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Generalized penalty method for history-dependent variational–hemivariational inequalities

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ABSTRACT

We consider a history-dependent variational–hemivariational inequality with unilateral constraints in a reflexive Banach space. The unique solvability of the inequality follows from an existence and uniqueness result obtained in Sofonea and Migórski (2016, 2018). In this current paper we introduce and study a generalized penalty method associated to the inequality. To this end we consider a sequence of generalized penalty problems, governed by a parameter λ_n and an operator P_n. We prove the unique solvability of the penalty problems as well as the convergence of corresponding solutions sequence to the solution of original problem. These results extend the previous results in Sofonea et al. (2018) and Xiao and Sofonea (2019). Finally, we illustrate them in the study of a history-dependent problem with unilateral boundary conditions which describes the quasistatic evolution of a rod–spring system under the action of given applied force.

1. Introduction

A large number of processes which arise in Mechanics, Physics and Engineering Sciences are described by boundary value problems which, in a weak formulation, lead to mathematical models expressed in terms of variational or hemivariational inequalities. Variational inequalities refer to those inequality problems which have a convex structure. They have been studied extensively for over half a century since 1960s, both theoretically and numerically, by using arguments of convex analysis. Representative references in the field include [1–3] and, more recently, [4–10]. The notion of hemivariational inequality was introduced in [11] in the study of engineering problems involving non-smooth, non-monotone and possibly multivalued

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relations for deformable bodies. Since then, the theory of hemivariational inequalities grew rapidly. It uses as main ingredient the properties of the subdifferential in the sense of Clarke, defined for locally Lipschitz functions which may be nonconvex. Comprehensive references in the area include [12–14] and, more recently, [15–20]. Finally, variational–hemivariational inequalities represent a special class of inequalities, driven by both convex and nonconvex functions. They represent a powerful tool in the study of a wide range of nonlinear boundary value problems with or without unilateral constraints, as shown in [21–25].

History-dependent operators represent a class of nonlinear operators defined on spaces of vector-valued continuous or Lebesgue integrable functions. They arise in Contact Mechanics and describe various memory effects which appear either in the material's behaviour or in the contact conditions. Variational–hemivariational inequalities involving in their structure a history-dependent operator are called history-dependent variational–hemivariational inequalities. They have been intensively studied in the recent literature. General existence and uniqueness results can be found in [26,27], together with various applications in Contact Mechanics. A convergence result which shows the continuous dependence of the solution with respect to the data was obtained in [28]. The numerical analysis of history-dependent variational–hemivariational inequalities can be found in [29,30]. There, numerical schemes have been considered and error estimates have been derived. Additional results on inequality problems with history-dependent operators can be found in [26,31–33].

The current paper represents a continuation of [34,35]. Indeed, the paper [34] was devoted to the study of a penalty method for history-dependent variational–hemivariational inequalities. There, the corresponding unconstrained problems have been constructed with a given penalty operator P and a convergence result was proved. In [35] we studied a generalized penalty method in the study of elliptic variational–hemivariational inequalities, i.e., time-independent variational–hemivariational inequalities. There, in contrast to [34], the unconstrained problems have been constructed with a sequence of penalty operators, denoted $\{P_n\}$.

The aim of the current paper is twofold. The first one is to use the generalized penalty method in the study of history-dependent variational–hemivariational inequalities. Thus, our main convergence result, Theorem 16, extends our previous work in [34,35], since these results can be obtained in the particular cases when $P_n = P$ for each $n \in \mathbb{N}$ and when the time is removed, respectively. Recall also that the proof of Theorem 16 is based on assumptions on the locally Lipschitz function j which are less restrictive than that used in [34,35]. This ingredient represents one of the traits of novelty of this contribution, as we mention in Remark 20. The second aim of the current paper is to illustrate the use of Theorem 16 in the study of a new mathematical model of contact and to provide the corresponding mechanical interpretations.

The outline of the paper is as follows. Basic notation and preliminary material needed in the rest of the paper are recalled in Section 2. In Section 3 we state the original inequality problem and the penalty problems, together with their unique solvability. Then, in Section 4 we state and prove our main result, Theorem 16, which states that the sequence of solutions of the generalized penalty problems converges to the solution of the original problem. The proof is carried out in several steps, based on arguments of compactness, pseudomonotonicity and the properties of the Clarke subdifferential. Finally, in Section 5 we illustrate the use of this abstract convergence result in the study of a nonlinear boundary value problem which describes the quasistatic evolution of a rod–spring system with unilateral constraints.

2. Background material

In this section we shortly recall some notation and preliminaries which are needed in the rest of the paper. For more details on the material presented below we refer the reader to [10,12,36–38].

Everywhere in this paper X represents a reflexive Banach space with dual X^* and $\langle\cdot,\cdot\rangle$ denotes the duality between X^* and X. We use the notations $\|\cdot\|_X$ and $\|\cdot\|_{X^*}$ for the norm on the spaces X and X^*, respectively, and 0_{X^*} for the zero element of X^*. Throughout the paper all the limits, upper and lower limits below are considered as $n \to \infty$, even if we do not mention it explicitly. Also, the symbols "$\rightharpoonup$" and "$\to$" stand for the weak and the strong convergence in various spaces, respectively, which will be specified. Finally, we denote by 2^{X^*} the set of all subsets of X^*.

We start by recalling the following definitions concerning single valued operators.

Definition 1. An operator $A : X \to X^*$ is said to be:

(a) monotone if $\langle Au - Av, u - v\rangle \geq 0$, for all $u, v \in X$;
(b) strongly monotone, if there exists $m_A > 0$ such that

$$\langle Au - Av, u - v\rangle \geq m_A \|u - v\|_X^2, \quad \text{for all } u,\, v \in X;$$

(c) bounded, if A maps bounded sets of X into bounded sets of X^*;
(d) pseudomonotone, if it is bounded and $u_n \rightharpoonup u$ in X with $\limsup \langle Au_n, u_n - u\rangle \leq 0$ implies $\liminf \langle Au_n, u_n - v\rangle \geq \langle Au, u - v\rangle$ for all $v \in X$;
(e) demicontinuous, if $u_n \to u$ in X implies $Au_n \rightharpoonup Au$ in X^*.

We shall use the following properties of pseudomonotone operators.

Proposition 2.

(a) *If the operator $A\colon X \to X^*$ is bounded, demicontinuous and monotone, then A is pseudomonotone.*
(b) *If A, $B\colon X \to X^*$ are pseudomonotone operators, then the sum $A + B\colon X \to X^*$ is pseudomonotone.*

We now recall the notions of the pseudomonotonicity and generalized pseudomonotonicity for multivalued operators.

Definition 3. An operator $A : X \to 2^{X^*}$ is said to be pseudomonotone if:

(a) for each $v \in X$, the set $Av \subset X^*$ is nonempty, bounded, closed and convex;
(b) A is upper semicontinuous from each finite dimensional subspace of X to X^* endowed with the weak topology;
(c) for any sequences $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ such that

$$u_n \rightharpoonup u \text{ in } X,\ \ u_n^* \in Au_n \text{ for all } n \in \mathbb{N} \text{ and } \limsup \langle u_n^*, u_n - u\rangle \leq 0,$$

and any $v \in X$, there exists $u^*(v) \in Au$ such that

$$\langle u^*(v), u - v\rangle \leq \liminf \langle u_n^*, u_n - v\rangle.$$

Definition 4. An operator $A\colon X \to 2^{X^*}$ is said to be generalized pseudomonotone, if for any sequences $\{u_n\} \subset X$, $\{u_n^*\} \subset X^*$ with $u_n^* \in Au_n$, $u_n \rightharpoonup u$ in X, $u_n^* \rightharpoonup u^*$ in X^* and

$$\limsup \langle u_n^*, u_n - u\rangle \leq 0,$$

we have $u^* \in Au$ and $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$.

It is well known that every pseudomonotone operator is generalized pseudomonotone, while the converse holds under an additional boundedness condition, see [17, Proposition 3.58].

For real valued functions defined on X we recall the following definition.

Definition 5. Let $\varphi\colon X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. Then:

(a) the function φ is proper, if it is not identically equal to $+\infty$, i.e., the effective domain $\operatorname{dom}\varphi$ is nonempty, where $\operatorname{dom}\varphi = \{v \in X \mid \varphi(v) < +\infty\}$.
(b) the function φ is (sequentially) lower semicontinuous (l.s.c., for short), if $v_n \to v$ in X implies $\varphi(v) \leq \liminf \varphi(v_n)$.

Moreover, the following property holds.

Proposition 6. *Let $\varphi\colon X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be a convex l.s.c. function such that $\operatorname{dom}\varphi = X$. Then, φ is continuous.*

We now recall the notion of generalized (Clarke) subdifferential for a locally Lipschitz function.

Definition 7. A function $j\colon X \to \mathbb{R}$ is called to be locally Lipschitz continuous if, for each $u \in X$, there exist a neighbourhood O_u of u and a constant $L_u > 0$ such that

$$|j(w) - j(v)| \leq L_u \|w - v\|_X \quad \text{for all} \quad w, v \in O_u.$$

Given a locally Lipschitz function $j\colon X \to \mathbb{R}$, we denote by $j^0(u; v)$ the generalized (Clarke) directional derivative of J at the point $u \in X$ in the direction $v \in X$ defined by

$$j^0(u; v) = \limsup_{\lambda \to 0^+,\, w \to u} \frac{j(w + \lambda v) - j(w)}{\lambda}.$$

The generalized (Clarke) gradient of $j\colon X \to \mathbb{R}$ at $u \in X$ is given by

$$\partial j(u) = \{\, \xi \in X^* \mid j^0(u; v) \geq \langle \xi, v \rangle \quad \text{for all} \quad v \in X \,\}.$$

Moreover, the function j is said to be regular (in the sense of Clarke) at the point $u \in X$ if, for all $v \in X$, the one-sided directional derivative $j'(u; v)$ exists and $j^0(u; v) = j'(u; v)$.

The generalized gradient and generalized directional derivative of a locally Lipschitz functional enjoy a number of properties that we gather in the following result, which corresponds to [17, Proposition 3.23].

Proposition 8. *Assume that $j\colon X \to \mathbb{R}$ is a locally Lipschitz function. Then:*

(a) *for every $x \in X$, the function $X \ni v \mapsto j^0(x; v) \in \mathbb{R}$ is positively homogeneous and subadditive, i.e., $j^0(x; \lambda v) = \lambda j^0(x; v)$ for all $\lambda \geq 0$, $v \in X$ and $j^0(x; v_1 + v_2) \leq j^0(x; v_1) + j^0(x; v_2)$ for all v_1, $v_2 \in X$.*
(b) *for every $v \in X$ it holds $j^0(x; v) = \max\{\, \langle \xi, v \rangle \ : \ \xi \in \partial j(x) \,\}$.*
(c) *the function $X \times X \ni (u, v) \mapsto j^0(u; v) \in \mathbb{R}$ is upper semicontinuous.*

Furthermore, we review the notion of the penalty operator.

Definition 9. An operator $P\colon X \to X^*$ is said to be a penalty operator of the set $K \subset X$ if P is bounded, demicontinuous, monotone and $K = \{x \in X \mid Px = 0_{X^*}\}$.

We recall that any reflexive Banach space X can be always considered as equivalently renormed strictly convex space. Therefore, the duality map $J\colon X \to 2^{X^*}$, defined by

$$J(x) = \{\, x^* \in X^* \mid \langle x^*, x \rangle = \|x\|_X^2 = \|x^*\|_{X^*}^2 \,\}, \quad \text{for all} \quad x \in X,$$

is a single-valued operator. Moreover, if K is a nonempty, closed and convex subset of reflexive Banach space X, then the operator $P = J(I - P_K)$ is a penalty operator of K, where I is the identity map on X, and $P_K\colon X \to K$ is the projection operator of K (see [37, Proposition 1.3.27]). We conclude that, under the assumptions above, the penalty operator always exists.

Assume now that Y is a normed space endowed with the norm $\|\cdot\|_Y$. We denote by $C(\mathbb{R}_+;X)$ and $C(\mathbb{R}_+;Y)$ the space of continuous functions defined on $\mathbb{R}_+$ with values on X and Y, respectively. Moreover, for $K \subset X$, we denote by $C(\mathbb{R}_+;K)$ the set of functions defined on $\mathbb{R}_+$ with values on K. We now recall the following definition used in [27,39], for instance.

Definition 10. An operator $\mathcal{S}\colon C(\mathbb{R}_+;X) \to C(\mathbb{R}_+;Y)$ is said to be a history-dependent operator if, for each $n \in \mathbb{N}$, there exists $s_n > 0$ satisfying

$$\|(\mathcal{S}u_1)(t) - (\mathcal{S}u_2)(t)\|_Y \le s_n \int_0^t \|u_1(s) - u_2(s)\|_X\, ds$$

for all $u_1, u_2 \in C(\mathbb{R}_+;X)$ and all $t \in [0,n]$.

Note that basic properties of history-dependent operators together with various examples in Solid and Contact Mechanics can be found in [27, Ch.3].

3. Problem statement and well-posedness results

We now turn to the inequality problem we consider in this paper. The functional framework is the following. Let K be a subset of X and let $(Y, \|\cdot\|_Y)$ be a normed space. Given two operators $A\colon X \to X^*$ and $\mathcal{S}\colon C(\mathbb{R}_+;X) \to C(\mathbb{R}_+;Y)$, a function $\varphi\colon Y \times X \times X \to \mathbb{R}$, a locally Lipschitz function $j\colon X \to \mathbb{R}$ and a function $f\colon \mathbb{R}_+ \to X^*$, we consider the following history-dependent variational–hemivariational inequality.

Problem 11. Find $u \in C(\mathbb{R}_+;K)$ such that, for all $t \in \mathbb{R}_+$,

$$\begin{aligned} &\langle Au(t), v - u(t)\rangle + \varphi((\mathcal{S}u)(t), u(t), v) - \varphi((\mathcal{S}u)(t), u(t), u(t)) \\ &\quad + j^0(u(t); v - u(t)) \ge \langle f(t), v - u(t)\rangle \quad \text{for all } v \in K. \end{aligned} \tag{3.1}$$

Note that Problem 11 is governed by a constraint set K. Therefore, for both theoretical and numerical reasons, it is useful to approximate it by using a penalty method. The classical penalty method replaces Problem 11 by a sequence of unconstrained inequality problems which, for every $n \in \mathbb{N}$, can be formulated as follows.

Problem 12. Find $u_n \in C(\mathbb{R}_+;X)$ such that, for all $t \in \mathbb{R}_+$,

$$\begin{aligned} &\langle Au_n(t), v - u_n(t)\rangle + \frac{1}{\lambda_n}\langle Pu_n(t), v - u_n(t)\rangle + \varphi((\mathcal{S}u_n)(t), u_n(t), v) \\ &\quad - \varphi((\mathcal{S}u_n)(t), u_n(t), u_n(t)) + j^0(u_n(t); v - u_n(t)) \\ &\quad \ge \langle f(t), v - u_n(t)\rangle \quad \text{for all } v \in X. \end{aligned} \tag{3.2}$$

Note that Problem 12 is formally obtained from Problem 11 by removing the constraint $u \in K$ and including a penalty term governed by a parameter $\lambda_n > 0$ and an operator $P : X \to X^*$. Penalty methods have been used as an approximation tool to treat constraints in variational inequalities, as shown in [2,3,39,40], and variational–hemivariational inequalities, as shown in [27,34,41], for instance.

An extension of Problem 12 can be obtained by replacing in (3.2) the operator P with an operator $P_n : X \to X^*$ which depends on $n \in \mathbb{N}$. It leads to the following generalized penalty problem associated with Problem 11.

Problem 13. Find $u_n \in C(\mathbb{R}_+; X)$ such that, for all $t \in \mathbb{R}_+$,

$$\begin{aligned} &\langle Au_n(t), v - u_n(t)\rangle + \frac{1}{\lambda_n}\langle P_n u_n(t), v - u_n(t)\rangle + \varphi((\mathcal{S}u_n)(t), u_n(t), v) \\ &-\varphi((\mathcal{S}u_n)(t), u_n(t), u_n(t)) + j^0(u_n(t); v - u_n(t)) \\ &\geq \langle f(t), v - u_n(t)\rangle \quad \text{for all } v \in X. \end{aligned} \tag{3.3}$$

Likewise, Problem 13 is formally obtained from Problem 11 by removing the constraint $u \in K$ and including a penalty term governed by a parameter $\lambda_n > 0$ and an operator $P_n : X \to X^*$ which, in contrast to (3.2), depends on n. Moreover, note that Problem 12 is a particular case of Problem 13, obtained when $P_n = P$ for all $n \in \mathbb{N}$.

In order to state existence and uniqueness results for Problems 11 and 13, we impose the following assumptions on the data.

$\underline{H(K)}$: K is a nonempty, closed and convex subset of X.

$\underline{H(A)}$: $A\colon X \to X^*$ is pseudomonotone and strongly monotone with constant $m_A > 0$.

$\underline{H(\mathcal{S})}$: $\mathcal{S}\colon C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$ is a history-dependent operator with constant $s_n > 0$, for each $n \in \mathbb{N}$.

$\underline{H(\varphi)}$: $\varphi\colon Y \times X \times X \to \mathbb{R}$ is a function such that:

(a) $\varphi(y, u, \cdot)\colon X \to \mathbb{R}$ is convex and l.s.c., for all $y \in Y$ and all $u \in X$;

(b) there exist constants $\alpha_\varphi \geq 0$ and $\beta_\varphi \geq 0$ such that

$$\begin{aligned} &\varphi(y_1, u_1, v_2) - \varphi(y_1, u_1, v_1) + \varphi(y_2, u_2, v_1) - \varphi(y_2, u_2, v_2) \\ &\quad\leq \alpha_\varphi \|u_1 - u_2\|_X \, \|v_1 - v_2\|_X + \beta_\varphi \|y_1 - y_2\|_Y \, \|v_1 - v_2\|_X, \end{aligned}$$

for all $y_1, y_2 \in Y$ and all $u_1, u_2, v_1, v_2 \in X$.

$\underline{H(j)}$: $j\colon X \to \mathbb{R}$ is a function such that:

(a) j is locally Lipschitz continuous;

(b) there exist constants $c_0 \geq 0$ and $c_1 > 0$ such that

$$\|\xi\|_{X^*} \leq c_0 + c_1\|v\|_X, \quad \text{for all } \xi \in \partial j(v) \text{ and } v \in X;$$

(c) there exists $\alpha_j \geq 0$ such that

$$j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \leq \alpha_j \|v_1 - v_2\|_X^2, \quad \text{for all } v_1, v_2 \in X.$$

$\underline{H(s)}$: $\alpha_\varphi + \alpha_j < m_A$.

$\underline{H(f)}$: $f \in C(\mathbb{R}_+; X^*)$.

We have the following existence and uniqueness result.

Theorem 14. *Let X be a reflexive Banach space, Y a normed space and assume $H(K)$, $H(A)$, $H(\mathcal{S})$, $H(\varphi)$, $H(j)$, $H(s)$, $H(f)$. Then Problem 11 has a unique solution $u \in C(\mathbb{R}_+; K)$.*

The proof of Theorem 14 can be found in [27]. It is obtained by using arguments of elliptic variational–hemivariational inequalities and a fixed point property for history-dependent operators. The unique solvability of Problem 13 is obtained under the following additional assumptions.

$\underline{H(\lambda_n)}$: $\lambda_n > 0$ for all $n \in \mathbb{N}$.

$\underline{H(P_n)}$: $P_n : X \to X^*$ is bounded, demicontinuous and monotone for all $n \in \mathbb{N}$.

Theorem 15. *Let X be a reflexive Banach space, Y a normed space and assume $H(A)$, $H(\mathcal{S})$, $H(\varphi)$, $H(j)$, $H(s)$, $H(f)$, $H(\lambda_n)$, $H(P_n)$. Then, for every $n \in \mathbb{N}$, Problem 13 has a unique solution $u_n \in C(\mathbb{R}_+; X)$.*

Proof. Let $n \in \mathbb{N}$. Assumptions $H(\lambda_n)$, $H(P_n)$ and Proposition 2(a) imply that the operator $\frac{1}{\lambda_n} P_n : X \to X^*$ is pseudomonotone. Therefore, assumption $H(A)$ on the operator A and Proposition 2(b) show that the operator $A_n : X \to X^*$ defined by $A_n = A + \frac{1}{\lambda_n} P_n$ is pseudomonotone, too. Moreover, since P_n is monotone and $\lambda_n > 0$, using assumption $H(A)$ we deduce that A_n is strongly monotone with constant m_A. We conclude from above that the operator A_n satisfies condition $H(A)$, too. This allows us to use Theorem 14 with X and A_n instead of K and A, respectively. In this way we obtain the unique solvability of the inequality (3.3) which concludes the proof. □

4. A convergence result

In the section we move to the convergence of solution to generalized penalty problem. To this end, besides the assumptions introduced in the previous section, we consider the following assumptions.

$\underline{(H_0)}$: $\lambda_n \to 0$ as $n \to \infty$.

$\underline{(H_1)}$: For each $v \in K$, there exists a sequence $\{v_n\} \subset X$ such that

$$P_n v_n = 0_{X^*} \text{ for each } n \in \mathbb{N} \text{ and } v_n \to v \text{ in } X \text{ as } n \to \infty. \tag{4.1}$$

$\underline{(H_2)}$: There exists an operator $P\colon X \to X^*$ such that:

(a) for any sequence $\{u_n\}$ satisfying $u_n \rightharpoonup u$ in X and $\limsup \langle P_n u_n, u_n - u\rangle \le 0$ we have

$$\liminf \langle P_n u_n, u_n - v\rangle \ge \langle Pu, u - v\rangle \quad \text{for all } v \in X;$$

(b) $Pu = 0_{X^*}$ implies $u \in K$.

$\underline{(H_3)}$: There exists a continuous function $c_\varphi\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(y, u, v_1) - \varphi(y, u, v_2) \le c_\varphi(\|y\|_Y, \|u\|_X)\, \|v_1 - v_2\|_X$$

for all $y \in Y$ and all $u, v_1, v_2 \in X$.

We now state and prove our main result in this paper.

Theorem 16.

Assume $H(A)$, $H(\mathcal{S})$, $H(\varphi)$, $H(j)$, $H(s)$, $H(f)$, $H(\lambda_n)$, $H(P_n)$, (H_0), (H_1), (H_2), (H_3). Then the solution u_n of Problem 13 converges to the solution u of Problem 11 in the following sense:

$$u_n(t) \to u(t) \quad \text{in } X \quad \text{as} \quad n \to \infty, \quad \text{for each } t \in \mathbb{R}_+. \tag{4.2}$$

The proof of Theorem 16 will be carried out in several steps. For the sake of convenience, we suppose, in what follows, that assumptions of Theorem 16 hold. We start by introducing the following intermediate problem.

Problem 17. Find $\widetilde{u}_n \in C(\mathbb{R}_+; X)$ such that, for all $t \in \mathbb{R}_+$,

$$\begin{aligned} &\langle A\widetilde{u}_n(t), v - \widetilde{u}_n(t)\rangle + \frac{1}{\lambda_n}\langle P_n\widetilde{u}_n(t), v - \widetilde{u}_n(t)\rangle + \varphi((\mathcal{S}u)(t), u(t), v) \\ &\quad -\varphi((\mathcal{S}u)(t), u(t), \widetilde{u}_n(t)) + j^0(\widetilde{u}_n(t); v - \widetilde{u}_n(t)) \\ &\qquad \ge \langle f(t), v - \widetilde{u}_n(t)\rangle \qquad \text{for all } v \in X. \end{aligned} \tag{4.3}$$

Note that in contrast to inequality (3.3), which is a history-dependent variational–hemivariational inequality, inequality (4.3) is a time-dependent variational–hemivariational inequality, since there $\mathcal{S}u$ is a given function. We have the following existence, uniqueness and convergence result.

Lemma 18.

(a) *For each $n \in \mathbb{N}$, Problem 17 admits a unique solution $\widetilde{u}_n \in C(\mathbb{R}_+; X)$.*
(b) *The following convergence holds:*

$$\widetilde{u}_n(t) \to u(t) \quad \text{in } X \quad \text{as} \quad n \to \infty, \quad \text{for each } t \in \mathbb{R}_+. \tag{4.4}$$

Proof. (a) The unique solvability of Problem 17 follows from arguments similar to those used to prove Theorem 15. For this reason we do not present the details.

(b) The proof of the assertion is divided into the three claims that we state and prove in what follows.

Claim 1. *For each $t \in \mathbb{R}_+$ the sequence $\{\widetilde{u}_n(t)\}$ is bounded.*

Let $t \in \mathbb{R}_+$ and $v \in K$. Condition (H_1) implies that there exists a sequence $\{v_n\} \subset X$ such that (4.1) holds. Assume now that $n \in \mathbb{N}$ is fixed. Then, assumptions $H(j)$(b), (c) and Proposition 8(b) guarantee that

$$\begin{aligned} & j^0(\widetilde{u}_n(t); v_n - \widetilde{u}_n(t)) \\ & = j^0(\widetilde{u}_n(t); v_n - \widetilde{u}_n(t)) + j^0(v_n; \widetilde{u}_n(t) - v_n) - j^0(v_n; \widetilde{u}_n(t) - v_n) \\ & \le j^0(\widetilde{u}_n(t); v_n - \widetilde{u}_n(t)) + j^0(v_n; \widetilde{u}_n(t) - v_n) + |j^0(v_n; \widetilde{u}_n(t) - v_n)| \\ & \le \alpha_j \|\widetilde{u}_n(t) - v_n\|_X^2 + |\max\{\langle \xi_n, \widetilde{u}_n(t) - v_n\rangle \,:\, \xi_n \in \partial j(v_n)\}| \\ & \le \alpha_j \|\widetilde{u}_n(t) - v_n\|_X^2 + (c_0 + c_1\|v_n\|_X)\|\widetilde{u}_n(t) - v_n\|_X. \end{aligned} \tag{4.5}$$

On the other hand, we use assumption (H_3) to see that

$$\begin{aligned} & \varphi((\mathcal{S}u)(t), u(t), v_n) - \varphi((\mathcal{S}u)(t), u(t), \widetilde{u}_n(t)) \\ & \qquad \le c_\varphi(\|(\mathcal{S}u)(t)\|_Y, \|u(t)\|_X)\|v_n - \widetilde{u}_n(t)\|_X. \end{aligned} \tag{4.6}$$

Next, we test with $v = v_n$ in (4.3) and take into account the fact that $P_n v_n = 0_{X^*}$ and hypothesis $H(A)$ to see that

$$\begin{aligned} & m_A \|\widetilde{u}_n(t) - v_n\|_X^2 \le \langle A\widetilde{u}_n(t) - Av_n, \widetilde{u}_n(t) - v_n\rangle \\ & = \langle A\widetilde{u}_n(t), \widetilde{u}_n(t) - v_n\rangle - \langle Av_n, \widetilde{u}_n(t) - v_n\rangle \\ & \le \frac{1}{\lambda_n}\langle P_n\widetilde{u}_n(t), v_n - \widetilde{u}_n(t)\rangle + \varphi((\mathcal{S}u)(t), u(t), v_n) - \varphi(\mathcal{S}u(t), u(t), \widetilde{u}_n(t)) \\ & \quad + j^0(\widetilde{u}_n(t); v_n - \widetilde{u}_n(t)) + \langle Av_n - f(t), v_n - \widetilde{u}_n(t)\rangle \\ & = -\frac{1}{\lambda_n}\langle P_n v_n - P_n\widetilde{u}_n(t), v_n - \widetilde{u}_n(t)\rangle \\ & \quad + \varphi((\mathcal{S}u)(t), u(t), v_n) - \varphi(\mathcal{S}u(t), u(t), \widetilde{u}_n(t)) \\ & \quad + j^0(\widetilde{u}_n(t); v_n - \widetilde{u}_n(t)) + \langle Av_n - f(t), v_n - \widetilde{u}_n(t)\rangle. \end{aligned} \tag{4.7}$$

By virtue of (4.5)–(4.7), the monotonicity of P_n and the Cauchy–Schwarz inequality, we find that

$$\begin{aligned} & (m_A - \alpha_j)\,\|\widetilde{u}_n(t) - v_n\|_X^2 \le c_\varphi(\|(\mathcal{S}u)(t)\|_Y, \|u(t)\|_X)\|v_n - \widetilde{u}_n(t)\|_X \\ & \quad + (c_0 + c_1\|v_n\|_X)\|\widetilde{u}_n(t) - v_n\|_X + \|Av_n - f(t)\|_{X^*}\|\widetilde{u}_n(t) - v_n\|_X. \end{aligned} \tag{4.8}$$

Recall that A is pseudomonotone, so it is a bounded operator. The latter combined with the convergence $v_n \to v$ in X, as $n \to \infty$, ensures that there exists a constant $d_1 > 0$ which does not depend on n such that

$$\max\{\|v_n\|_X, \|Av_n - f(t)\|_{X^*}\} \leq d_1. \tag{4.9}$$

We now combine inequalities (4.8) and (4.9), then use the smallness assumption $H(s)$ to deduce that

$$\|\widetilde{u}_n(t) - v_n\|_X \leq C(t), \tag{4.10}$$

where $C(t)$ is defined by

$$C(t) = \frac{1}{m_A - \alpha_j}\Big(c_\varphi(\|(\mathcal{S}u)(t)\|_Y, \|u(t)\|_X) + (c_0 + c_1 d_1 + d_1)\Big). \tag{4.11}$$

Next, we use inequalities (4.9) and (4.10) to conclude that

$$\|\widetilde{u}_n(t)\|_X \leq C(t) + d_1, \tag{4.12}$$

which proves the claim.

Claim 2. *For each $t \in \mathbb{R}_+$, the whole sequence $\{\widetilde{u}_n(t)\}$ converges weakly to $u(t)$ in X.*

Let $t \in \mathbb{R}_+$ be fixed. From Claim 1 and reflexivity of X, we are able to find an element $\widetilde{u}(t) \in X$ such that, passing to a subsequence if necessary,

$$\widetilde{u}_n(t) \rightharpoonup \widetilde{u}(t) \quad \text{in} \quad X. \tag{4.13}$$

We shall prove that $\widetilde{u}(t) \in K$. To this end, we use (4.3) to see that

$$\begin{aligned}
&\frac{1}{\lambda_n}\langle P_n\widetilde{u}_n(t), \widetilde{u}_n(t) - v\rangle \\
&\quad \leq \langle A\widetilde{u}_n(t) - Av, v - \widetilde{u}_n(t)\rangle + \langle Av - f(t), v - \widetilde{u}_n(t)\rangle \\
&\qquad + \varphi((\mathcal{S}u)(t), u(t), v) - \varphi((\mathcal{S}u)(t), u(t), \widetilde{u}_n(t)) + j^0(\widetilde{u}_n(t); v - \widetilde{u}_n(t)).
\end{aligned}$$

This inequality combined with hypotheses $H(\varphi)$, $H(j)$(b), (H_3) and the monotonicity of A infer

$$\begin{aligned}
\frac{1}{\lambda_n}\langle P_n\widetilde{u}_n(t), \widetilde{u}_n(t) - v\rangle &\leq \|Av - f(t)\|_{X^*}\|\widetilde{u}_n(t) - v\|_X \\
&+ \Big(c_\varphi(\|(\mathcal{S}u)(t)\|_Y, \|u(t)\|_X) + c_0 + c_1\|\widetilde{u}_n(t)\|_X\Big)\|\widetilde{u}_n(t) - v\|_X.
\end{aligned}$$

We now use the bounds (4.10) and (4.12) to see that

$$\frac{1}{\lambda_n}\langle P_n\widetilde{u}_n(t), \widetilde{u}_n(t) - v\rangle \leq \widetilde{C}(t, u, v), \tag{4.14}$$

where $\widetilde{C}(t, u, v)$ is a positive constant which depends on t, u and v but is independent of n.

Notice that $\lambda_n \to 0$ as $n \to \infty$ and, therefore, (4.14) yields

$$\limsup\, \langle P_n\widetilde{u}_n(t), \widetilde{u}_n(t) - v\rangle \leq 0. \tag{4.15}$$

We now take $v = \widetilde{u}(t)$ in (4.15) to deduce that

$$\limsup\, \langle P_n\widetilde{u}_n(t), \widetilde{u}_n(t) - \widetilde{u}(t)\rangle \leq 0. \tag{4.16}$$

Then we use (4.13), (4.16) and condition (H_2)(a) to find that

$$\langle P\widetilde{u}(t), \widetilde{u}(t) - v\rangle \leq \liminf \langle P_n\widetilde{u}_n(t), \widetilde{u}_n(t) - v\rangle. \tag{4.17}$$

Finally, we combine inequalities (4.15) and (4.17) to see that $\langle P\widetilde{u}(t), \widetilde{u}(t) - v\rangle \leq 0$. Hence, choosing $v = \widetilde{u}(t) \pm w$ with $w \in X$, we get $\langle P\widetilde{u}(t), w\rangle = 0$ for all $w \in X$. Therefore, $P\widetilde{u}(t) = 0_{X^*}$ and, using assumption (H_2)(b) we deduce that $\widetilde{u}(t) \in K$, as claimed.

Next, we prove that $\widetilde{u}(t) = u(t)$. To this end, let $v \in K$ be fixed. Hypothesis (H_1) guarantees that there exists a sequence $\{v_n\}$ in X such that (4.1) holds. Inserting $v = v_n$ into (4.3) and using the identity $P_n v_n = 0_{X^*}$ yield

$$\begin{aligned}&\langle A\widetilde{u}_n(t), \widetilde{u}_n(t) - v_n\rangle\\ &\quad \leq \langle f(t), \widetilde{u}_n(t) - v_n\rangle + \frac{1}{\lambda_n}\langle P_n v_n - P_n\widetilde{u}_n(t), \widetilde{u}_n(t) - v_n\rangle\\ &\qquad +\varphi((\mathcal{S}u)(t), u(t), v_n) - \varphi((\mathcal{S}u)(t), u(t), \widetilde{u}_n(t)) + j^0(\widetilde{u}_n(t); v_n - \widetilde{u}_n(t)).\end{aligned}$$

Then, the monotonicity of P_n reveals that

$$\begin{aligned}&\langle A\widetilde{u}_n(t), \widetilde{u}_n(t) - v_n\rangle \leq \langle f(t), \widetilde{u}_n(t) - v_n\rangle\\ &\quad +\varphi((\mathcal{S}u)(t), u(t), v_n) - \varphi((\mathcal{S}u)(t), u(t), \widetilde{u}_n(t)) + j^0(\widetilde{u}_n(t); v_n - \widetilde{u}_n(t)).\end{aligned} \tag{4.18}$$

Besides, we use Proposition 8(b) to see that, for each $n \in \mathbb{N}$, we are able to find an element $\xi_n(t) \in \partial j(\widetilde{u}_n(t))$ such that

$$j^0(\widetilde{u}_n(t); v_n - \widetilde{u}_n(t)) = \langle \xi_n(t), v_n - \widetilde{u}_n(t)\rangle. \tag{4.19}$$

We now combine inequalities (4.18) and (4.19) to see that

$$\begin{aligned}&\langle A\widetilde{u}_n(t) + \xi_n(t), \widetilde{u}_n(t) - v_n\rangle \leq \langle f(t), \widetilde{u}_n(t) - v_n\rangle\\ &\quad +\varphi((\mathcal{S}u)(t), u(t), v_n) - \varphi((\mathcal{S}u)(t), u(t), \widetilde{u}_n(t)).\end{aligned} \tag{4.20}$$

Note that the function $v \mapsto \varphi((\mathcal{S}u)(t), u(t), v)$ is convex and lower semicontinuous. Therefore, using Proposition 6, by virtue of convergences $v_n \to v$ in X as $n \to \infty$ and (4.13), we conclude that

$$\begin{aligned}&\limsup\big[\varphi((\mathcal{S}u)(t), u(t), v_n) - \varphi((\mathcal{S}u)(t), u(t), \widetilde{u}_n(t))\big]\\ &\quad \leq \limsup \varphi((\mathcal{S}u)(t), u(t), v_n) - \liminf \varphi((\mathcal{S}u)(t), u(t), \widetilde{u}_n(t))\\ &\quad \leq \varphi((\mathcal{S}u)(t), u(t), v) - \varphi((\mathcal{S}u)(t), u(t), \widetilde{u}(t)).\end{aligned} \tag{4.21}$$

Additionally, it is obvious to see that

$$\langle f(t), v_n - \widetilde{u}_n(t)\rangle \to \langle f(t), v - \widetilde{u}(t)\rangle \tag{4.22}$$

as $n \to \infty$. We now pass to the upper limit in inequality (4.20) and take into account (4.21), (4.22) to deduce that

$$\begin{aligned}&\limsup \langle A\widetilde{u}_n(t) + \xi_n(t), \widetilde{u}_n(t) - v_n\rangle\\ &\quad \leq \langle f(t), \widetilde{u}(t) - v\rangle + \varphi((\mathcal{S}u)(t), u(t), v) - \varphi((\mathcal{S}u)(t), u(t), \widetilde{u}(t)).\end{aligned} \tag{4.23}$$

On the other hand, keeping in mind that A is a bounded operator, from Claim 1 and hypothesis $H(j)$(b) we find a constant $d_3 > 0$ such that

$$\|A\widetilde{u}_n(t) + \xi_n(t)\|_{X^*} \leq d_3 \quad \text{for all } n \in \mathbb{N}. \tag{4.24}$$

which implies that

$$\lim \langle A\widetilde{u}_n(t) + \xi_n(t), v - v_n\rangle = 0.$$

Therefore, writing

$$\begin{aligned}&\langle A\widetilde{u}_n(t) + \xi_n(t), \widetilde{u}_n(t) - v_n\rangle\\ &\quad = \langle A\widetilde{u}_n(t) + \xi_n(t), \widetilde{u}_n(t) - v\rangle + \langle A\widetilde{u}_n(t) + \xi_n(t), v - v_n\rangle,\end{aligned}$$

it follows that

$$\begin{aligned}&\limsup \langle A\widetilde{u}_n(t) + \xi_n(t), \widetilde{u}_n(t) - v_n\rangle\\ &\quad = \limsup \langle A\widetilde{u}_n(t) + \xi_n(t), \widetilde{u}_n(t) - v\rangle.\end{aligned} \tag{4.25}$$

We now combine inequalities (4.23) and (4.25) to find that

$$\begin{aligned}&\limsup \langle A\widetilde{u}_n(t) + \xi_n(t), \widetilde{u}_n(t) - v\rangle\\ &\quad \le \langle f(t), \widetilde{u}(t) - v\rangle + \varphi((\mathcal{S}u)(t), u(t), v) - \varphi((\mathcal{S}u)(t), u(t), \widetilde{u}(t)).\end{aligned} \tag{4.26}$$

Recall that $v \in K$ is arbitrary and $\widetilde{u}(t) \in K$. Therefore, testing with $v = \widetilde{u}(t)$ in (4.26) yields

$$\limsup \langle A\widetilde{u}_n(t) + \xi_n(t), \widetilde{u}_n(t) - \widetilde{u}(t)\rangle \le 0. \tag{4.27}$$

Next, we use the bound (4.24) to see that, passing to a subsequence if necessary, we may suppose that

$$A\widetilde{u}_n(t) + \xi_n(t) \rightharpoonup \eta(t) \quad \text{in} \quad X^*, \ \text{as } n \to \infty,$$

with $\eta(t) \in X^*$. Furthermore, as proved in [41, Lemma 20], we know that the multivalued operator $A+\partial j \colon X \to 2^{X^*}$ is generalized pseudomonotone. Exploiting now Definition 4, from conditions $\{\widetilde{u}_n(t)\} \subset X$, $\{A\widetilde{u}_n(t) + \xi_n(t)\} \subset X^*$ with $A\widetilde{u}_n(t) + \xi_n(t) \in A\widetilde{u}_n(t) + \partial j(\widetilde{u}_n(t))$, $\widetilde{u}_n(t) \rightharpoonup \widetilde{u}(t)$ in X, $A\widetilde{u}_n(t) + \xi_n(t) \rightharpoonup \eta(t)$ in X^* and (4.27) we deduce that

$$\eta(t) \in A\widetilde{u}(t) + \partial j(\widetilde{u}(t)) \ \text{ and } \ \langle A\widetilde{u}_n(t) + \xi_n(t), \widetilde{u}_n(t)\rangle \to \langle \eta(t), \widetilde{u}(t)\rangle.$$

Hence, it follows that $\eta(t) = A\widetilde{u}(t) + \xi(t)$ with $\xi(t) \in \partial j(\widetilde{u}(t))$ and

$$\lim \langle A\widetilde{u}_n(t) + \xi_n(t), \widetilde{u}_n(t) - v\rangle = \langle A\widetilde{u}(t) + \xi(t), \widetilde{u}(t) - v\rangle \text{ for all } v \in K. \tag{4.28}$$

Combining (4.26) and (4.28) we have

$$\begin{aligned}&\langle A\widetilde{u}(t) + \xi(t), \widetilde{u}(t) - v\rangle \le \langle f(t), \widetilde{u}(t) - v\rangle + \varphi((\mathcal{S}u)(t), u(t), v)\\ &\quad -\varphi((\mathcal{S}u)(t), u(t), \widetilde{u}(t)) \quad \text{for all } v \in K\end{aligned}$$

with $\xi(t) \in \partial j(\widetilde{u}(t))$. Therefore, using the definition of Clarke subgradient, it follows that

$$\begin{aligned}&\langle A\widetilde{u}(t), v - \widetilde{u}(t)\rangle + \varphi((\mathcal{S}u)(t), u(t), v) - \varphi((\mathcal{S}u)(t), u(t), \widetilde{u}(t))\\ &\quad + j^0(\widetilde{u}(t); v - \widetilde{u}(t)) \ge \langle f(t), v - \widetilde{u}(t)\rangle \quad \text{for all } v \in K.\end{aligned} \tag{4.29}$$

We now take $v = u(t)$ in (4.29) and $v = \widetilde{u}(t)$ in (3.1), then we sum the resulting inequalities to obtain

$$\langle A\widetilde{u}(t) - Au(t), \widetilde{u}(t) - u(t)\rangle \le j^0(\widetilde{u}(t); u(t) - \widetilde{u}(t)) + j^0(u(t); \widetilde{u}(t) - u(t)).$$

We now use assumptions $H(A)$ and $H(j)$(c) to deduce that

$$\big(m_A - \alpha_j\big)\|\widetilde{u}(t) - u(t)\|_X^2 \le 0$$

and, therefore, the smallness condition $m_A > \alpha_j$ guarantees that $u(t) = \widetilde{u}(t)$.

The previous analysis reveals that any weakly convergent subsequence of $\{\widetilde{u}_n(t)\}$ has the same limit in X which coincides with the unique solution $u(t)$ of Problem 11 at $t \in \mathbb{R}_+$. On the other side, the sequence $\{\widetilde{u}_n(t)\}$ is bounded in X, as proved in Claim 1. The proof of Claim 2 follows now from a standard argument ([39, Theorem 1.20], for instance).

Claim 3. *For each $t \in \mathbb{R}_+$, the convergence (4.4) holds.*

Let $\xi(t) \in \partial j(u(t))$ and, for each $n \in \mathbb{N}$, let $\xi_n(t) \in \partial j(\widetilde{u}_n(t))$. Then, using Definition 7 and assumption $H(j)(c)$ it follows that

$$\begin{aligned}\langle \xi_n(t) - \xi(t), \widetilde{u}_n(t) - u(t)\rangle &= -\langle \xi_n(t), u(t) - \widetilde{u}_n(t)\rangle - \langle \xi(t), \widetilde{u}_n(t) - u(t)\rangle \\ &\geq -j^0(\widetilde{u}_n(t); u(t) - \widetilde{u}_n(t)) - j^0(u(t); \widetilde{u}_n(t) - u(t))\end{aligned}$$

and, therefore, assumption $H(j)(c)$ yields

$$\langle \xi_n(t) - \xi(t), \widetilde{u}_n(t) - u(t)\rangle \geq -\alpha_j \|\widetilde{u}_n(t) - u(t)\|_X^2.$$

We combine this inequality with assumption $H(A)$ to deduce that

$$\big(m_A - \alpha_j\big)\|\widetilde{u}_n(t) - u(t)\|_X^2 \leq \langle A\widetilde{u}_n(t) + \xi_n(t) - Au(t) - \xi(t), \widetilde{u}_n(t) - u(t)\rangle.$$

We now pass to the upper limit in this inequality and use the convergence (4.28) with $v = u(t) \in K$ and the convergence $\widetilde{u}_n(t) \rightharpoonup u(t)$ in X to find that

$$\limsup \big(m_A - \alpha_j\big)\|\widetilde{u}_n(t) - u(t)\|_X^2 \leq 0.$$

This inequality combined with assumption $H(s)$ implies (4.4) which completes the proof of the lemma. □

We now have all the ingredients needed to provide the proof of our main result.

Proof of Theorem 16. Let $t \in \mathbb{R}_+$ be fixed and let $m \in \mathbb{N}$ be such that $t \in [0, m]$. We take $v = u_n(t)$ in (4.3) and $v = \widetilde{u}_n(t)$ in (3.3), then we add the resulting inequalities to obtain

$$\begin{aligned}&\langle A\widetilde{u}_n(t) - Au_n(t), \widetilde{u}_n(t) - u_n(t)\rangle \\ &\quad\leq \frac{1}{\lambda_n}\langle P_n\widetilde{u}_n(t) - P_nu_n(t), u_n(t) - \widetilde{u}_n(t)\rangle + \varphi((\mathcal{S}u)(t), u(t), u_n(t)) \\ &\qquad -\varphi((\mathcal{S}u)(t), u(t), \widetilde{u}_n(t)) + \varphi((\mathcal{S}u_n)(t), u_n(t), \widetilde{u}_n(t)) - \varphi((\mathcal{S}u_n)(t), u_n(t), u_n(t)) \\ &\qquad\quad + j^0(\widetilde{u}_n(t); u_n(t) - \widetilde{u}_n(t)) + j^0(u_n(t); \widetilde{u}_n(t) - u_n(t)).\end{aligned}$$

We now use assumptions $H(A)$, $H(j)$(c) and $H(\varphi)$(b) to find that

$$\begin{aligned}\big(m_A - \alpha_j\big)\|\widetilde{u}_n(t) - u_n(t)\|_X^2 &\leq \alpha_\varphi \|u_n(t) - u(t)\|_X \|\widetilde{u}_n(t) - u_n(t)\|_X \\ &\quad + \beta_\varphi \|(\mathcal{S}u_n)(t) - (\mathcal{S}u)(t)\|_Y \|\widetilde{u}_n(t) - u_n(t)\|_X\end{aligned}$$

and, moreover,

$$\begin{aligned}&\|\widetilde{u}_n(t) - u_n(t)\|_X \\ &\quad\leq \frac{\alpha_\varphi}{m_A - \alpha_j}\|u_n(t) - u(t)\|_X + \frac{\beta_\varphi}{m_A - \alpha_j}\|(\mathcal{S}u_n)(t) - (\mathcal{S}u)(t)\|_Y.\end{aligned}$$

Next, we use condition $H(\mathcal{S})$ and the inequality

$$\|u_n(t) - u(t)\|_X - \|\widetilde{u}_n(t) - u_n(t)\|_X \leq \|\widetilde{u}_n(t) - u(t)\|_X,$$

to conclude that

$$\left(1-\frac{\alpha_\varphi}{m_A-\alpha_j}\right)\|u_n(t)-u(t)\|_X \le \|\widetilde{u}_n(t)-u(t)\|_X + \frac{\beta_\varphi s_m}{m_A-\alpha_j}\int_0^t \|u_n(s)-u(s)\|_X\,ds. \tag{4.30}$$

Denote $k := 1-\frac{\alpha_\varphi}{m_A-\alpha_j}$ and note that condition $H(s)$ guarantees that $k>0$. Then, using (4.30) and the Gronwall inequality yields

$$\|u_n(t)-u(t)\|_X \le \frac{1}{k}\|\widetilde{u}_n(t)-u(t)\|_X + \frac{\beta_\varphi s_m}{k^2(m_A-\alpha_j)}e^{\frac{\beta_\varphi m s_m}{k(m_A-\alpha_j)}}\int_0^t \|\widetilde{u}_n(s)-u(s)\|_X\,ds. \tag{4.31}$$

Recall that the sequence $\{\widetilde{u}_n(t)\}$ is uniformly bounded on $[0,m]$, as it follows from (4.12), (4.11) and assumption (H_3). Moreover, recall the convergence (4.4). These ingredients allow us to use Lebesgue's convergence theorem to obtain that

$$\int_0^t \|\widetilde{u}_n(s)-u(s)\|_X\,ds \to 0 \quad \text{as} \quad n\to\infty. \tag{4.32}$$

The convergence (4.2) is now a direct consequence of (4.31), (4.4) and (4.32). □

We now state and prove the following consequence of Theorems 15 and 16.

Corollary 19. *Assume $H(K)$, $H(A)$, $H(\mathcal{S})$, $H(\varphi)$, $H(j)$, $H(s)$, $H(f)$, $H(\lambda_n)$, (H_0) and (H_3). Moreover, assume that*

$$P : X \to X^* \quad \text{is a penalty operator for the set } K. \tag{4.33}$$

Then, for every $n\in\mathbb{N}$, Problem 12 has a unique solution $u_n \in C(\mathbb{R}_+;X)$. Moreover, the solution u_n of Problem 12 converges to the solution u of Problem 11 in the following sense:

$$u_n(t)\to u(t) \quad \text{in } X \quad \text{as} \quad n\to\infty, \quad \text{for each } t\in\mathbb{R}_+.$$

Proof. Recall that Problem 12 is a particular case of Problem 13 with $P_n = P$ for all $n\in\mathbb{N}$. We now use assumption (4.33), Definition 9 and Proposition 2(a) to see that in this case condition $H(P_n)$ is satisfied. Moreover, Definition 9 shows that assumption (H_1) is satisfied. In addition, since $P_n = P$, assumption (4.33) combined with Proposition 2(a) and Definition 1(d) shows that condition (H_2) is satisfied, too. Corollary 19 is a direct consequence of Theorems 15 and 16. □

We end this section with the following remarks.

Remark 20. A proof of Corollary 19 was given in [27,34] under the following additional assumption

$$u_n \rightharpoonup u \text{ in } X \implies \limsup j^0(u_n; v-u_n) \le j^0(u;v-u) \text{ for all } v\in X.$$

Note that in the present paper we use arguments of generalized pseudomonotonicity and properties of Clarke's subdifferential operator to remove this assumption. This represents the first of trait of novelty of the current paper. The second one consists in the fact that, in contrast to Problem 12 (governed by a given penalty operator), here we consider more general penalty problem, Problem 13 (governed by an operator which depends on $n\in\mathbb{N}$).

Remark 21. Note that in applications it is quite difficult to verify condition (H_2)(a). For this reason, it is important to provide sufficient conditions which could be verified easily to guarantee (H_2)(a). Such conditions were considered in [35]. There, it was proved that if $P\colon X \to X^*$ is a pseudomonotone operator such that

$$\|P_n u_n - P u_n\|_{X^*} \to 0 \quad \text{as} \quad n \to \infty$$

whenever $\{u_n\}$ is a weakly convergent sequence in X, then the operator P satisfies condition (H_2)(a).

Remark 22. Another useful result which is convenient in applications is the following. Assume that X is a Hilbert space, K satisfies condition $H(K)$ and, for each $n \in \mathbb{N}$, $P_n = J(I - \widetilde{P}_n) : X \to X^*$ where $J : X \to X^*$ is the canonical isometry, $I : X \to X$ is the identity map and $\widetilde{P}_n$ is the projection operator on a set K_n, assumed to satisfy condition $H(K)$. In addition, assume that the Hausdorff distance of the sets K_n and K, denoted $\mathcal{H}(K_n, K)$, satisfies the condition

$$\mathcal{H}(K_n, K) \to 0, \quad \text{as} \quad n \to \infty.$$

Then it was proved in [35] that conditions (P_n), (H_0) and (H_2) are satisfied.

5. A spring-rod system with unilateral constraints

The abstract results presented in Sections 3–4 in this paper are useful in the study of various mathematical models which describe the equilibrium of viscoelastic or viscoplastic bodies in unilateral contact with a foundation. In this section we present a one-dimensional example which illustrates the applicability of these results.

Consider a viscoelastic rod which occupies the interval $[0, L]$ on the Ox axis with $L > 0$. The rod is fixed at its end $x = 0$ and its extremity $x = L$ is situated at the distance g from a rigid obstacle. The gap between the rod and the obstacle is filled in a nonlinear spring which is attached to the obstacle. The natural length of the spring is $g \geq 0$ and the system is in equilibrium when no forces are acting on the rod. Assume now that the rod is submitted to the action of a time-dependent body force of density f_0, which acts along the Ox axis. The time interval of interest is $\mathbb{R}_+ = [0, \infty)$ and, everywhere below, we use the prime to denote the derivative with respect the spatial variable, i.e. $u' = \frac{du}{dx}$ and $\sigma' = \frac{d\sigma}{dx}$. Then, the problem of finding the equilibrium of the rod in the physical setting above can be formulated as follows.

Problem 23. Find a displacement field $u\colon [0, L] \times \mathbb{R}_+ \to \mathbb{R}$ and a stress field $\sigma\colon [0, L] \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\sigma(x,t) = \mathcal{F}(u'(x,t)) + \int_0^t b(t-s)u'(x,s)\,ds \quad \text{for} \quad (x,t) \in (0,L) \times \mathbb{R}_+, \tag{5.1}$$

$$\sigma'(x,t) + f_0(x,t) = 0 \quad \text{for} \quad (x,t) \in (0,L) \times \mathbb{R}_+, \tag{5.2}$$

$$u(0,t) = 0 \quad \text{for} \quad t \in \mathbb{R}_+, \tag{5.3}$$

$$\left\{ \begin{array}{ll} u(L,t) \leq g, & \\ -\sigma(L,t) = \theta(u(L,t)) \quad \text{if} \quad u(L,t) < g, & \quad \text{for} \quad t \in \mathbb{R}_+. \\ -\sigma(L,t) \geq \theta(u(L,t)) \quad \text{if} \quad u(L,t) = g. & \end{array} \right. \tag{5.4}$$

A brief description of the equations and conditions in Problem 23 is the following. First, Eq. (5.1) represents the viscoelastic constitutive law in which $\mathcal{F}$ and b are given functions which will be described below. Eq. (5.2) is the equilibrium equation and condition (5.3) represents the displacement condition which shows that the rod is assumed to be fixed at the end $x = 0$ during the deformation process. Conditions (5.4) represent the contact condition in which θ is a given positive function which vanishes for a negative argument. Our interest lies in this condition and, for this reason, we describe it in detail.

Let $t \in \mathbb{R}_+$ be given. First, inequality $u(L,t) \le g$ shows that the displacement in $x = L$ is subjected to an unilateral constraint. This constraint arises from the physical setting, since the obstacle is assumed to be rigid and, therefore, its penetration is not allowed. Equality $u(L,t) = g$ corresponds to the case when the spring is completely compressed. Next, condition

$$-\sigma(L,t) = \theta(u(L,t)) \quad \text{if} \quad u(L,t) < g$$

shows that when the spring is partially compressed, then the stress in $x = L$ depends on the displacement field in $x = L$. When $0 < u(L,t) < g$ the rod moves towards the spring, the last one is in compression, and its reaction is in the negative direction of the Ox axis (since $\sigma(L,t) < 0$). When $u(L,t) < 0$ then there is separation between the rod and the spring and the reaction of the spring vanishes (since, in this case $\theta(u(L,t)) = 0$ and, therefore, $\sigma(L,t) = 0$).

Assume now that $u(L,t) = g$, i.e., the spring is totally compressed between the rod and the obstacle. Then (5.4) implies that $\sigma(L,t) + \theta(u(L,t)) \le 0$ and, moreover, the properties of the function θ show that $-\theta(u(L,t)) \le 0$. Therefore,

$$\sigma(L,t) = \sigma_e(L,t) + \sigma_a(L,t) \tag{5.5}$$

where

$$\sigma_e(L,t) = -\theta(u(L,t)) \le 0, \quad \sigma_a(L,t) = \sigma(L,t) + \theta(u(L,t)) \le 0. \tag{5.6}$$

We conclude from (5.5)–(5.6) that, when the spring is totally compressed, then the pressure it acts on the rod in $x = L$, $\sigma(L,t)$, is decomposed into two parts: an elastic one, $\sigma_e(L,t)$, and an additional one, $\sigma_a(L,t)$, both negative. The additional pressure prevents the displacement of the extremity $x = L$ of the rod in such a way that the constraint $u(L,t) \le g$ is satisfied.

In the study of Problem 23 we assume that the constitutive functions $\mathcal{F}$ and b are such that:

$$\left\{\begin{array}{l} \text{(a) } \mathcal{F}:\mathbb{R}\to\mathbb{R}. \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \qquad |\mathcal{F}(r_1) - \mathcal{F}(r_2)| \le L_F\,|r_1 - r_2| \quad \forall\, r_1,\, r_2 \in \mathbb{R}. \\ \text{(c) There exists } m_F > 0 \text{ such that} \\ \qquad (\mathcal{F}(r_1) - \mathcal{F}(r_2))(r_1 - r_2) \ge m_F |r_1 - r_2|^2 \quad \forall\, r_1,\, r_2 \in \mathbb{R}. \\ \text{(d) } \mathcal{F}(0) = 0. \end{array}\right. \tag{5.7}$$

$$b \in C(\mathbb{R}_+, \mathbb{R}). \tag{5.8}$$

We also assume that the density of the body forces f_0 and the function θ are such that

$$f_0 \in C(\mathbb{R}_+, L^2(0,L)). \tag{5.9}$$

$$\left\{\begin{array}{l} \text{(a) } \theta:\mathbb{R}\to\mathbb{R} \text{ is a continuous function.} \\ \text{(b) There exist } c_0 > 0,\ c_1 > 0 \text{ such that} \\ \qquad |\theta(r)| \le c_0 + c_1|r| \quad \forall\, r \in \mathbb{R}. \\ \text{(c) There exists } L_\theta > 0 \text{ such that} \\ \qquad (\theta(r_1) - \theta(r_2))(r_1 - r_2) \ge -L_\theta\,|r_1 - r_2|^2 \quad \forall\, r_1,\, r_2 \in \mathbb{R}. \\ \text{(d) } \theta(r) = 0 \text{ for all } r \le 0. \end{array}\right. \tag{5.10}$$

Finally, we recall that

$$g \ge 0. \tag{5.11}$$

We denote by $q\colon \mathbb{R} \to \mathbb{R}$ the locally Lipschitz function defined by

$$q(r) = \int_0^r \theta(s)\, ds \qquad \text{for all} \quad r \in \mathbb{R}. \tag{5.12}$$

Note that the function θ is not assumed to be monotone. Therefore, the function q could be nonconvex. Nevertheless, it is regular and it satisfies the equality

$$q^0(s;r) = \theta(s)\, r \qquad \text{for all} \;\; r,\; s \in \mathbb{R}, \tag{5.13}$$

where $q^0(s;r)$ denotes the generalized directional derivative of q at the point s in the direction r.

We use the space

$$V = \{\, v \in H^1(0,L) \mid v(0) = 0 \,\}$$

which is a real Hilbert space with the inner product

$$(u,v) = \int_0^L u'v'\,dx.$$

and the associated norm $\|\cdot\|_V$. Recall that the completeness of the space $(V, \|\cdot\|_V)$ follows from the Friedrichs–Poincaré inequality. Moreover, using the identity

$$v(x) = \int_0^x v'\,dx$$

and the Cauchy–Schwarz inequality it follows that

$$|v(L)| \le \sqrt{L}\,\|v\|_V \qquad \forall\, v \in V, \tag{5.14}$$

$$\|v\|_{L^2(0,L)} \le \frac{L}{\sqrt{2}}\,\|v\|_V \qquad \forall\, v \in V. \tag{5.15}$$

We denote by V^* and $\langle\cdot,\cdot\rangle$ the dual of V and the duality pairing between V^* and V, respectively. In addition, we define the set K, the operators $A\colon V \to V^*$, $\mathcal{S}\colon C(\mathbb{R}_+, V) \to C(\mathbb{R}_+, L^2(0,L))$ and the functions $\varphi\colon L^2(0,L)\times V\times V \to \mathbb{R}$, $j\colon V \to \mathbb{R}$, $f\colon \mathbb{R}_+ \to V^*$ by equalities

$$K = \{u \in V \mid u(L) \le g\}, \tag{5.16}$$

$$\langle Au, v\rangle = \int_0^L \mathcal{F}(u')\,v'\,dx \qquad \text{for all} \;\; u, v \in V, \tag{5.17}$$

$$(\mathcal{S}u)(t) = \int_0^t b(t-s)u'(s)\,dx \qquad \text{for all} \;\; u \in C(\mathbb{R}_+, V),\; t \in \mathbb{R}_+, \tag{5.18}$$

$$\varphi(y,u,v) = \int_0^L yv'\,dx \qquad \text{for all} \;\; y \in L^2(0,L),\; u,\, v \in V, \tag{5.19}$$

$$j(v) = q(v(L)) \qquad \text{for all} \;\; v \in V, \tag{5.20}$$

$$\langle f(t), v\rangle = \int_0^L f_0(t)v\,dx \qquad \text{for all} \;\; v \in V,\; t \in \mathbb{R}_+. \tag{5.21}$$

Note that sometimes, here and below, we do not mention the dependence of some functions on the spatial variable $x \in [0,L]$.

To derive the variational formulation of Problem 23 we assume in what follows that (u,σ) are sufficiently smooth functions which satisfy (5.1)–(5.4). Let $v \in K$ and $t \in \mathbb{R}_+$. First, we perform an integration by parts and use the equilibrium equation (5.2) to see that

$$\int_0^L \sigma(t)(v' - u'(t))\,dx = \int_0^L f_0(t)(v - u(t))\,dx$$
$$+\sigma(L,t)(v(L) - u(L,t)) - \sigma(0,t)(v(0) - u(0,t)).$$

Next, since $v(0) = u(0,t) = 0$, we deduce that

$$\int_0^L \sigma(t)(v' - u'(t))\,dx = \int_0^L f_0(t)(v - u(t))\,dx + \sigma(L,t)(v(L) - u(L,t)). \tag{5.22}$$

Moreover, we write

$$\sigma(L,t)(v(L)-u(L,t)) = (\sigma(L,t)+\theta(u(L,t)))\,(v(L)-g) \\ +(\sigma(L,t)+\theta(u(L,t)))(g-u(L,t))-\theta(u(L,t))(v(L)-u(L,t)),$$

then we use the contact condition (5.4) and the definition (5.16) of the set K to deduce that

$$\sigma(L,t)(v(L)-u(L,t)) \geq -\theta(u(L,t))(v(L)-u(L,t)). \tag{5.23}$$

We now combine (5.22), (5.23) and use (5.13), (5.21) to find that

$$\int_0^L \sigma(t)(v'-u'(t))dx + q^0(u(L,t);v(L)-u(L,t)) \geq \langle f(t), v-u(t)\rangle. \tag{5.24}$$

On the other hand, a simple computation based on (5.20) shows that

$$j^0(u;v) = q^0(u(L);v(L)) \qquad \text{for all } \ u,\, v\in V, \tag{5.25}$$

where $j^0(u;v)$ denotes the generalized directional derivative of j at the point u in the direction v. We now substitute the constitutive law (5.1) in (5.24), then we use definitions (5.17)–(5.19) and equality (5.25) to obtain the following variational formulation of Problem 23.

Problem 24. Find a displacement field $u:\mathbb{R}_+\to K$ such that, for all $t\in\mathbb{R}_+$,

$$\langle Au(t), v-u(t)\rangle + \varphi((\mathcal{S}u)(t),u(t),v)-\varphi((\mathcal{S}u)(t),u(t),u(t)) \\ +j^0(u(t);v-u(t)) \geq \langle f(t), v-u(t)\rangle \quad \text{for all } \ v\in K. \tag{5.26}$$

In order to construct the penalty problems associated to the constrained inequality (5.26) we consider a function p such that

$$\left\{\begin{array}{l} \text{(a) } p:\mathbb{R}\to\mathbb{R}. \\ \text{(b) There exists } L_p>0 \text{ such that} \\ \qquad |p(r_1)-p(r_2)|\leq L_p\,|r_1-r_2| \quad \forall\, r_1,\, r_2\in\mathbb{R}. \\ \text{(c) } (p(r_1)-p(r_2))(r_1-r_2)\geq 0 \quad \forall\, r_1,\, r_2\in\mathbb{R}. \\ \text{(d) } p(r)>0 \ \text{ if } \ r>0 \ \text{ and } \ p(r)=0 \ \text{ if } \ r\leq 0. \end{array}\right. \tag{5.27}$$

A typical example of such function is $p(r)=r^+=max\,\{0,r\}$ for all $r\in\mathbb{R}$. Moreover, we consider a sequence $\{g_n\}\subset\mathbb{R}$ such that

$$g_n>0 \qquad \forall\, n\in\mathbb{N}, \tag{5.28}$$

$$g_n\to g \quad \text{as} \quad n\to\infty \tag{5.29}$$

and, for each $n\in\mathbb{N}$, we define the operator $P_n:V\to V^*$ by equality

$$\langle P_n u, v\rangle = p(u(L)-g_n)v(L) \qquad \text{for all } \ u,v\in V. \tag{5.30}$$

Then, for the sequence $\{\lambda_n\}\subset\mathbb{R}$ satisfying condition $H(\lambda_n)$, we consider the following penalty problem.

Problem 25. Find $u_n:\mathbb{R}_+\to V$ such that, for all $t\in\mathbb{R}_+$,

$$\langle Au_n(t), v-u_n(t)\rangle + \frac{1}{\lambda_n}\langle P_n u_n(t), v-u_n(t)\rangle + \varphi((\mathcal{S}u_n)(t),u_n(t),v) \\ -\varphi((\mathcal{S}u_n)(t),u_n(t),u_n(t)) + j^0(u_n(t);v-u_n(t)) \\ \geq \langle f(t), v-u_n(t)\rangle \quad \text{for all } \ v\in V. \tag{5.31}$$

Our main result in this section is the following.

Theorem 26. *Assume (5.7), (5.8), (5.9), (5.10), (5.11), (5.27), (5.28), (5.29), $H(\lambda_n)$, (H_0). Moreover, assume that*

$$L_\theta L < m_F. \tag{5.32}$$

Then:

(a) *Problem 24 has a unique solution $u \in C(\mathbb{R}_+; K)$.*
(b) *For each $n \in \mathbb{N}$ Problem 25 has a unique solution $u_n \in C(\mathbb{R}_+; V)$.*
(c) *The solution u_n of Problem 25 converges to the solution u of Problem 24 in the following sense:*

$$u_n(t) \to u(t) \quad \text{in } V \quad \text{as} \quad n \to \infty, \quad \text{for each } \ t \in \mathbb{R}_+. \tag{5.33}$$

Proof. We check point by point the validity of conditions of Theorems 14–16 on the spaces $X = V$ and $Y = L^2(0, L)$.

First, we use definition (5.16) to see condition $H(K)$ is satisfied. Next, we use the properties (5.7) of the function $\mathcal{F}$ to see that the operator (5.17) satisfies condition $H(A)$ with $m_A = m_F$. Moreover, assumption (5.8) guarantees that the operator (5.18) is history-dependent and, therefore, condition $H(\mathcal{S})$ holds, too. In addition, using (5.15), it is easy to see that the function (5.19) satisfies condition $H(\varphi)$ with $\alpha_\varphi = 0$ and $\beta_\varphi = 1$. Moreover, assumption (5.10) combined with inequality (5.14) implies that the function j defined by (5.20) satisfies condition $H(j)$ with $\alpha_j = L_\theta L$. In addition, condition $H(s)$ is a direct consequence of the smallness assumption (5.32). Next, the regularity (5.9) implies that the function f defined by (5.21) satisfies assumption $H(f)$.

On the other hand, note that the properties (5.27) of the function p and assumption (5.28) show that, for each $n \in \mathbb{N}$, the operator P_n defined by (5.30) is bounded, continuous and monotone and, therefore, condition $H(P_n)$ is satisfied.

Assume now that $v \in K$ and, for each $n \in \mathbb{N}$, denote

$$v_n = \begin{cases} \frac{g_n}{g}\, v & \text{if} \quad g > 0, \\ v & \text{if} \quad g = 0. \end{cases}$$

Then, using assumptions (5.27) and (5.29) it is easy to see that the sequence $\{v_n\}$ satisfies (4.1). We conclude from here that condition (H_1) is satisfied.

Next, we define the operator $P : V \to V^*$ by equality

$$\langle Pu, v\rangle = p(u(L) - g)v(L) \qquad \text{for all} \ \ u, v \in V \tag{5.34}$$

and assume that $u_n \rightharpoonup u$ in V which, obviously, implies that $u_n(L) \to u(L)$. Therefore, using the properties (5.27) of the function p as well as the convergence (5.28) we deduce that

$$\begin{aligned} \langle P_n u_n, v - u_n\rangle &= p(u_n(L) - g_n)(v(L) - u_n(L)) \\ &\to p(u(L) - g)(v(L) - u(L)) = \langle Pu, v - u\rangle. \end{aligned}$$

This shows that condition (H_2)(a) holds. Moreover, assumption (5.27)(d) shows that condition (H_2)(b) holds, too.

Finally, using (5.15) it is easy to check that the function (5.19) satisfies condition (H_3) with $c_\varphi(r, s) = r$.

The properties described above show that we are in a position to apply Theorems 14–16 in order to conclude the proof of Theorem 26. □

We end this section with the following mechanical interpretation. First, Problem 25 represents the variational formulation of a contact problem similar to Problem 23, in which the obstacle is assumed to be deformable and the natural length of the spring is g_n. The function p is the so-called normal compliance function which describes the reaction of the obstacle and λ_n represents a deformability coefficient. Besides the existence and uniqueness results, Theorem 26 is important for the convergence result (5.33) which can be interpreted as follows: at each time moment $t \in \mathbb{R}_+$ the weak solution of the contact problem with a rigid obstacle can be approached by the weak solution of the problem with a deformable obstacle, provided that the deformability coefficient of the obstacle is small enough and the natural length of the springs in the two problems is close enough.

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