|_V^2) + c.$$

Let $e_n := k\sum_{i=1}^{n}(\|\boldsymbol{v}_i^{hk}\|_V^2 + \|\boldsymbol{u}_i^{hk}\|_V^2 + \|\boldsymbol{\sigma}_i^{hk}\|_Q^2)$. Then, the above inequality can be rewritten as

$$e_n \le ck\sum_{i=1}^{n}e_i + c.$$

From the Gronwall inequality, we have

$$e_N = k\sum_{i=1}^{N}(\|\boldsymbol{v}_i^{hk}\|_V^2 + \|\boldsymbol{u}_i^{hk}\|_V^2 + \|\boldsymbol{\sigma}_i^{hk}\|_Q^2) \le c. \tag{34}$$

Use (34) in (31) to obtain

$$\max_{1\le i\le N}\|\boldsymbol{v}_n^{hk}\|_H^2 + \sum_{i=1}^{N}\|\boldsymbol{v}_i^{hk} - \boldsymbol{v}_{i-1}^{hk}\|_H^2 \le c. \tag{35}$$

Hence, the proof ends with (34) and (35). ■

To derive error estimates, we additionally assume the Lipschitz continuity of the operator $\mathcal{A}$

$$\|\mathcal{A}(x,\boldsymbol{\varepsilon}_1)-\mathcal{A}(x,\boldsymbol{\varepsilon}_2)\|_{\mathbb{S}^d}\le L_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d}. \tag{36}$$

Additional regularities of the solution are also assumed

$$\boldsymbol{v}\in W^{1,1}(0,T;V)\cap H^2(0,T;V^*)\cap C(0,T;H^2(\Omega;\mathbb{R}^d)), \tag{37}$$

$$\boldsymbol{v}|_{\Gamma_3}\in C(0,T;H^2(\Gamma_3;\mathbb{R}^d)), \tag{38}$$

$$\boldsymbol{\sigma}\in W^{1,1}(0,T;Q)\cap C(0,T;H^1(\Omega;\mathbb{S}^d)). \tag{39}$$

Then, the first term $\langle\boldsymbol{v}'(t),\boldsymbol{w}\rangle$ in (18) can be replaced by $(\boldsymbol{v}'(t),\boldsymbol{w})_H$. We set $t=t_n$ in (16)–(18) and deduce that

$$\boldsymbol{u}_n=\boldsymbol{u}_0+\int_0^{t_n}\boldsymbol{v}(s)\,\mathrm{d}s, \tag{40}$$

$$\boldsymbol{\sigma}_n=\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n)+\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n)+\int_0^{t_n}\mathcal{G}(\boldsymbol{\sigma}(s)-\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(s)),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,\mathrm{d}s, \tag{41}$$

$$(\boldsymbol{v}_n',\boldsymbol{w})_H+(\boldsymbol{\sigma}_n,\boldsymbol{\varepsilon}(\boldsymbol{w}))_Q+\int_{\Gamma_3}j^0(v_{n\nu};w_\nu)\,\mathrm{d}\Gamma\ge\langle\boldsymbol{f}_n,\boldsymbol{w}\rangle. \tag{42}$$

Let $\Pi^h\boldsymbol{v}(t)\in V^h$ be the finite element interpolant of $\boldsymbol{v}(t)$ in V^h. Since $\boldsymbol{v}(t)\in H^2(\Omega;\mathbb{R}^d)\cap V$, we have the interpolation error estimate

$$\|\boldsymbol{v}(t)-\Pi^h\boldsymbol{v}(t)\|_H+h\|\boldsymbol{v}(t)-\Pi^h\boldsymbol{v}(t)\|_{H^1(\Omega;\mathbb{R}^d)}\le ch^2|\boldsymbol{v}(t)|_{H^2(\Omega;\mathbb{R}^d)}. \tag{43}$$

The initial values $\boldsymbol{u}_0^h\in V^h$ and $\boldsymbol{v}_0^h\in V^h$ are chosen to be the finite element interpolants of $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$. Then,

$$\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V\le ch,\quad \|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_V\le ch,\quad \|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_H\le ch. \tag{44}$$

Moreover, from definition of $\boldsymbol{\sigma}_0^{hk}$, we have

$$\begin{aligned}\|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0^{hk}\|_Q&\le\|\boldsymbol{\sigma}_0-\mathcal{P}_Q^h\boldsymbol{\sigma}_0\|_Q+\|\mathcal{P}_Q^h\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0^{hk}\|_Q\\&\le ch|\boldsymbol{\sigma}_0|_{H^1(\Omega;\mathbb{S}^d)}+\left\|\mathcal{P}_{Q^h}(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_0)-\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_0^h))\right\|_Q+\left\|\mathcal{P}_{Q^h}(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^{hk}))\right\|_Q\\&\le ch.\end{aligned} \tag{45}$$

From [21],

$$\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V^2\le c(h^2+k^2)+ck\sum_{j=1}^n\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2, \tag{46}$$

$$\|\boldsymbol{u}_n-\boldsymbol{u}_{n-1}^{hk}\|_V^2\le c(h^2+k^2)+ck\sum_{j=1}^{n-1}\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2. \tag{47}$$

Denote $\boldsymbol{\theta}_n=\int_0^{t_n}\mathcal{G}(\boldsymbol{\sigma}(s)-\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(s)),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,\mathrm{d}s-k\sum_{j=0}^{n-1}\mathcal{G}(\boldsymbol{\sigma}_j^{hk}-\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_j^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk}))$. Then

$$\|\boldsymbol{\theta}_n\|_Q\le c\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(\|\boldsymbol{\sigma}(s)-\boldsymbol{\sigma}_j\|_Q+\|\boldsymbol{v}(s)-\boldsymbol{v}_j\|_V+\|\boldsymbol{u}(s)-\boldsymbol{u}_j\|_V)\,\mathrm{d}s$$

$$+ck\sum_{j=0}^{n-1}(\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V+\|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\|_V)$$

$$\leq ck(\|\boldsymbol{\sigma}'\|_{L^1(0,T;Q)}+\|\boldsymbol{v}'\|_{L^1(0,T;V)}+\|\boldsymbol{u}'\|_{L^1(0,T;V)})$$

$$+ck\sum_{j=0}^{n-1}(\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V+\|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\|_V)$$

$$\leq ck+ck\sum_{j=0}^{n-1}(\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V+\|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\|_V).$$

Here, as an intermediate step of the derivation, we used

$$\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\|\boldsymbol{\sigma}(s)-\boldsymbol{\sigma}_j\|_Q=\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\|\int_{t_j}^{s}\frac{\mathrm{d}}{\mathrm{d}\tau}(\boldsymbol{\sigma}(\tau))\,\mathrm{d}\tau\|_Q\,\mathrm{d}s$$

$$\leq k\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\|\boldsymbol{\sigma}'(\tau)\|_Q d\tau=k\int_0^T\|\boldsymbol{\sigma}'(\tau)\|_Q\,\mathrm{d}\tau.$$

Similar arguments can be applied to $\|\boldsymbol{v}(s)-\boldsymbol{v}_j\|_V$ and $\|\boldsymbol{u}(s)-\boldsymbol{u}_j\|_V$. Take the squares of $\|\boldsymbol{\theta}_n\|_Q$ and use (46), we obtain

$$\|\boldsymbol{\theta}_n\|_Q^2\leq ck^2+ck\sum_{j=0}^{n-1}(\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q^2+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2+\|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\|_V^2)$$

$$\leq c(h^2+k^2)+ck\sum_{j=0}^{n-1}(\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q^2+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2). \tag{48}$$

Write $\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}=(\boldsymbol{\sigma}_n-\mathcal{P}_{Q^h}\boldsymbol{\sigma}_n)+(\mathcal{P}_{Q^h}\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk})$. Then, we have

$$\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\|_Q\leq\|\boldsymbol{\sigma}_n-\mathcal{P}_{Q^h}\boldsymbol{\sigma}_n\|_Q+\Big\|\mathcal{P}_{Q^h}(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n)-\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})+\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n)-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})$$

$$+\int_0^{t_n}\mathcal{G}(\boldsymbol{\sigma}(s)-\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(s)),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,\mathrm{d}s+k\sum_{j=0}^{n-1}\mathcal{G}(\boldsymbol{\sigma}_j^{hk}-\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_j^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk})))\Big\|_Q$$

$$\leq ch+c\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_V++c\|\boldsymbol{u}_n-\boldsymbol{u}_{n-1}^{hk}\|_V+\|\boldsymbol{\theta}_n\|_Q.$$

Take the squares of both sides and use (47) and (48),

$$\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\|_Q^2\leq c(h^2+k^2)+c\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_V^2+ck\sum_{j=0}^{n-1}(\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q^2+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2$$

over i from 1 to n,

$$\sum_{i=1}^{n}\|\boldsymbol{\sigma}_i-\boldsymbol{\sigma}_i^{hk}\|_Q^2\leq c(h^2+k^2)+ck\sum_{i=1}^{n}\|\boldsymbol{v}_i-\boldsymbol{v}_i^{hk}\|_V^2+ck\sum_{i=1}^{n}k\sum_{j=0}^{i-1}(\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q^2+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2). \tag{49}$$

$$m_{\mathcal{A}}\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_V^2\leq(\mathcal{A}\varepsilon(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}),\varepsilon(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Q$$

$$= (\mathcal{A}\varepsilon(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \varepsilon(\boldsymbol{v}_n - \boldsymbol{w}^h))_Q + (\mathcal{A}\varepsilon(\boldsymbol{v}_n), \varepsilon(\boldsymbol{w}^h - \boldsymbol{v}_n))_Q \\ + (\mathcal{A}\varepsilon(\boldsymbol{v}_n), \varepsilon(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q + (\mathcal{A}\varepsilon(\boldsymbol{v}_n^{hk}), \varepsilon(\boldsymbol{v}_n^{hk} - \boldsymbol{w}^h))_Q \tag{50}$$

hold for any $\boldsymbol{w}^h \in V^h$. By (41) and (42),

$$(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n), \varepsilon(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q \\ \leq (\boldsymbol{v}_n', \boldsymbol{v}_n^{hk} - \boldsymbol{v}_n)_H + \int_{\Gamma_3} j^0(v_{n\nu}; v_{n\nu}^{hk} - v_{n\nu})\,\mathrm{d}\Gamma - \left\langle \boldsymbol{f}_n, \boldsymbol{v}_n^{hk} - \boldsymbol{v}_n \right\rangle \\ + (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \varepsilon(\boldsymbol{v}_n^{hk} - \boldsymbol{v}_n))_Q + \left(\int_0^{t_n} \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,\mathrm{d}s, \varepsilon(\boldsymbol{v}_n^{hk} - \boldsymbol{v}_n) \right)_Q.$$

By (24) and (25),

$$(\mathcal{A}\varepsilon(\boldsymbol{v}_n^{hk}), \varepsilon(\boldsymbol{v}_n^{hk} - \boldsymbol{w}^h))_Q \\ \leq \left(\frac{\boldsymbol{v}_n^{hk} - \boldsymbol{v}_{n-1}^{hk}}{k}, \boldsymbol{w}^h - \boldsymbol{v}_n^{hk} \right)_H + \int_{\Gamma_3} j^0(v_{n\nu}^{hk}; w_\nu^h - v_{n\nu}^{hk})\,\mathrm{d}\Gamma - \left\langle \boldsymbol{f}_n, \boldsymbol{w}^h - \boldsymbol{v}_n^{hk} \right\rangle \\ + (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \varepsilon(\boldsymbol{w}^h - \boldsymbol{v}_n^{hk}))_Q + \left(k \sum_{j=0}^{n-1} \mathcal{G}\left(\boldsymbol{\sigma}_j^{hk} - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_j^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk}) \right), \varepsilon(\boldsymbol{w}^h - \boldsymbol{v}_n^{hk}) \right)_Q.$$

Then (50) can be rewritten as

$$m_{\mathcal{A}} \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 \leq (\mathcal{A}\varepsilon(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \varepsilon(\boldsymbol{v}_n - \boldsymbol{w}^h))_Q + R_n(\boldsymbol{w}^h - \boldsymbol{v}_n) + I_1 + I_2 + I_3, \tag{51}$$

where

$$I_1 = (\boldsymbol{v}_n', \boldsymbol{v}_n^{hk} - \boldsymbol{w}^h)_H + \left(\frac{\boldsymbol{v}_n^{hk} - \boldsymbol{v}_{n-1}^{hk}}{k}, \boldsymbol{w}^h - \boldsymbol{v}_n^{hk} \right)_H, \tag{52}$$

$$I_2 = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \varepsilon(\boldsymbol{v}_n^{hk} - \boldsymbol{w}^h))_Q - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \varepsilon(\boldsymbol{v}_n^{hk} - \boldsymbol{w}^h))_Q \\ + \left(\int_0^{t_n} \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,\mathrm{d}s, \varepsilon(\boldsymbol{v}_n^{hk} - \boldsymbol{w}^h) \right)_Q \\ - \left(k \sum_{j=0}^{n-1} \mathcal{G}(\boldsymbol{\sigma}_j^{hk} - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_j^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk})), \varepsilon(\boldsymbol{v}_n^{hk} - \boldsymbol{w}^h) \right)_Q, \tag{53}$$

$$I_3 = \int_{\Gamma_3} j^0(v_{n\nu}; v_{n\nu}^{hk} - v_{n\nu}) + j^0(v_{n\nu}^{hk}; w_\nu^h - v_{n\nu}^{hk}) - j^0(v_{n\nu}; w_\nu^h - v_{n\nu})\,\mathrm{d}\Gamma, \tag{54}$$

and the residual $R_n(\boldsymbol{w})$ is defined by

$$R_n(\boldsymbol{w}) = (\boldsymbol{v}_n', \boldsymbol{w})_H + \int_{\Gamma_3} j^0(v_{n\nu}; w_\nu)\,\mathrm{d}\Gamma - \langle \boldsymbol{f}_n, \boldsymbol{w} \rangle \\ + (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n), \varepsilon(\boldsymbol{w}))_Q + (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \varepsilon(\boldsymbol{w}))_Q \\ + \left(\int_0^{t_n} \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,\mathrm{d}s, \varepsilon(\boldsymbol{w}) \right)_Q. \tag{55}$$

Firstly,

$$(\mathcal{A}\varepsilon(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \varepsilon(\boldsymbol{v}_n - \boldsymbol{w}^h))_Q \le L_{\mathcal{A}} \|\varepsilon(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk})\|_Q \|\varepsilon(\boldsymbol{v}_n - \boldsymbol{w}^h)\|_Q$$
$$\le \epsilon \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 + c\|\boldsymbol{v}_n - \boldsymbol{w}^h\|_V^2. \tag{56}$$

Secondly, we have the following bound for I_1.

$$\begin{aligned}
I_1 &= -\left\langle \frac{(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}) - (\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk})}{k}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk} \right\rangle - \left\langle \boldsymbol{v}_n' - \frac{\boldsymbol{v}_n - \boldsymbol{v}_{n-1}}{k}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk} \right\rangle \\
&\quad + \left\langle \frac{(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}) - (\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk})}{k}, \boldsymbol{v}_n - \boldsymbol{w}^h \right\rangle + \left\langle \boldsymbol{v}_n' - \frac{\boldsymbol{v}_n - \boldsymbol{v}_{n-1}}{k}, \boldsymbol{v}_n - \boldsymbol{w}^h \right\rangle \\
&\le -\frac{1}{2k}(\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2) + \|E_n\|_{V^*}\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V \\
&\quad + \left\langle \frac{(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}) - (\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk})}{k}, \boldsymbol{v}_n - \boldsymbol{w}^h \right\rangle + \|E_n\|_{V^*}\|\boldsymbol{v}_n - \boldsymbol{w}^h\|_V \\
&\le -\frac{1}{2k}(\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2) + c\|E_n\|_{V^*}^2 + \epsilon\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 \\
&\quad + \epsilon\|\boldsymbol{v}_n - \boldsymbol{w}^h\|_V^2 + \left(\frac{(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}) - (\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk})}{k}, \boldsymbol{v}_n - \boldsymbol{w}^h \right)_H ,
\end{aligned} \tag{57}$$

with

$$E_n := \boldsymbol{v}_n' - \frac{\boldsymbol{v}_n - \boldsymbol{v}_{n-1}}{k}. \tag{58}$$

Thirdly, we bound I_2 by using (47) and (48).

$$\begin{aligned}
|I_2| &\le c\|\boldsymbol{v}_n^{hk} - \boldsymbol{w}^h\|_V \|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V + c\|\boldsymbol{v}_n^{hk} - \boldsymbol{w}^h\|_V \, \|\boldsymbol{\theta}_n\|_Q \\
&\le \epsilon\|\boldsymbol{v}_n^{hk} - \boldsymbol{v}_n\|_V^2 + \epsilon\|\boldsymbol{v}_n - \boldsymbol{w}^h\|_V^2 + c\|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\|_V^2 + c\|\boldsymbol{\theta}_n\|_Q^2 \\
&\le c(h^2 + k^2) + \epsilon\|\boldsymbol{v}_n^{hk} - \boldsymbol{v}_n\|_V^2 + \epsilon\|\boldsymbol{v}_n - \boldsymbol{w}^h\|_V^2 \\
&\quad + ck\sum_{j=0}^{n-1}(\|\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j^{hk}\|_Q^2 + \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2).
\end{aligned} \tag{59}$$

Finally, it follows from (3) that

$$\begin{aligned}
I_3 &\le \int_{\Gamma_3} j^0(v_{n\nu}; v_{n\nu}^{hk} - v_{n\nu}) + j^0(v_{n\nu}; v_{n\nu} - w_\nu^h) + j^0(v_{n\nu}; w_\nu^h - v_{n\nu})\, \mathrm{d}\Gamma \\
&\quad + \int_{\Gamma_3} j^0(v_{n\nu}^{hk}; v_{n\nu} - v_{n\nu}^{hk}) + j^0(v_{n\nu}^{hk}; w_\nu^h - v_{n\nu})\, \mathrm{d}\Gamma - \int_{\Gamma_3} j^0(v_{n\nu}; w_\nu^h - v_{n\nu})\, \mathrm{d}\Gamma \\
&= \int_{\Gamma_3} j^0(v_{n\nu}; v_{n\nu}^{hk} - v_{n\nu}) + j^0(v_{n\nu}^{hk}; v_{n\nu} - v_{n\nu}^{hk})\, \mathrm{d}\Gamma \\
&\quad + \int_{\Gamma_3} j^0(v_{n\nu}; v_{n\nu} - w_\nu^h) + j^0(v_{n\nu}^{hk}; w_\nu^h - v_{n\nu})\, \mathrm{d}\Gamma.
\end{aligned}$$

Then, we have the following bound for I_3 from (2), H(j)(iii) and assumption (37).

$$I_3 \le m_j \, \|\gamma\|^2 \, \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 + \int_{\Gamma_3} c_j(2 + |v_{n\nu}| + |v_{n\nu}^{hk}|)|v_{n\nu} - w_\nu^h|\, \mathrm{d}\Gamma$$

$$\leq m_j \,\|\gamma\|^2 \,\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 + c(1 + \|\boldsymbol{v}_n^{hk}\|_V) \,\|\boldsymbol{v}_n - \boldsymbol{w}^h\|_{L^2(\Gamma_3;\mathbb{R}^d)}. \tag{60}$$

By applying (56), (57), (59) and (60) in (51),

$$\begin{aligned}
&\frac{1}{2k}(\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2) + (m_{\mathcal{A}} - m_j\|\gamma\|^2 - 3\epsilon)\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 \\
&\quad \leq c\|\boldsymbol{v}_n - \boldsymbol{w}^h\|_V^2 + |R_n(\boldsymbol{w}^h - \boldsymbol{v}_n)| + c\|E_n\|_{V^*}^2 \\
&\qquad + \left(\frac{(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}) - (\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk})}{k}, \boldsymbol{v}_n - \boldsymbol{w}^h\right)_H + c(h^2 + k^2) \\
&\qquad + ck\sum_{j=0}^{n-1}(\|\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j^{hk}\|_Q^2 + \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2) + c(1 + \|\boldsymbol{v}_n^{hk}\|_V) \,\|\boldsymbol{v}_n - \boldsymbol{w}^h\|_{L^2(\Gamma_3;\mathbb{R}^d)}.
\end{aligned}$$

Choosing $\epsilon = (m_{\mathcal{A}} - m_j\|\gamma\|^2)/6$, we obtain the following inequality, with $\boldsymbol{w}^h \in V^h$ renamed as $\boldsymbol{w}_n^h \in V^h$.

$$\begin{aligned}
&(\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2) + k\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 \\
&\quad \leq ck\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_V^2 + ck|R_n(\boldsymbol{w}_n^h - \boldsymbol{v}_n)| + ck\|E_n\|_{V^*}^2 \\
&\qquad + c((\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}) - (\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}), \boldsymbol{v}_n - \boldsymbol{w}_n^h)_H + ck(h^2 + k^2) \\
&\qquad + ck^2\sum_{j=0}^{n-1}(\|\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j^{hk}\|_Q^2 + \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2) + ck(1 + \|\boldsymbol{v}_n^{hk}\|_V) \,\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_{L^2(\Gamma_3;\mathbb{R}^d)}.
\end{aligned}$$

Replace n with i in the above inequality and make a summation over i from 1 to n,

$$\begin{aligned}
&\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + k\sum_{i=1}^{n}\|\boldsymbol{v}_i - \boldsymbol{v}_i^{hk}\|_V^2 \\
&\quad \leq ck\sum_{i=1}^{n}\|\boldsymbol{v}_i - \boldsymbol{w}_i^h\|_V^2 + ck\sum_{i=1}^{n}|R_i(\boldsymbol{w}_i^h - \boldsymbol{v}_i)| + ck\sum_{i=1}^{n}\|E_i\|_{V^*}^2 \\
&\qquad + c\sum_{i=1}^{n}((\boldsymbol{v}_i - \boldsymbol{v}_i^{hk}) - (\boldsymbol{v}_{i-1} - \boldsymbol{v}_{i-1}^{hk}), \boldsymbol{v}_i - \boldsymbol{w}_i^h)_H \\
&\qquad + c(h^2 + k^2) + ck\sum_{i=1}^{n}k\sum_{j=0}^{i-1}(\|\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j^{hk}\|_Q^2 + \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2) \\
&\qquad + ck\sum_{i=1}^{n}(1 + \|\boldsymbol{v}_i^{hk}\|_V) \,\|\boldsymbol{v}_i - \boldsymbol{w}_i^h\|_{L^2(\Gamma_3;\mathbb{R}^d)}.
\end{aligned} \tag{61}$$

For the term E_n defined by (58), we rewrite it as $E_n = \frac{1}{k}\int_{t_{n-1}}^{t_n}(t - t_{n-1})\boldsymbol{v}''(t)\,\mathrm{d}t$. Then,

$$\|E_n\|_{V^*}^2 \leq \frac{1}{k^2}\int_{t_{n-1}}^{t_n}(t - t_{n-1})^2\,\mathrm{d}t \int_{t_{n-1}}^{t_n}\|\boldsymbol{v}''(t)\|_{V^*}^2\,\mathrm{d}t = \frac{k}{3}\int_{t_{n-1}}^{t_n}\|\boldsymbol{v}''(t)\|_{V^*}^2\,\mathrm{d}t.$$

By regularity assumption (37), we obtain

$$\sum_{i=1}^{n}\|E_i\|_{V^*}^2 \leq \frac{k}{3}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\|\boldsymbol{v}''(t)\|_{V^*}^2\,\mathrm{d}t = \frac{k}{3}\|\boldsymbol{v}''\|_{L^2(0,T;V^*)} \leq ck.$$

For the term $\sum_{i=1}^{n}((\boldsymbol{v}_i-\boldsymbol{v}_i^{hk})-(\boldsymbol{v}_{i-1}-\boldsymbol{v}_{i-1}^{hk}),\boldsymbol{v}_i-\boldsymbol{w}_i^h)_H$, combined with (44), we have

$$\begin{aligned}&\sum_{i=1}^{n}((\boldsymbol{v}_i-\boldsymbol{v}_i^{hk})-(\boldsymbol{v}_{i-1}-\boldsymbol{v}_{i-1}^{hk})\,,\,\boldsymbol{v}_i-\boldsymbol{w}_i^h)_H\\&\quad=(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}_n^h)_H+\sum_{i=1}^{n-1}(\boldsymbol{v}_i-\boldsymbol{v}_i^{hk},(\boldsymbol{v}_i-\boldsymbol{w}_i^h)-(\boldsymbol{v}_{i-1}-\boldsymbol{w}_{i-1}^h))_H\\&\qquad-(\boldsymbol{v}_0-\boldsymbol{v}_0^{hk}\,,\,\boldsymbol{v}_1-\boldsymbol{w}_1^h)_H\\&\quad\le\epsilon\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+c\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_H^2+\frac{k}{2}\sum_{i=1}^{n-1}\|\boldsymbol{v}_i-\boldsymbol{v}_i^{hk}\|_H^2\\&\qquad+\frac{1}{2k}\sum_{i=1}^{n-1}\|(\boldsymbol{v}_i-\boldsymbol{w}_i^h)-(\boldsymbol{v}_{i+1}-\boldsymbol{w}_{i+1}^h)\|_H^2+\frac{1}{2}\|\boldsymbol{v}_1-\boldsymbol{w}_1^h\|_H^2+ch^2.\end{aligned}$$

Also note that $k\sum_{i=1}^{n}(1+\|\boldsymbol{v}_i^{hk}\|_V)^2\le 2k\sum_{i=1}^{n}(1+\|\boldsymbol{v}_i^{hk}\|_V^2)\le c$ from Theorem 4.1. Applying the above inequalities in (61), we deduce that

$$\begin{aligned}&\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+k\sum_{i=1}^{n}\|\boldsymbol{v}_i-\boldsymbol{v}_i^{hk}\|_V^2\\&\quad\le ck\sum_{i=1}^{n}\|\boldsymbol{v}_i-\boldsymbol{w}_i^h\|_V^2+ck\sum_{i=1}^{n}|R_i(\boldsymbol{w}_i^h-\boldsymbol{v}_i)|+c\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_H^2\\&\qquad+ck\sum_{i=1}^{n-1}\|\boldsymbol{v}_i-\boldsymbol{v}_i^{hk}\|_H^2+\frac{c}{k}\sum_{i=1}^{n-1}\|(\boldsymbol{v}_i-\boldsymbol{w}_i^h)-(\boldsymbol{v}_{i+1}-\boldsymbol{w}_{i+1}^h)\|_H^2\\&\qquad+c\|\boldsymbol{v}_1-\boldsymbol{w}_1^h\|_H^2+c(h^2+k^2)\\&\qquad+ck\sum_{i=1}^{n}k\sum_{j=0}^{i-1}(\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q^2+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2)+c\left[k\sum_{i=1}^{n}\|\boldsymbol{v}_i-\boldsymbol{w}_i^h\|_{L^2(\Gamma_3;\mathbb{R}^d)}^2\right]^{1/2}.\end{aligned}\tag{62}$$

Combine (49) and (62), we have

$$\begin{aligned}&\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+k\sum_{i=1}^{n}(\|\boldsymbol{\sigma}_i-\boldsymbol{\sigma}_i^{hk}\|_Q^2+\|\boldsymbol{v}_i-\boldsymbol{v}_i^{hk}\|_V^2)\\&\quad\le ck\sum_{i=1}^{n}\|\boldsymbol{v}_i-\boldsymbol{w}_i^h\|_V^2+ck\sum_{i=1}^{n}|R_i(\boldsymbol{w}_i^h-\boldsymbol{v}_i)|+c\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_H^2\\&\qquad+\frac{c}{k}\sum_{i=1}^{n-1}\|(\boldsymbol{v}_i-\boldsymbol{w}_i^h)-(\boldsymbol{v}_{i+1}-\boldsymbol{w}_{i+1}^h)\|_H^2+c\|\boldsymbol{v}_1-\boldsymbol{w}_1^h\|_H^2+c(h^2+k^2)\\&\qquad+c\left[k\sum_{i=1}^{n}\|\boldsymbol{v}_i-\boldsymbol{w}_i^h\|_{L^2(\Gamma_3;\mathbb{R}^d)}^2\right]^{1/2}+ck\sum_{i=1}^{n-1}\|\boldsymbol{v}_i-\boldsymbol{v}_i^{hk}\|_H^2\\&\qquad+ck\sum_{i=1}^{n}k\sum_{j=0}^{i-1}(\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q^2+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2).\end{aligned}\tag{63}$$

By the Gronwall inequality, we have the following theorem.

Theorem 4.2: *Let* $\{\boldsymbol{v}_n^{hk}\}_{n=0}^N$ *and* $\{\boldsymbol{\sigma}_n^{hk}\}_{n=0}^N$ *be the unique solution of Problem* P_V^{hk} *. Let* $\boldsymbol{v}$ *and* $\boldsymbol{\sigma}$ *be the unique solution of Problem* P_V *. Then under the additional assumptions* (36)–(39)*, we have*

$$\max_{1\le n\le N} \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + k\sum_{i=1}^N (\|\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_i^{hk}\|_Q^2 + \|\boldsymbol{v}_i - \boldsymbol{v}_i^{hk}\|_V^2)$$
$$\le c(h^2+k^2) + c\max_{1\le n\le N} \tilde{E}_n, \tag{64}$$

where

$$\tilde{E}_n = \inf_{w^h\in V^h}\Bigg\{k\sum_{i=1}^n \|\boldsymbol{v}_i - \boldsymbol{w}_i^h\|_V^2 + k\sum_{i=1}^n |R_i(\boldsymbol{w}_i^h - \boldsymbol{v}_i)|$$
$$+\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_H^2 + \frac{1}{k}\sum_{i=1}^{n-1}\|(\boldsymbol{v}_i - \boldsymbol{w}_i^h) - (\boldsymbol{v}_{i+1} - \boldsymbol{w}_{i+1}^h)\|_H^2$$
$$+\|\boldsymbol{v}_1 - \boldsymbol{w}_1^h\|_H^2 + \left[k\sum_{i=1}^n \|\boldsymbol{v}_i - \boldsymbol{w}_i^h\|_{L^2(\Gamma_3;\mathbb{R}^d)}^2\right]^{1/2}\Bigg\}. \tag{65}$$

We furthermore derive the optimal error estimate. Choose $\boldsymbol{w}_i^h \in V^h$ to be the finite element interpolation of $\boldsymbol{v}_i$, $1\le i\le N$. We apply the standard finite element interpolation error estimates ([6]),

$$\|\boldsymbol{v}_i - \boldsymbol{w}_i^h\|_V \le ch\|\boldsymbol{v}_i\|_{H^2(\Omega;\mathbb{R}^d)},$$
$$\|\boldsymbol{v}_i - \boldsymbol{w}_i^h\|_H \le ch\|\boldsymbol{v}_i\|_V,$$
$$\|\boldsymbol{v}_i - \boldsymbol{w}_i^h\|_{L^2(\Gamma_3;\mathbb{R}^d)} \le ch^2\|\boldsymbol{v}_i\|_{H^2(\Gamma_3;\mathbb{R}^d)}.$$

Since $\boldsymbol{w}_i^h - \boldsymbol{w}_{i+1}^h$ is the finite element interpolation of $\boldsymbol{v}_i - \boldsymbol{v}_{i+1}$, we have

$$\sum_{i=1}^{n-1}\|(\boldsymbol{v}_i - \boldsymbol{w}_i^h) - (\boldsymbol{v}_{i+1} - \boldsymbol{w}_{i+1}^h)\|_H^2 \le ch^2\sum_{i=1}^{n-1}\|\boldsymbol{v}_i - \boldsymbol{v}_{i+1}\|_V^2$$
$$\le ch^2k\sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}\|\boldsymbol{v}'(t)\|_V^2\,\mathrm{d}t$$
$$\le ch^2k.$$

Then

$$\tilde{E}_n \le ch^2 + k\sum_{i=1}^n |R_i(\boldsymbol{w}_i^h - \boldsymbol{v}_i)|.$$

Similar to [11], it can be shown that under the stated regularity assumptions, for a.e. $t\in(0,T)$

$$\boldsymbol{v}'(t) - \mathrm{Div}\,\boldsymbol{\sigma}(t) = \boldsymbol{f}_0(t) \text{ a.e. in } \Omega, \tag{66}$$
$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \text{ a.e. on } \Gamma_2. \tag{67}$$

We multiply Equation (66) by an arbitrary function $\boldsymbol{w} \in V$, integrate over Ω and perform integration by parts,

$$\begin{aligned}&\langle \boldsymbol{v}'(t), \boldsymbol{w}\rangle + (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q \\ &\quad + \left(\int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,\mathrm{d}s, \boldsymbol{\varepsilon}(\boldsymbol{w})\right)_Q - \int_{\Gamma_3} \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{w}\,\mathrm{d}\Gamma \ \geq\ \langle \boldsymbol{f}(t), \boldsymbol{w}\rangle.\end{aligned}$$

Then $R_n(\boldsymbol{w})$ can be rewritten as

$$R_n(\boldsymbol{w}) = \int_{\Gamma_3} j^0(v_{n\nu}; w_\nu)\,\mathrm{d}\Gamma + \int_{\Gamma_3} \boldsymbol{\sigma}_n\boldsymbol{\nu}\cdot\boldsymbol{w}\,\mathrm{d}\Gamma. \tag{68}$$

Note that $\boldsymbol{\sigma} \in C(0,T;H^1(\Omega;\mathbb{S}^d))$ implies $\boldsymbol{\sigma}\boldsymbol{\nu} \in C(0,T;L^2(\Gamma_3;\mathbb{R}^d))$. Then,

$$R_n(\boldsymbol{w}) \leq c\|\boldsymbol{w}\|_{L^2(\Gamma_3;\mathbb{R}^d)} \quad \text{for all } \boldsymbol{w} \in V.$$

Thus,

$$|R_i(\boldsymbol{w}_i^h - \boldsymbol{v}_i)| \leq c\|\boldsymbol{w}_i^h - \boldsymbol{v}_i\|_{L^2(\Gamma_3;\mathbb{R}^d)} \leq ch^2\|\boldsymbol{v}_i\|_{H^2(\Gamma_3;\mathbb{R}^d)}. \tag{69}$$

Corollary 4.3: *Let $\{\boldsymbol{v}_n^{hk}\}_{n=0}^N$ and $\{\boldsymbol{\sigma}_n^{hk}\}_{n=0}^N$ be the unique solution of Problem P_V^{hk} . Let $\boldsymbol{v}$ and $\boldsymbol{\sigma}$ be the unique solution of Problem P_V . Under the regularity assumptions in Theorem 4.2, we have*

$$\max_{1\leq n\leq N}\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + k\sum_{i=1}^N(\|\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_i^{hk}\|_Q^2 + \|\boldsymbol{v}_i - \boldsymbol{v}_i^{hk}\|_V^2) \leq c(h^2+k^2). \tag{70}$$

Finally, we bound the error estimate for the displacement field. From (46), we can deduce that

$$\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 \leq c(h^2+k^2) + ck\sum_{j=1}^n\|v_j - v_j^{hk}\|_V^2 \leq c(h^2+k^2).$$

The corresponding optimal-order error estimate for the displacement field is as follows.

Corollary 4.4: *Let $\{\boldsymbol{v}_n^{hk}\}_{n=0}^N$ and $\{\boldsymbol{\sigma}_n^{hk}\}_{n=0}^N$ be the unique solution of Problem P_V^{hk} . Define the corresponding $\{\boldsymbol{u}_n^{hk}\}_{n=0}^N$ by* (23)*. Let $\boldsymbol{v}$ and $\boldsymbol{\sigma}$ be the unique solution of Problem P_V. Define the corresponding $\boldsymbol{u}$ by* (16)*. Under the regularity assumptions in Theorem* 4.2*, we have*

$$\max_{1\leq n\leq N}\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V \leq c(h+k). \tag{71}$$

5. Numerical results

In this section, we present some numerical simulations, which are based on the primal–dual active set method [3].

The physical setting is depicted in Figure 1. The deformable body $\Omega = (0, L_1)\times(0, L_2)$ is a rectangle with the boundary Γ divided into three parts

$$\Gamma_1 = \{0\}\times[0, L_1], \Gamma_2 = (\{L_1\}\times(0, L_2])\cup((0, L_1)\times\{L_2\}), \Gamma_3 = (0, L_1]\times\{0\}.$$

It represents the cross-section of a three-dimensional viscoplastic body, and a plane stress hypothesis is assumed.

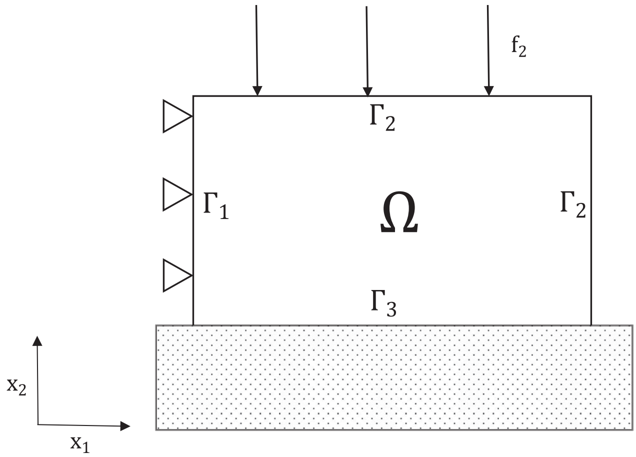

Figure 1. Initial configuration of the two-dimensional example.

The frictionless contact boundary conditions on Γ_3 are given by

$$-\sigma_\nu = \begin{cases} -c_\nu^3(v_\nu + r_2) - c_\nu^2(r_1 - r_2) + c_\nu^1 r_1 & \text{if } v_\nu < -r_2, \quad \text{(S1)} \\ -c_\nu^2(v_\nu + r_1) + c_\nu^1 r_1 & \text{if } -r_2 \le v_\nu < -r_1, \quad \text{(S2)} \\ -c_\nu^1 v_\nu & \text{if } -r_1 \le v_\nu < 0, \quad \text{(S3)} \\ c_\nu^1 v_\nu & \text{if } 0 \le v_\nu < r_1, \quad \text{(S4)} \\ c_\nu^2(v_\nu - r_1) + c_\nu^1 r_1 & \text{if } r_1 \le v_\nu < r_2, \quad \text{(S5)} \\ c_\nu^3(v_\nu - r_2) + c_\nu^2(r_2 - r_1) + c_\nu^1 r_1 & \text{if } r_2 \le v_\nu, \quad \text{(S6)} \end{cases} \tag{72}$$

$$\boldsymbol{\sigma}_\tau = \mathbf{0}. \tag{73}$$

In normal direction, it is a piecewise function with six segments, corresponding to six states of the velocity value (S1)–(S6). Figure 2 shows the dependence of $-\sigma_\nu$ as a function of the normal velocity v_ν.

The viscosity tensor $\mathcal{A}$ and the elasticity tensor $\mathcal{E}$ are characterized by

$$(\mathcal{A}\tau)_{ij} = \mu_1(\tau_{11} + \tau_{22})\delta_{ij} + \mu_2\tau_{ij}, \quad 1 \le i,j \le 2,$$

$$(\mathcal{E}\tau)_{ij} = \frac{E\kappa}{1-\kappa^2}(\tau_{11} + \tau_{22})\delta_{ij} + \frac{E}{1+\kappa}\tau_{ij}, \quad 1 \le i,j \le 2.$$

The coefficients μ_1 and μ_2 are viscosity constants, E and κ are Young's modulus and Poisson's ratio of the material, respectively. δ_{ij} denotes the Kronecker symbol. Assume that the nonlinear viscoplastic constitutive law $\mathcal{G}$ is of the classical Perzyna type [8]

$$\mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) = -\frac{1}{2\lambda}\mathcal{E}(\boldsymbol{\sigma} - \mathcal{P}_K\boldsymbol{\sigma}),$$

where $\lambda > 0$ is a constant and $\mathcal{P}_K$ is the orthogonal projection operator (with respect to the norm $\|\boldsymbol{\tau}\| = (\mathcal{E}\boldsymbol{\tau}, \boldsymbol{\tau})^{1/2}$) over the convex subset $K \subset \mathbb{S}^2$. The subset K is given by

$$K = \{\boldsymbol{\tau} \in \mathbb{S}^2 \ : \|\boldsymbol{\tau}\|_{VM} \le \sigma_Y\},$$

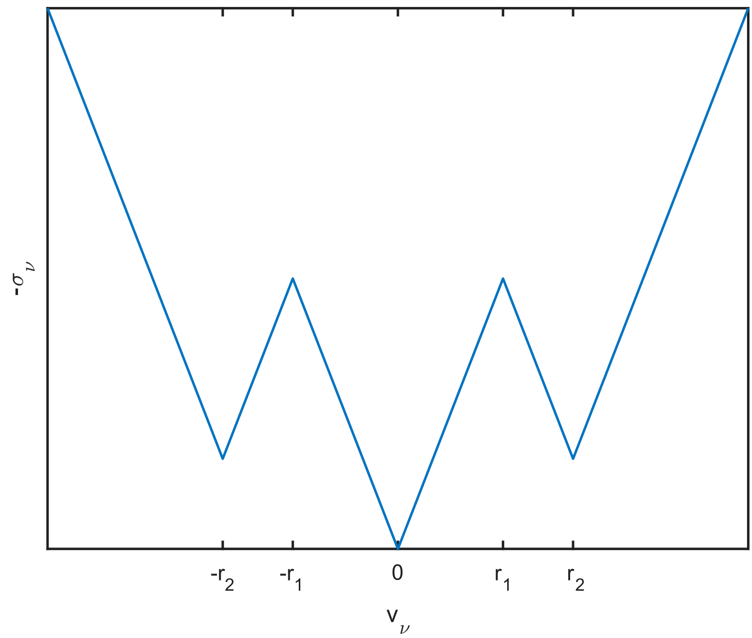

Figure 2. Dependence of $-\sigma_\nu$ on v_ν.

where $\|\cdot\|_{VM}$ represents the von Mises norm defined by

$$\|\boldsymbol{\tau}\|_{VM}^2 = \tau_{11}^2 + \tau_{22}^2 - \tau_{11}\tau_{22} + 3\tau_{12}^2,$$

and σ_Y is the uniaxial yield stress.

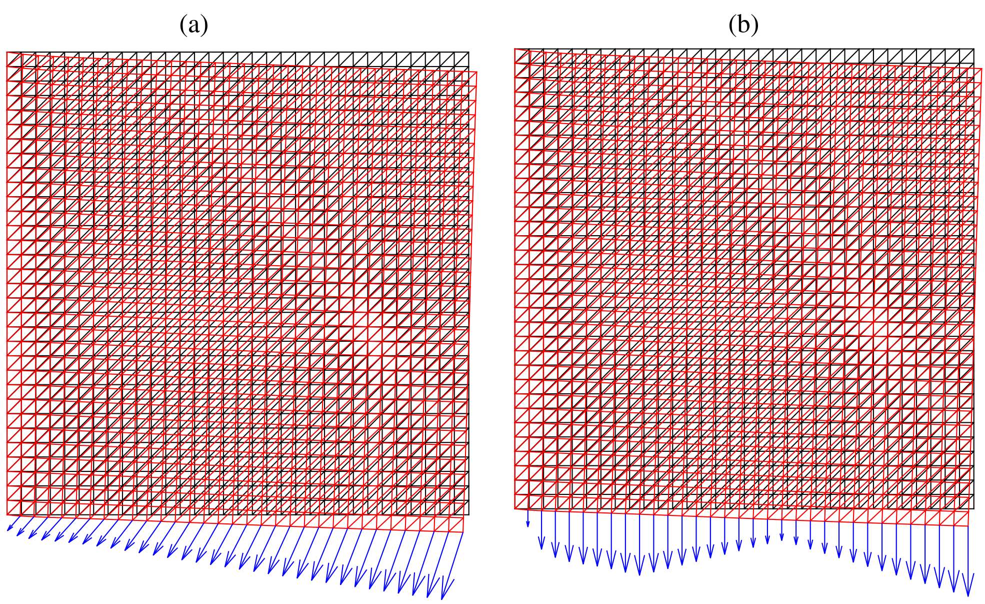

Figure 3. Deformed meshes, contact velocity and interface forces at $T = 1$. (a) Contact velocity at $T = 1$. (b) Interface forces at $T = 1$.

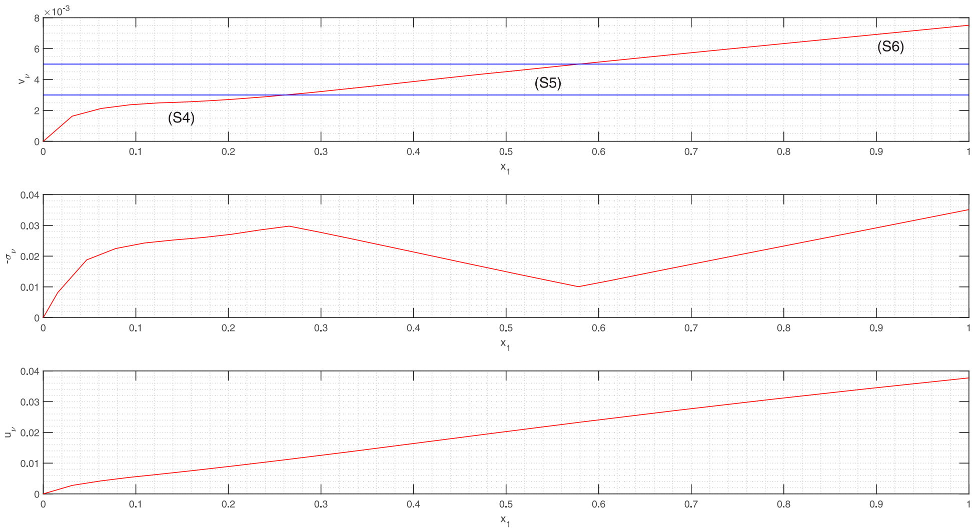

Figure 4. Normal velocity, interface forces, and normal displacement at $T = 1$ on Γ_3.

Choose the density of mass to be $\rho = 1\text{kg/m}^2$. We use the following parameters:

$$
\begin{aligned}
&L_1 = 1\text{m}\,, \quad L_2 = 1\text{m}\,, \quad T = 1\text{s}\,,\\
&\mu_1 = 25\text{Ns/m}^2, \quad \mu_2 = 50\text{Ns/m}^2, \quad E = 1000\text{N/m}^2, \quad \kappa = 0.4,\\
&\lambda = 100\text{Ns/m}^2, \quad \sigma_Y = 19\text{N/m}^2,\\
&r_1 = 0.003\text{m/s}, \quad r_2 = 0.005\text{m/s},\\
&c_\nu^1 = 10\text{Ns/m}^2, \quad c_\nu^2 = -10\text{Ns/m}^2, \quad c_\nu^3 = 10\text{Ns/m}^2,\\
&\boldsymbol{f}_0(t) = (0,0)\text{N/m}^2,
\end{aligned}
$$

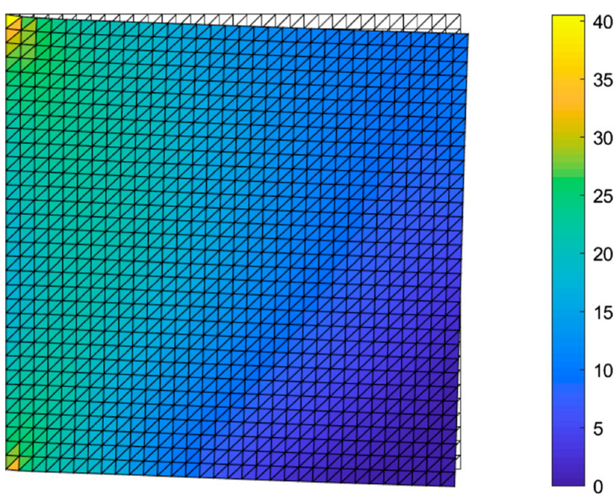

Figure 5. The von Mises norm of the stress $\boldsymbol{\sigma}$ at $T = 1$ in Ω.

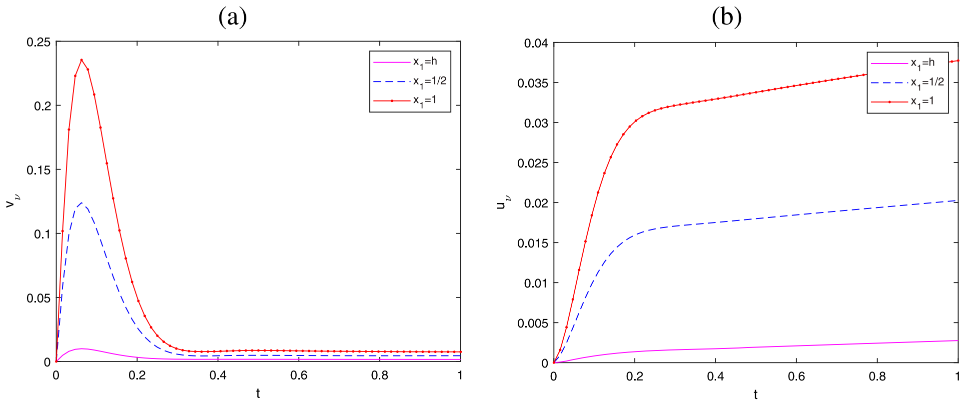

Figure 6. Evolution of normal velocity and normal displacement. (a) Evolution of the normal velocity v_ν. (b) Evolution of the normal displacement u_ν.

$$f_2(t) = \begin{cases} (0, -10)\ \text{N/m} & \text{on } (0, L_1) \times \{L_2\}, \\ (0, 0)\ \text{N/m} & \text{on } \{L_1\} \times (0, L_2], \end{cases}$$

$$\boldsymbol{u}_0 = \mathbf{0}\text{m}, \quad \boldsymbol{v}_0 = \mathbf{0}\text{m/s}.$$

The time step and spatial step are chosen to be $h = \frac{1}{32}, k = \frac{1}{64}$. Our numerical results are presented in Figures 3–6 and Table 1.

In Figure 3, the deformed configuration, the contact velocity, and the interface forces are plotted at $T = 1s$. When x_1 goes from 0 to 1, the normal velocity is increasing while the interface forces are not changing monotonically. Note that, since $\boldsymbol{\sigma}_\tau = \mathbf{0}$ on Γ_3, the stress $\boldsymbol{\sigma}$ is decided by the normal component σ_ν. As σ_ν takes the non-positive value, we introduce the interface forces $-\sigma_\nu$ on Γ_3.

Table 1. Error estimates and convergence order.

h	τ	$\max_{1\le n\le N} \Vert \boldsymbol{v}_n - \boldsymbol{v}_n^{hk} \Vert_H$	Convergence order
1/4	1/8	0.0573	\
1/8	1/16	0.0204	1.4900
1/16	1/32	0.0124	0.7182
1/32	1/64	0.0051	1.2818
h	τ	$\left(k \sum_{i=1}^{N} \Vert \boldsymbol{\sigma}_i - \boldsymbol{\sigma}_i^{hk} \Vert_Q^2\right)^{1/2}$	Convergence order
1/4	1/8	4.6970	\
1/8	1/16	2.6479	0.8269
1/16	1/32	1.3486	0.9734
1/32	1/64	0.5847	1.2057
h	τ	$\left(k \sum_{i=1}^{N} \Vert \boldsymbol{v}_i - \boldsymbol{v}_i^{hk} \Vert_V^2\right)^{1/2}$	Convergence order
1/4	1/8	$8.7006 * 10^{-4}$	\
1/8	1/16	$1.6826 * 10^{-4}$	2.3704
1/16	1/32	$7.6837 * 10^{-5}$	1.1308
1/32	1/64	$3.1544 * 10^{-5}$	1.2844
h	τ	$\max_{1\le n\le N} \Vert \boldsymbol{u}_n - \boldsymbol{u}_n^{hk} \Vert_V$	Convergence order
1/4	1/8	$1.0693 * 10^{-4}$	\
1/8	1/16	$5.6726 * 10^{-5}$	0.9146
1/16	1/32	$3.4261 * 10^{-5}$	0.7274
1/32	1/64	$1.7603 * 10^{-5}$	0.9607

In fact, as x_1 goes from 0 to 1, Figure 4 shows that the normal velocity is firstly in the state of (S4), then (S5), and finally (S6) on Γ_3. The interface forces are firstly increasing with respect to the normal velocity when in the state of (S4), then decreasing in the state of (S5), and finally increasing in the state of (S6). This agrees with the theory since we have the nonmonotone normal boundary condition (71). We also plot the normal displacement at $T = 1$ on contact boundary Γ_3 in Figure 4.

In Figure 5, we show the von Mises norm of the stress at $T = 1$ in Ω. Due to the twisting forces, when spatial point approaches Γ_1, the $\|\boldsymbol{\sigma}\|_{VM}$ gradually increases.

In Figure 6, we plot the evolution of normal velocity and normal displacement at three points $x_1 = h, 0.5, 1$ on the boundary Γ_3.

We also compute the numerical errors for several values of discretization parameter h and k, see Table 1. Since the exact solution is unknown, we take the numerical solution corresponding to $h = \frac{1}{64}, k = \frac{1}{128}$ as the 'reference' solution $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\sigma})$. The theoretically predicted first-order convergence of the numerical solution can be observed.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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