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Minimization arguments in analysis of variational–hemivariational inequalities

Mircea Sofonea and Weimin Han

Abstract. In this paper, an alternative approach is provided in the well-posedness analysis of elliptic variational–hemivariational inequalities in real Hilbert spaces. This includes the unique solvability and continuous dependence of the solution on the data. In most of the existing literature on elliptic variational–hemivariational inequalities, well-posedness results are obtained by using arguments of surjectivity for pseudomonotone multivalued operators, combined with additional compactness and pseudomonotonicity properties. In contrast, following (Han in Nonlinear Anal B Real World Appl 54:103114, 2020; Han in Numer Funct Anal Optim 42:371–395, 2021), the approach adopted in this paper is based on the fixed point structure of the problems, combined with minimization principles for elliptic variational–hemivariational inequalities. Consequently, only elementary results of functional analysis are needed in the approach, which makes the theory of elliptic variational–hemivariational inequalities more accessible to applied mathematicians and engineers. The theoretical results are illustrated on a representative example from contact mechanics.

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1. Introduction

Variational–hemivariational inequalities represent a special class of inequalities which arise in the study of nonsmooth boundary value problems. They are governed by two real-valued functions, say φ and j, such that φ is convex, whereas j is locally Lipschitz and is generally nonconvex. The case with a vanishing j corresponds to a pure variational inequality, and the case with a vanishing φ corresponds to a pure hemivariational inequality. For this reason, the study of variational inequalities is carried out by using arguments of nonlinear and convex analysis, while the analysis of hemivariational and variational–hemivariational inequalities requires additional knowledge on nonsmooth analysis.

Hemivariational inequalities were introduced by Panagiotopoulous in early eighties in the context of applications to engineering problems. The mathematical literature in the field concerns well-posedness, regularity, convergence and error analysis of numerical approximations, among others. The area has grown rapidly in the past few decades, motivated by a wide variety of applications in Physics, Mechanics and Engineering Sciences. Representative books in the field include [14,19,21]. Detailed mathematical analysis of variational–hemivariational inequalities, together with relevant applications in contact mechanics, can be found in [24]. Numerical analysis of various classes of variational–hemivariational inequalities was carried out in a number of papers, including [6,11,12] and the survey work [10].

The mathematical theory of contact mechanics deals with the study of systems of partial differential equations which describe processes of contact with different constitutive laws, different loadings and different interface laws. Such kind of processes is commonly seen in industry and daily life, and, therefore, a lot of effort has been observed in their modeling, analysis and numerical simulations. The literature in this field is extensive. It deals with analysis of various models of contact, which are expressed in

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terms of strongly nonlinear elliptic, time-dependent or evolutionary boundary value problems. Some references in the field include [3,4,9,15,20,21] and, more recently, [2,16,23,24]. There, various existence and uniqueness results have been proved, by using arguments of variational, hemivariational and variational–hemivariational inequalities. Once existence and uniqueness of solutions have been established, related important questions arise, such as the behavior of the solution, which provides the continuous dependence of the solution with respect to the data and the link between the solutions of different contact models.

As is shown in [11,12,24], a number of relevant mathematical models of contact mechanics lead, in a weak formulation, to variational–hemivariational inequalities of the form

$$u \in K, \quad \langle Au, v-u\rangle + \varphi(u,v) - \varphi(u,u) + j^0(u; v-u) \geq \langle f, v-u\rangle \quad \forall\, v \in K. \tag{1.1}$$

Here, $K \subset X$, X is a real reflexive Banach space, $\langle\cdot,\cdot\rangle$ represents the duality pairing between X and its dual X^*, $A\colon X \to X^*$, $\varphi : X \times X \to \mathbb{R}$, $j : X \to \mathbb{R}$ is a locally Lipschitz function and $f \in X^*$. Moreover, $j^0(u; v)$ denotes the Clarke directional derivative of j at the point u in the direction v. Existence and uniqueness result in the study of inequality problems of the form (1.1) has been obtained in [17,24], under several assumptions on the data, including that the operator A is strongly monotone and pseudomonotone. The proof in these references was based on an application of a surjectivity result for pseudomonotone multivalued operators followed by the Banach fixed point argument. Stability of the solution of (1.1) with respect to the data is shown in [25,26].

In the case when X is a real Hilbert space, thanks to the Riesz representation theorem, the inequality (1.1) can be written in an equivalent form as

$$u \in K, \quad (Au, v-u)_X + \varphi(u,v) - \varphi(u,u) + j^0(u; v-u) \geq (f, v-u)_X \quad \forall\, v \in K. \tag{1.2}$$

Here, $(\cdot,\cdot)_X$ represents the inner product on X, $A : X \to X$ and $f \in X$. Solution existence and uniqueness of the problem (1.2) are shown in [8] through the use of elementary knowledge in functional analysis, namely, convex minimization and Banach fixed-point argument. Compared to the solution existence and uniqueness result in [17,24], the operator A is assumed to be strongly monotone and Lipschitz continuous; meanwhile, a linear growth assumption on the generalized gradient of j is removed. The starting point of the new approach adopted in [8] is a minimization principle established in [7] for a special case of (1.2) where A is a potential operator and the function φ depends on only one argument, i.e., $\varphi(u, v) = \varphi(v)$. This new approach of analysis of variational–hemivariational inequalities eliminates the need of the notion of pseudomonotonicity and the application of an abstract surjectivity result for a pseudomonotone operator, and is thus more accessible to applied mathematicians, numerical analysts and engineers.

The current paper represents a continuation of [8] and deals with the analysis of the variational–hemivariational inequality (1.2). First, a new proof is provided on the existence and uniqueness result presented in [8]. The novelty of the proof is that in contrast to [8], the Banach fixed point argument is used only once, in a different space and with different operators. The idea of this proof is useful also in the stability study of the problem (1.2). Finally, the theoretical results are illustrated in the study of a mathematical model of contact mechanics.

The rest of the paper is organized as follows. In Sect. 2, we introduce some preliminary material and recall results in [7] needed later in the paper. In Sect. 3, we provide a new proof of the solution existence and uniqueness result for the variational–hemivariational inequality (1.2). The proof is based on the minimization principle established in [7] and a new fixed point argument. In Sect. 4, we use the fixed point structure of the inequality (1.2) to show the stability of the solution of (1.2) with respect to perturbations in the data. Section 5 is devoted to discussion of specializing the general results in previous sections to some particular cases useful in applications. Finally, in Sect. 6 we illustrate applications of the theoretical results in the study of a mathematical model of contact.

2. Preliminaries

We will use X to stand for a real Hilbert space, unless stated otherwise. We denote by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$ the inner product and the associated norm on X. Moreover, we will use the product space $X\times X$ endowed with the inner product

$$(\eta,\xi)_{X\times X}=(\eta_1,\xi_1)_X+(\eta_2,\xi_2)_X\quad\forall\,\eta=(\eta_1,\eta_2),\ \xi=(\xi_1,\xi_2)\in X\times X$$

and the associated norm $\|\cdot\|_{X\times X}$. We denote by "$\to$" and "$\rightharpoonup$" the strong and weak convergence in X and in $X\times X$. For a sequence indexed by n, all the limits, upper limits and lower limits are understood to be when $n\to\infty$, even if this is not stated explicitly.

We start by recalling some basic definitions.

Definition 1. A function $j:X\to\mathbb{R}$ is said to be locally Lipschitz if for any $x\in X$ there exist a neighborhood U_x of x and a constant L_x such that

$$|j(u)-j(v)|\le L_x\|u-v\|_X\quad\forall\,u,v\in U_x.$$

The Clarke directional derivative of the locally Lipschitz function $j:X\to\mathbb{R}$ at the point $u\in X$ in the direction $v\in X$ is defined by

$$j^0(u;v)=\limsup_{w\to u,\lambda\downarrow 0}\frac{j(w+\lambda v)-j(w)}{\lambda}.$$

Definition 2. A function $J:X\to\mathbb{R}$ is said to be strongly convex if there exists $\beta>0$ such that

$$(1-t)\,J(u)+t\,J(v)-J((1-t)u+tv)\ge\beta\,t\,(1-t)\,\|u-v\|_X^2\quad\forall\,u,v\in X,\ t\in[0,1].\tag{2.1}$$

Definition 3. A function $J:X\to\mathbb{R}$ is said to be coercive if $J(v_n)\to\infty$ for any sequence $\{v_n\}\subset X$ such that $\|v_n\|_X\to\infty$.

A sufficient and necessary condition for the local Lipschitz continuity of a convex function is stated in the following result [5, Corollary 2.4, p. 12].

Lemma 4. *Let $\psi:X\to\mathbb{R}$ be a convex function over a normed space X. Then, ψ is locally Lipschitz continuous on X if and only if ψ is bounded above on a nonempty open set in X.*

Let us recall definitions for properties of nonlinear operators.

Definition 5. An operator $A:X\to X$ is said to be strongly monotone if there exists $m_A>0$ such that

$$(Au-Av,u-v)_X\ge m_A\|u-v\|_X^2\quad\forall\,u,v\in X.\tag{2.2}$$

The operator A is Lipschitz continuous if there exists $L_A>0$ such that

$$\|Au-Av\|_X\le L_A\|u-v\|_X\quad\forall\,u,v\in X.\tag{2.3}$$

Remark 6. Note that if $A:X\to X$ is a strongly monotone Lipschitz continuous operator, then inequalities (2.2) and (2.3) imply that $m_A\le L_A$.

The following result is proved in [23, p. 22] and will be applied in Sect. 3.

Lemma 7. *Let $A:X\to X$ be a strongly monotone and Lipschitz continuous operator with constants m_A and L_A, respectively, and let $\rho>0$. Then, for the operator $B_\rho:X\to X$ defined by*

$$B_\rho u=u-\rho Au,\quad u\in X,\tag{2.4}$$

we have the inequality

$$\|B_\rho u-B_\rho v\|_X\le k(\rho)\|u-v\|_X,\ \ k(\rho)=(1-2\rho m_A+\rho^2L_A^2)^{1/2}\quad\forall\,u,v\in X.\tag{2.5}$$

Consequently, for $\rho\in\big(0,2\,m_A/L_A^2\big)$, $0\le k(\rho)<1$ and B_ρ is a contraction on X.

Next, we recall the notion of set convergence in the sense of Mosco [18] that will be needed for stability analysis in Sect. 4.

Definition 8. Let X be a normed space, $\{K_n\}$ a sequence of nonempty subsets of X and K a nonempty subset of X. The sequence $\{K_n\}$ is said to converge to K in the sense of Mosco, written $K_n \xrightarrow{M} K$ in X, if the following conditions hold.

(a) For each $u \in K$, there exists a sequence $\{u_n\}$ such that $u_n \in K_n$ for any $n \in \mathbb{N}$ and $u_n \to u$ in X.

(b) For each sequence $\{u_n\}$ such that $u_n \in K_n$ for any $n \in \mathbb{N}$ and $u_n \rightharpoonup u$ in X, we have $u \in K$.

We introduce a particular variational–hemivariational inequality of the form (1.2).

Problem $\mathcal{P}^0$. *Find an element* $u \in K$ *such that*

$$(u, v-u)_X + \psi(v) - \psi(u) + h^0(u; v-u) \geq (f, v-u)_X \quad \forall\, v \in K. \tag{2.6}$$

In the study of this problem, we consider the following assumptions.

$\underline{H(K)}$ K is a nonempty, closed and convex subset of X.

$\underline{H(\psi)}$ $\psi : X \to \mathbb{R}$ is convex and bounded above on a nonempty open set in X.

$\underline{H(h)}$ $h \colon X \to \mathbb{R}$ is locally Lipschitz continuous and there exists $\alpha_h \in [0,1)$ such that

$$h^0(v_1; v_2 - v_1) + h^0(v_2; v_1 - v_2) \leq \alpha_h \|v_1 - v_2\|_X^2 \quad \forall\, v_1, v_2 \in X. \tag{2.7}$$

$\underline{H(f)}$ $f \in X$.

Moreover, define an energy functional

$$J(v) = \frac{1}{2}\,\|v\|_X^2 + \psi(v) + h(v) - (f, v)_X \quad \forall\, v \in X \tag{2.8}$$

and a minimization problem:

Problem $\mathcal{P}^M$. *Find an element* $u \in K$ *such that*

$$J(u) \leq J(v) \quad \forall\, v \in K. \tag{2.9}$$

The following existence, uniqueness and equivalence results have been proved in [7].

Theorem 9. *Assume* $H(K)$, $H(\psi)$, $H(h)$ *and* $H(f)$. *Then, Problem* $\mathcal{P}^M$ *has a unique solution, which is also the unique solution of Problem* $\mathcal{P}^0$.

Note that this theorem shows the equivalence between Problems $\mathcal{P}^0$ and $\mathcal{P}^M$ as well as their unique solvability. Its proof is based on an elementary result for convex minimization (e.g., [1, Theorem 3.3.12]) and properties of the subdifferential in the sense of Clarke. We now quote the following lemma from [7] that is used in proving Theorem 9.

Lemma 10. *Assume* $H(K)$, $H(\psi)$, $H(h)$ *and* $H(f)$. *Then, the function* J *is locally Lipschitz and satisfies condition* (2.1) *with* $\beta = (1-\alpha_h)/2$, *i.e., it is strongly convex on* X.

Under the assumptions $H(K)$, $H(\psi)$, $H(h)$, Theorem 9 allows us to define an operator $\Theta(K,\psi,h) : X \to K$ as follows: for $f \in X$,

$$u = \Theta(K,\psi,h)f \quad \Longleftrightarrow \quad u \text{ is a solution to Problem } \mathcal{P}^0. \tag{2.10}$$

Properties of this operator will be explored and will be used in the analysis of Problem $\mathcal{P}$ in the next two sections.

3. A new proof of existence and uniqueness result

In this section, we provide a new proof of a unique solvability result on the variational–hemivariational inequality (1.2). For convenience, we restate the problem as follows.
Problem $\mathcal{P}$. *Find an element* $u \in K$ *such that*

$$(Au, v-u)_X + \varphi(u, v) - \varphi(u, u) + j^0(u; v-u) \geq (f, v-u)_X \quad \forall\, v \in K. \tag{3.1}$$

In the study of this problem, besides the assumptions $H(K)$ and $H(f)$ already introduced in Sect. 2, we consider the following assumptions on the operator A, the function φ and the function j.
$\underline{H(A)}$ A is a strongly monotone Lipschitz continuous operator with constants m_A and $L_A \geq 0$, i.e., it satisfies inequalities (2.2) and (2.3).
$\underline{H(\varphi)}$ $\varphi : X \times X \to \mathbb{R}$, for any $u \in X$ the function $\varphi(u, \cdot) : X \to \mathbb{R}$ is convex and bounded above on a nonempty open set, and there exists $\alpha_\varphi > 0$ such that

$$\begin{cases} \varphi(\eta_1, v_2) - \varphi(\eta_1, v_1) + \varphi(\eta_2, v_1) - \varphi(\eta_2, v_2) \\ \leq \alpha_\varphi \|\eta_1 - \eta_2\|_X \, \|v_1 - v_2\|_X \quad \forall\, \eta_1, \eta_2, v_1, v_2 \in X. \end{cases} \tag{3.2}$$

$\underline{H(j)}$ $j \colon X \to \mathbb{R}$ is locally Lipschitz continuous and there exists $\alpha_j > 0$ such that

$$j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \leq \alpha_j \|v_1 - v_2\|_X^2 \quad \forall\, v_1, v_2 \in X. \tag{3.3}$$

Moreover, we consider the following smallness condition involving the constants α_φ, α_j and m_A in assumptions $H(\varphi)$, $H(j)$, and $H(A)$, respectively.
$\underline{H(s)}$ $\alpha_\varphi + \alpha_j < m_A$.

An existence and uniqueness result on Problem $\mathcal{P}$ was proved in [17, 24] in the framework of a reflexive Banach space X, under the more general assumption on the operator A that it is strongly monotone and pseudomonotone, and under an additional linear growth assumption on the generalized gradient of j. There, the proof was carried out by using an abstract surjectivity result for pseudomonotone operators and the Banach fixed-point theorem. The result in the form below (Theorem 11) is proved in [8], using existence of a minimizer of a convex minimization problem and the Banach fixed-point theorem. The main purpose of this section is to provide a new proof of the result. The proof method will be also useful in the stability analysis of Problem $\mathcal{P}$ in Sect. 4.

Theorem 11. *Assume* $H(K)$, $H(A)$, $H(\varphi)$, $H(j)$, $H(s)$ *and* $H(f)$. *Then, Problem* $\mathcal{P}$ *has a unique solution.*

Proof. Let $\rho \in (0, 1/\alpha_j)$ be a positive parameter, to be determined later. Note that the function $h = \rho j$ satisfies condition $H(h)$ with $\alpha_h = \rho\alpha_j$. For any fixed $w \in X$, the function $\psi : X \to \mathbb{R}$ defined by $\psi(v) = \rho\varphi(w, v)$ for $v \in X$ satisfies condition $H(\psi)$. These properties allow us to define three operators R, S and Λ as follows; recall the definition of B_ρ in (2.4).

$$R : X \times K \to K, \quad R\xi = \Theta(K, \rho\varphi(\xi_2, \cdot), \rho j)\xi_1 \quad \forall\, \xi = (\xi_1, \xi_2) \in X \times K; \tag{3.4}$$

$$S : K \to X \times K, \quad Su = (S_1 u, u),\ S_1 u = B_\rho u + \rho f \quad \forall\, u \in K; \tag{3.5}$$

$$\Lambda : K \to K, \quad \Lambda u = RSu \quad \forall\, u \in K. \tag{3.6}$$

Note that the operator S depends on ρ and the operators R and Λ depend on K, φ, j and ρ. Nevertheless, for simplicity, in this section we do not mention this dependence.

We now proceed in three steps, as follows.
(i) *We prove that for all* $\eta = (\eta_1, \eta_2)$ *and* $\xi = (\xi_1, \xi_2) \in X \times K$, *the following bound holds:*

$$\|R\eta - R\xi\|_X \leq \frac{1}{1 - \rho\alpha_j} \|\eta_1 - \xi_1\|_X + \frac{\rho\alpha_\varphi}{1 - \rho\alpha_j} \|\eta_2 - \xi_2\|_X. \tag{3.7}$$

Indeed, from the definition (3.4) of the operator R and (2.10) of the operator Θ, we know that $R\eta, R\xi \in K$, and

$$(R\eta, v - R\eta)_X + \rho\varphi(\eta_2, v) - \rho\varphi(\eta_2, R\eta) + \rho j^0(R\eta; v - R\eta) \geq (\eta_1, v - R\eta)_X,$$
$$(R\xi, v - R\xi)_X + \rho\varphi(\xi_2, v) - \rho\varphi(\xi_2, R\xi) + \rho j^0(R\xi; v - R\xi) \geq (\xi_1, v - R\xi)_X$$

for all $v \in K$. Take $v = R\xi$ in the first inequality, $v = R\eta$ in the second one, and add the two resulting inequalities to find that

$$\begin{aligned}\|R\eta - R\xi\|_X^2 \leq \rho\,[\varphi(\eta_2, R\xi) - \varphi(\eta_2, R\eta) + \varphi(\xi_2, R\eta) - \varphi(\xi_2, R\xi)] \\ + \rho\left[j^0(R\eta; R\xi - R\eta) + j^0(R\xi; R\eta - R\xi)\right] + (\eta_1 - \xi_1, R\eta - R\xi)_X.\end{aligned}$$

Then, we use assumptions $H(\varphi)$ and $H(j)$ on the functions φ and j to deduce that

$$\|R\eta - R\xi\|_X^2 \leq \rho\alpha_\varphi\|\eta_2 - \xi_2\|_X\|R\eta - R\xi\|_X + \rho\alpha_j\|R\eta - R\xi\|_X^2 + \|\eta_1 - \xi_1\|_X\|R\eta - R\xi\|_X.$$

Hence, (3.7) holds.

(ii) *We prove that for $\rho > 0$ sufficiently small, the operator $\Lambda : K \to K$ is a contraction.* Indeed, for $u, v \in K$, by the definition (3.6),

$$\Lambda u - \Lambda v = R(Su) - R(Sv).$$

Applying (3.7),

$$\|\Lambda u - \Lambda v\|_X \leq \frac{1}{1 - \rho\alpha_j}\|S_1 u - S_1 v\|_X + \frac{\rho\alpha_\varphi}{1 - \rho\alpha_j}\|u - v\|_X,$$

i.e.,

$$\|\Lambda u - \Lambda v\|_X \leq \frac{1}{1 - \rho\alpha_j}\|B_\rho u - B_\rho v\|_X + \frac{\rho\alpha_\varphi}{1 - \rho\alpha_j}\|u - v\|_X.$$

We now use assumption $H(A)$ and Lemma 7 to deduce that

$$\|\Lambda u - \Lambda v\|_X \leq \frac{k(\rho) + \rho\alpha_\varphi}{1 - \rho\alpha_j}\,\|u - v\|_X, \tag{3.8}$$

where $k(\rho) = (1 - 2\rho m_A + \rho^2 L_A^2)^{1/2}$.

Consider the real-valued function

$$F(\rho) = k(\rho) + \rho\alpha_\varphi + \rho\alpha_j = (1 - 2\rho m_A + \rho^2 L_A^2)^{1/2} + \rho\alpha_\varphi + \rho\alpha_j \tag{3.9}$$

for ρ in a neighborhood of 0 and $\rho < 1/\alpha_j$. Then, $F'(0) = \alpha_\varphi + \alpha_j - m_A < 0$ thanks to the smallness assumption $H(s)$. Thus, F is strictly decreasing in a neighborhood of the origin. Note that $F(0) = 1$. So for $\rho > 0$ sufficiently small, we have $F(\rho) < 1$ and the inequality (3.8) indicates that the operator Λ is a contraction.

(iii) *We now prove the existence of a unique solution to Problem $\mathcal{P}$.* By the definitions (3.4)–(3.6),

$$\Lambda u = R(Su) = R(\rho f - \rho A u + u, u) = \Theta(K, \rho\varphi(u, \cdot), \rho j)(\rho f - \rho A u + u).$$

Therefore, from (2.10) for the definition of the operator Θ, we know that $u = \Lambda u$ if and only if

$$u \in K, \quad (u, v - u)_X + \rho\varphi(u, v) - \rho\varphi(u, u) + \rho j^0(u; v - u) \geq (\rho f - \rho A u + u, v - u)_X \quad \forall\, v \in K,$$

or if and only if u is a solution to Problem $\mathcal{P}$. By the argument in Step (ii), for $\rho > 0$ sufficiently small, the operator Λ is a contraction on K and thus admits a unique fixed-point according to the Banach fixed point theorem. Hence, Problem $\mathcal{P}$ has a unique solution. □

4. A convergence result

The solution of Problem $\mathcal{P}$ depends on the data K, A, φ, j and f. In this section, we study its continuous dependence with respect to these data. To this end, we start with a convergence result for the auxiliary problem $\mathcal{P}^0$. We assume in what follows that $H(K)$, $H(\psi)$, $H(h)$ and $H(f)$ hold. Moreover, we consider sequences $\{K_n\}$, $\{\psi_n\}$, $\{h_n\}$ and $\{f_n\}$ such that for each $n \in \mathbb{N}$, the following conditions hold.

$\underline{H(K_n)}$ K_n is a nonempty, closed and convex subset of X.

$\underline{H(\psi_n)}$ $\psi_n : X \to \mathbb{R}$ is convex and bounded above on a nonempty open set in X.

$\underline{H(h_n)}$ $h_n : X \to \mathbb{R}$ is locally Lipschitz continuous and there exists $\alpha_{h_n} \in [0,1)$ such that

$$h_n^0(v_1; v_2 - v_1) + h_n^0(v_2; v_1 - v_2) \le \alpha_{h_n} \|v_1 - v_2\|_X^2 \quad \forall\, v_1, v_2 \in X.$$

$\underline{H(f_n)}$ $f_n \in X$.

By Theorem 9, for each $n \in \mathbb{N}$ there exists a unique solution to the following inequality problem.

Problem $\mathcal{P}_n^0$. *Find an element $u_n \in K_n$ such that*

$$(u_n, v - u_n)_X + \psi_n(v) - \psi_n(u_n) + h_n^0(u_n; v - u_n) \ge (f_n, v - u_n) \quad \forall\, v \in K_n. \tag{4.10}$$

For each $n \in \mathbb{N}$, we define an energy functional

$$J_n(v) = \frac{1}{2}\|v\|_X^2 + \psi_n(v) + h_n(v) - (f_n, v)_X \quad \forall\, v \in X. \tag{4.11}$$

By Theorem 9, u_n is the solution of Problem $\mathcal{P}_n^0$ if and only if u_n is the solution to the following optimization problem.

Problem $\mathcal{P}_n^M$. *Find an element $u_n \in K_n$ such that*

$$J_n(u_n) \le J_n(v) \quad \forall\, v \in K_n. \tag{4.12}$$

For a relation between the solutions of Problem $\mathcal{P}^0$ and Problem $\mathcal{P}_n^0$, we consider the following conditions.

$$K_n \xrightarrow{M} K \text{ in } X. \tag{4.13}$$

$$\psi_n(v_n) - \psi(v_n) \to 0 \quad \text{for any weakly convergent sequence } \{v_n\} \subset X. \tag{4.14}$$

$$h_n(v_n) - h(v_n) \to 0 \quad \text{for any weakly convergent sequence } \{v_n\} \subset X. \tag{4.15}$$

$$\text{There exist } c_0,\, c_1 \in \mathbb{R} \text{ such that} \tag{4.16}$$

$$\psi_n(v) + h_n(v) \ge c_1\|v\|_X + c_0 \quad \forall\, v \in X,\ n \in \mathbb{N}.$$

$$f_n \to f \quad \text{in } X. \tag{4.17}$$

Recall that the symbol "$\xrightarrow{M}$" denotes the set convergence in the sense of Mosco, see Definition 8.

We have the following convergence result.

Theorem 12. *Assume $H(K)$, $H(\psi)$, $H(h)$ and $H(f)$ and, for each $n \in \mathbb{N}$ assume $H(K_n)$, $H(\psi_n)$, $H(h_n)$ and $H(f_n)$. Moreover, assume (4.13)–(4.17). Then, the solution u_n of Problem $\mathcal{P}_n^0$ converges to the solution u of Problem $\mathcal{P}^0$, i.e.,*

$$u_n \to u \quad \text{in } X. \tag{4.18}$$

Proof. The proof is split into three steps.

Step (i). We prove that functions J_n and J enjoy the following properties.

$$J_n(v_n) - J(v_n) \to 0 \quad \text{for any weakly convergent sequence } \{v_n\} \subset X. \tag{4.19}$$

$$J_n(v_n) \to J(v) \quad \text{for any } \{v_n\} \subset X \text{ such that } v_n \to v \text{ in } X. \tag{4.20}$$

Indeed, assume that $\{v_n\}$ is a sequence of elements of X such that

$$v_n \rightharpoonup v \quad \text{in } X. \tag{4.21}$$

From the definitions of J_n and J, cf. (4.11) and (2.8),

$$J_n(v_n) - J(v_n) = \big(\psi_n(v_n) - \psi(v_n)\big) + \big(h_n(v_n) - h(v_n)\big) + (f - f_n, v_n)_X.$$

By the assumptions (4.14), (4.15), (4.17), each of the three terms on the right side of the above equality converges to 0; thus, (4.19) holds.

Now, let $v_n \to v$ in X. Write

$$J(v_n) - J(v) = \frac{1}{2}\left(\|v_n\|_X^2 - \|v\|_X^2\right) + (\psi(v_n) - \psi(v)) + (h(v_n) - h(v)) - (f, v_n - v)_X.$$

By Lemma 4 and assumptions $H(\psi)$, $H(h)$, we know that both ψ and h are continuous. Thus,

$$J(v_n) - J(v) \to 0.$$

From (4.19), we also have

$$J_n(v_n) - J(v_n) \to 0.$$

Hence, (4.20) holds:

$$J_n(v_n) - J(v) = [J_n(v_n) - J(v_n)] + [J(v_n) - J(v)] \to 0.$$

Step (ii). We prove that the sequence $\{u_n\}$ converges weakly to u, that is

$$u_n \rightharpoonup u \quad \text{in } X. \tag{4.22}$$

First, we claim that the sequence $\{u_n\}$ is bounded in X. Arguing by contradiction, suppose $\{u_n\}$ is not bounded in X. Then, we can find a subsequence of the sequence $\{u_n\}$, again denoted by $\{u_n\}$, such that $\|u_n\|_X \to \infty$. Using now (4.16) and (4.17), we deduce that there exist $\widetilde{c}_1$, $c_0 \in \mathbb{R}$, independent of n, such that

$$J_n(u_n) \ge \frac{1}{2}\,\|u_n\|_X^2 + \widetilde{c}_1\|u_n\|_X + c_0 \quad \forall\, n \in \mathbb{N}.$$

Thus,

$$J_n(u_n) \to \infty. \tag{4.23}$$

Let v be a given element in K and note that condition (4.13) implies that there exists a sequence $\{v_n\}$ such that $v_n \in K_n$ for each $n \in \mathbb{N}$ and $v_n \to v$ in X. Since u_n is the solution of Problem $\mathcal{P}_n^0$ and $v_n \in K_n$, by Theorem 9,

$$J_n(u_n) \le J_n(v_n) \quad \forall\, n \in \mathbb{N}.$$

By (4.20),

$$J_n(v_n) \to J(v).$$

Therefore, the sequence $\{J_n(u_n)\}$ is bounded in $\mathbb{R}$, which contradicts (4.23).

We conclude from above that the sequence $\{u_n\}$ is bounded in X. Since X is reflexive, there exists a subsequence of the sequence $\{u_n\}$, again denoted by $\{u_n\}$, and an element $\widetilde{u} \in X$, such that

$$u_n \rightharpoonup \widetilde{u} \quad \text{in } X. \tag{4.24}$$

Let us prove that $\widetilde{u}$ is a solution of Problem $\mathcal{P}^M$. First, by the assumption (4.13),

$$\widetilde{u} \in K. \tag{4.25}$$

By Lemma 10, J is convex and continuous; hence, J is weakly lower semicontinuous. Consequently,

$$J(\widetilde{u}) \le \liminf J(u_n). \tag{4.26}$$

By the property (4.19),

$$\liminf J(u_n) = \liminf J_n(u_n). \tag{4.27}$$

Again, let $v \in K$ be an arbitrarily fixed element and let $v_n \in K_n$ such that $v_n \to v$ in X. Then, from (4.12) and (4.20) we have

$$J_n(u_n) \le J_n(v_n), \qquad J_n(v_n) \to J(v),$$

and, therefore,

$$\liminf J_n(u_n) \le J(v). \tag{4.28}$$

Combine (4.26)–(4.28) to see that

$$J(\widetilde{u}) \le J(v). \tag{4.29}$$

From (4.25) and (4.29), we see that $\widetilde{u}$ solves Problem $\mathcal{P}^M$. Now, since Problem $\mathcal{P}^M$ has a unique solution, denoted by u, it follows that $\widetilde{u} = u$. Thus, the limit $\widetilde{u}$ is the unique solution of Problem $\mathcal{P}^M$ and it is independent of the subsequence selected. Consequently, the whole sequence $\{u_n\}$ converges weakly in X to u, i.e., (4.22) holds.

Step (iii). We prove that the sequence $\{u_n\}$ converges strongly to u, i.e., (4.18) holds. By assumption (4.13), there exists a sequence $\{\widetilde{u}_n\}$ such that $\widetilde{u}_n \in K_n$ for each $n \in \mathbb{N}$ and

$$\widetilde{u}_n \to u \quad \text{in } X. \tag{4.30}$$

Then, using Lemma 10 and inequality (2.1) with $t = \frac{1}{2}$ we find that

$$\frac{\beta}{4}\,\|\widetilde{u}_n - u_n\|_X^2 \le \frac{1}{2}\Big[J(\widetilde{u}_n) - J\Big(\frac{\widetilde{u}_n + u_n}{2}\Big)\Big] + \frac{1}{2}\Big[J(u_n) - J\Big(\frac{\widetilde{u}_n + u_n}{2}\Big)\Big]. \tag{4.31}$$

Write

$$\begin{aligned} J(\widetilde{u}_n) - J\Big(\frac{\widetilde{u}_n + u_n}{2}\Big) &= [J(\widetilde{u}_n) - J(u_n)] + [J(u_n) - J_n(u_n)] \\ &\quad + \Big[J_n(u_n) - J_n\Big(\frac{\widetilde{u}_n + u_n}{2}\Big)\Big] \\ &\quad + \Big[J_n\Big(\frac{\widetilde{u}_n + u_n}{2}\Big) - J\Big(\frac{\widetilde{u}_n + u_n}{2}\Big)\Big]. \end{aligned} \tag{4.32}$$

Note that the convergences (4.22), (4.30) and the properties of J imply that

$$\limsup\,[J(\widetilde{u}_n) - J(u_n)] = J(u) - \liminf J(u_n) \le 0. \tag{4.33}$$

By the convergence relations (4.22), (4.30) and (4.19),

$$J(u_n) - J_n(u_n) \to 0, \quad J_n\Big(\frac{\widetilde{u}_n + u_n}{2}\Big) - J\Big(\frac{\widetilde{u}_n + u_n}{2}\Big) \to 0. \tag{4.34}$$

Finally, since u_n is a solution to Problem $\mathcal{P}_n^M$,

$$J_n(u_n) - J_n\Big(\frac{\widetilde{u}_n + u_n}{2}\Big) \le 0. \tag{4.35}$$

Therefore, (4.32)–(4.35) imply that

$$\limsup\Big[J(\widetilde{u}_n) - J\Big(\frac{\widetilde{u}_n + u_n}{2}\Big)\Big] \le 0. \tag{4.36}$$

On the other hand,

$$\begin{aligned} J(u_n) - J\Big(\frac{\widetilde{u}_n + u_n}{2}\Big) &= [J(u_n) - J_n(u_n)] + \Big[J_n\Big(\frac{\widetilde{u}_n + u_n}{2}\Big) - J\Big(\frac{\widetilde{u}_n + u_n}{2}\Big)\Big] \\ &\quad + \Big[J_n(u_n) - J_n\Big(\frac{\widetilde{u}_n + u_n}{2}\Big)\Big]. \end{aligned}$$

Thus, using (4.34) and (4.35) we find that

$$\limsup\Big[J(u_n) - J\Big(\frac{\widetilde{u}_n + u_n}{2}\Big)\Big] \le 0. \tag{4.37}$$

We now combine inequalities (4.31), (4.36) and (4.37) to deduce that

$$\limsup\,\|\widetilde{u}_n - u_n\|_X^2 = 0$$

or, equivalently,

$$\widetilde{u}_n - u_n \to 0 \quad \text{in} \quad X. \tag{4.38}$$

Finally, we write $u_n - u = (u_n - \widetilde{u}_n) + (\widetilde{u}_n - u)$ and combine the convergences (4.30) and (4.38) to see that $u_n \to u$ in X which concludes the proof. □

Remark 13. We can restate Theorem 12 by using the definition (2.10) of the operator Θ. Indeed, assume $H(K)$, $H(\psi)$, $H(h)$, $H(f)$ hold and, for each $n \in \mathbb{N}$ assume $H(K_n)$, $H(\psi_n)$, $H(h_n)$ and $H(f_n)$. Moreover, assume (4.13)–(4.17). Then, Theorem 12 states that the following convergence holds:

$$\Theta(K_n, \psi_n, h_n) f_n \to \Theta(K, \psi, h) f \quad \text{in } X. \tag{4.39}$$

We now move on to a convergence result in the study of Problem $\mathcal{P}$. To this end, we keep in what follows the assumptions of Theorem 11 and, besides the sequences $\{K_n\}$ and $\{f_n\}$, we consider the sequences $\{A_n\}$, $\{\varphi_n\}$ and $\{j_n\}$ such that for each $n \in \mathbb{N}$, the following conditions hold.

$\underline{H(A_n)}$ A_n is a strongly monotone and Lipschitz continuous operator with constants m_n and $L_n \geq 0$.

$\underline{H(\varphi_n)}$ $\varphi_n : X \times X \to \mathbb{R}$, satisfies condition $H(\varphi)$ with constant α_{φ_n}.

$\underline{H(j_n)}$ $j_n : X \to \mathbb{R}$, satisfies condition $H(j)$ with constant α_{j_n}.

$\underline{H(s_n)}$ $\alpha_{\varphi_n} + \alpha_{j_n} < m_n$.

Under these assumptions, it follows from Theorem 11 that for each $n \in \mathbb{N}$ there exists a unique solution to the following inequality problem.

Problem $\mathcal{P}_n$. *Find an element $u_n \in K_n$ such that*

$$\begin{aligned} &(A_n u_n, v - u_n)_X + \varphi_n(u_n, v) - \varphi_n(u_n, u_n) + j_n^0(u_n; v - u_n) \\ &\quad \geq (f_n, v - u_n) \quad \forall\, v \in K_n. \end{aligned} \tag{4.40}$$

We now consider the following additional assumptions.

$$A_n v \to Av \quad \text{in } X,\ \forall\, v \in X. \tag{4.41}$$

$$\varphi_n(u, v_n) - \varphi(u, v_n) \to 0 \quad \forall\, u \in X,\ \{v_n\} \subset X \ \text{ weakly convergent.} \tag{4.42}$$

$$j_n(v_n) - j(v_n) \to 0 \quad \forall\, \{v_n\} \subset X \ \text{ weakly convergent.} \tag{4.43}$$

$$\begin{aligned} &\text{There exist } c_0,\ c_1 \in \mathbb{R} \text{ such that} \\ &\qquad \varphi_n(u, v) + j_n(v) \geq c_1 \|v\|_X + c_0 \quad \forall\, u, v \in X,\ n \in \mathbb{N}. \end{aligned} \tag{4.44}$$

$$\begin{cases} \text{There exist } m > 0, L > 0 \text{ and } \alpha > 0 \text{ such that} \\ \text{(a)} \quad m \leq \min\{m_n, m_A\}, \quad L \geq \max\{L_n, L_A\} \quad \forall\, n \in \mathbb{N}. \\ \text{(b)} \quad \max\{\alpha_\varphi + \alpha_j, \alpha_{\varphi_n} + \alpha_{j_n}\} \leq \alpha \quad \forall\, n \in \mathbb{N}. \\ \text{(c)} \quad \alpha < \frac{L^2}{m}\Big(1 - \sqrt{1 - \frac{m^2}{L^2}}\Big). \end{cases} \tag{4.45}$$

Remark 14. It follows from Remark 6 and the condition (a) in (4.45) that $m \leq L$; then the expression in condition (c) is well-defined.

The main result in this section is the following convergence result.

Theorem 15. *Assume $H(K)$, $H(A)$, $H(\varphi)$, $H(j)$, $H(s)$ and $H(f)$ and, for each $n \in \mathbb{N}$ assume $H(K_n)$, $H(A_n)$, $H(\varphi_n)$, $H(j_n)$, $H(s_n)$ and $H(f_n)$. Moreover, assume (4.13), (4.17) and (4.41)–(4.45). Then, the solution u_n of Problem $\mathcal{P}_n$ converges to the solution u of Problem $\mathcal{P}$, i.e.,*

$$u_n \to u \quad \text{in } X. \tag{4.46}$$

Proof. The proof is split into three steps.

Step (i) Preliminary results. Let $n \in \mathbb{N}$ and denote

$$\rho_0 = \frac{m}{L^2} \tag{4.47}$$

where m and L are defined in (4.45). Note that the elementary inequality

$$1-\sqrt{1-\frac{m^2}{L^2}}\le\frac{m^2}{L^2}$$

combined with assumption (4.45)(c) implies that

$$\alpha<m. \tag{4.48}$$

Then, using the inequality $m\le L$ and assumption (4.45)(b) it follows that m/L^2 is strictly less than $1/\alpha_j$ and $1/\alpha_{j_n}$. Therefore,

$$\rho_0\in\Big(0,\frac{1}{\alpha_{j_n}}\Big),\qquad \rho_0\in\Big(0,\frac{1}{\alpha_j}\Big). \tag{4.49}$$

Since these inclusions are satisfied, it follows from the proof of Theorem 11 that we are in a position to define the operators R, S and Λ by equalities (3.4), (3.5), (3.6), with $\rho=\rho_0$. Similarly, for each $n\in\mathbb{N}$ we can consider the operators R_n, S_n, Λ_n defined by equalities

$$R_n:X\times K\to K,\quad R_n\xi=\Theta(K_n,\rho_0\varphi_n(\xi_2,\cdot),\rho_0 j_n)\xi_1\quad\forall\,\xi=(\xi_1,\xi_2)\in X\times K, \tag{4.50}$$

$$S_n:K\to X\times K,\quad S_nu=(\rho_0 f_n-\rho_0A_nu+u,u)\qquad\forall\,u\in K, \tag{4.51}$$

$$\Lambda_n:K\to K,\quad \Lambda_nu=R_nS_nu\qquad\forall\,u\in K. \tag{4.52}$$

Note that the solutions u_n and u of the variational–hemivariational inequalities (2.6) and (4.10), respectively, satisfy the equalities

$$u_n=\Lambda_nu_n,\qquad u=\Lambda u. \tag{4.53}$$

Step (ii) We prove that here exists a constant $k_0\in[0,1)$ *such that for any* $n\in\mathbb{N}$ *the following inequality holds:*

$$\|\Lambda_nu-\Lambda_nv\|_X\le k_0\|u-v\|_X\quad\forall\,u,\,v\in X. \tag{4.54}$$

Let $n\in\mathbb{N}$. Define

$$k_0=\sqrt{1-\frac{m^2}{L^2}}+\frac{\alpha m}{L^2}. \tag{4.55}$$

The smallness assumption (4.45)(c) guarantees that $k_0\in[0,1)$.

Assumptions $H(A_n)$ and (4.45)(a) imply that the operator A_n is strongly monotone and Lipschitz continuous with constants m and L. It follows from (3.8) that the operator Λ_n is Lipschitz continuous with the constant

$$k_n=\frac{k(\rho_0)+\rho_0\alpha_{\varphi_n}}{1-\rho_0\alpha_{j_n}} \tag{4.56}$$

where $k(\rho_0)=(1-2\rho_0m+\rho_0^2L^2)^{1/2}$. We now use (4.47), inequality $k_0<1$ and assumption (4.45)(b) to see that

$$k(\rho_0)+\rho_0\alpha_{\varphi_n}+k_0\rho_0\alpha_{j_n}\le\sqrt{1-\frac{m^2}{L^2}}+\frac{m}{L^2}(\alpha_{\varphi_n}+\alpha_{j_n})\le\sqrt{1-\frac{m^2}{L^2}}+\frac{\alpha m}{L^2}.$$

Thus, from (4.55),

$$k(\rho_0)+\rho_0\alpha_{\varphi_n}+k_0\rho_0\alpha_{j_n}\le k_0;$$

equivalently,

$$\frac{k(\rho_0)+\rho_0\alpha_{\varphi_n}}{1-\rho_0\alpha_{j_n}}\le k_0. \tag{4.57}$$

Combine (4.56) and (4.57) to see that $k_n\le k_0$. Since Λ_n is Lipschitz continuous with the constant k_n, it follows that (4.54) holds.

Step (iii) We prove the convergence (4.46). Let $n\in\mathbb{N}$. From equalities (4.53),

$$\|u_n-u\|_X=\|\Lambda_nu_n-\Lambda u\|_X\le\|\Lambda_nu_n-\Lambda_nu\|_X+\|\Lambda_nu-\Lambda u\|_X.$$

We use inequality (4.54) to find that

$$\|u_n - u\|_X \leq k_0 \|u_n - u\|_X + \|\Lambda_n u - \Lambda u\|_X;$$

equivalently,

$$\|u_n - u\|_X \leq \frac{1}{1-k_0}\, \|\Lambda_n u - \Lambda u\|_X. \tag{4.58}$$

Next,

$$\begin{aligned}\|S_n u - Su\|_{X\times X} &= \|(\rho f_n - \rho A_n u + u, u) - (\rho f - \rho A u + u, u)\|_{X\times X}\\ &= \|(\rho(f_n - f) - \rho(A_n u - Au)\|_X\\ &\leq \rho\|f_n - f\|_X + \rho\|A_n u - Au\|_X.\end{aligned}$$

By assumptions (4.41), (4.17), we have

$$S_n u \to Su \quad \text{in } X\times X \text{ as } n\to\infty. \tag{4.59}$$

On the other hand, the inequality (3.7) for the operator R_n yields

$$\|R_n\eta - R_n\xi\|_X \leq \frac{1}{1-\rho_0\alpha_{j_n}}\|\eta_1 - \xi_1\|_X + \frac{\rho_0\alpha_{\varphi_n}}{1-\rho_0\alpha_{j_n}}\|\eta_2 - \xi_2\|_X \tag{4.60}$$

for $\eta = (\eta_1, \eta_2)$, $\xi = (\xi_1, \xi_2) \in X \times K$. From the assumption (4.45),

$$\frac{1}{1-\rho_0\alpha_{j_n}} \leq \frac{1}{1-\rho_0\alpha}, \qquad \frac{\rho_0\alpha_{\varphi_n}}{1-\rho_0\alpha_{j_n}} \leq \frac{\rho_0\alpha}{1-\rho_0\alpha}.$$

Therefore, using (4.60) we deduce that there exists a constant $d_0 > 0$ which does not depend on n such that

$$\|R_n\eta - R_n\xi\|_X \leq d_0\|\eta - \xi\|_{X\times X} \quad \forall\, \eta = (\eta_1,\eta_2),\ \xi = (\xi_1,\xi_2) \in X\times K. \tag{4.61}$$

Write

$$\Lambda_n u - \Lambda u = R_n(S_n u) - R(Su) = R_n(S_n u) - R_n(Su) + R_n(Su) - R(Su).$$

By (4.61),

$$\|\Lambda_n u - \Lambda u\|_X \leq d_0\|S_n u - Su\|_{X\times X} + \|R_n(Su) - R(Su)\|_X. \tag{4.62}$$

We use the convergence (4.39) with $\psi_n = \rho_0\varphi_n(u,\cdot)$, $h_n = \rho_0 j_n$ and $f_n = f = Su$ to deduce that

$$R_n(Su) \to R(Su) \quad \text{in } X \text{ as } n\to\infty. \tag{4.63}$$

Therefore, (4.59), (4.63) and (4.62) imply

$$\|\Lambda_n u - \Lambda u\|_X \to 0. \tag{4.64}$$

We now combine inequality (4.58) with (4.64) to see that (4.46) holds, which concludes the proof. □

We end this section with a remark.

Remark 16. From the proof of Theorem 15, it can be seen that the assumptions $H(s)$ and $H(s_n)$ are implied by the assumption (4.45), cf. (4.48). Thus, in the statement of Theorem 15, the assumptions $H(s)$ and $H(s_n)$ can be formally removed.

5. Relevant particular cases and applications

In this section, we present some particular cases of the results obtained in Sects. 3 and 4. Our first particular case is when $K_n \xrightarrow{M} K$, $A_n = A$, $\varphi_n = \varphi$, $j_n = j$ and $f_n = f$. In this case, Theorems 11 and 15 lead to the following result.

Corollary 17. *Assume $H(K)$, $H(A)$, $H(\varphi)$, $H(j)$, $H(s)$, $H(f)$, $H(K_n)$ for each $n \in \mathbb{N}$, and $K_n \xrightarrow{M} K$ in X. Then, for each $n \in \mathbb{N}$, there exists a unique element $u_n \in K_n$ such that*

$$(Au_n, v - u_n)_X + \varphi(u_n, v) - \varphi(u_n, u_n) + j^0(u_n; v - u_n) \geq (f, v - u_n)_X \quad \forall\, v \in K_n.$$

Moreover, $u_n \to u$ in X, where u represents the solution of Problem $\mathcal{P}$ guaranteed by Theorem 11.

When we associate the index n with a discretization parameter, Corollary 17 may be interpreted as a convergence result for numerical solutions of Problem $\mathcal{P}$. For the setting, we consider a sequence of subsets $\{K_n\}$ in finite-dimensional spaces $\{X_n\}$ such that X_n is a finite element subspace of the space X, corresponding to a finite element partition of the spatial domain of the variational–hemivariational inequality (3.1). If we take $K_n = X_n \cap K$, then $K_n \subset K$ and Problem $\mathcal{P}_n$ represents an internal numerical approximations of Problem $\mathcal{P}$. We refer the reader to [11,12] for convergence results related to internal numerical approximations, and [13] for both internal and external numerical approximations of such inequalities. A comprehensive reference on the numerical analysis of Problem $\mathcal{P}$ can be found in the survey paper [10].

Our second particular case is when $K_n = K$. In this case, Theorems 11, 15 and Remark 16 lead to the following result.

Corollary 18. *Assume $H(K)$, $H(A)$, $H(\varphi)$, $H(j)$ and $H(f)$ and, for each $n \in \mathbb{N}$ assume $H(A_n)$, $H(\varphi_n)$, $H(j_n)$ and $H(f_n)$. Moreover, assume (4.17) and (4.41)–(4.45). Then, for each $n \in \mathbb{N}$, there exists a unique element $u_n \in K$ such that*

$$(A_n u_n, v - u_n)_X + \varphi_n(u_n, v) - \varphi_n(u_n, u_n) + j_n^0(u_n; v - u_n) \geq (f_n, v - u_n)_X \quad \forall\, v \in K.$$

Moreover, $u_n \to u$ in X.

This result provides the continuous dependence of the solution to Problem $\mathcal{P}$ with respect to the operator A, the functions φ, j, and the element f.

When j vanishes and $j_n \not\equiv 0$ for each $n \in \mathbb{N}$, Corollary 18 provides the convergence of the solution of a variational–hemivariational inequality to the solution of a pure variational inequality. Similarly, in the case φ vanishes and $\varphi_n \not\equiv 0$ for each $n \in \mathbb{N}$, Corollary 18 provides the convergence of the solution of a variational–hemivariational inequality to the solution of a pure hemivariational inequality. Moreover, Corollaries 17 and 18 reduce to corresponding results for pure variational inequalities if j_n and j are removed from consideration, and to that for pure hemivariational inequalities if φ_n and φ are removed from consideration.

We end this section with examples of sets, operators and functions which satisfy conditions stated in Theorem 15. We conclude that the convergence result (4.46) holds for the corresponding inequality problems.

Example 19. Assume $H(K)$ and let $\{a_n\}$, $\{b_n\}$ be two sequence of positive reals such that $a_n \to 1$ and $b_n \to 0$. Let $\theta \in X$ and assume that for each $n \in \mathbb{N}$ the set K_n is defined by

$$K_n = a_n K + b_n\{\theta\}.$$

It is easy to see that the sequence $\{K_n\}$ satisfies conditions $H(K_n)$ and (4.13).

Example 20. Assume $H(A)$. Let $\{\omega_n\}$ be a sequence of positive numbers such that $\omega_n \to 0$ and let $T : X \to X$ be a monotone and Lipschitz continuous operator, i.e., an operator which satisfies the inequalities

$$(Tu - Tv, u - v)_X \geq 0, \quad \|Tu - Tv\|_X \leq L_T \|u - v\|_X \quad \forall\, u, v \in X$$

with $L_T > 0$. For all $n \in \mathbb{N}$ define an operator $A_n : X \to X$ by $A_n v = Av + \omega_n T v$ for all $v \in X$. Then, it is easy to see that the sequence $\{A_n\}$ satisfies conditions $H(A_n)$ and (4.41).

Example 21. Assume $H(\varphi)$. Let $\{\omega_n\}$ be a sequence of positive numbers such that $\omega_n \to 0$ and let $\psi : X \to \mathbb{R}$ be convex Lipschitz continuous function. For all $n \in \mathbb{N}$ consider the function $\varphi_n : X \times X \to \mathbb{R}$ defined by $\varphi_n(u, v) = \varphi(u, v) + \omega_n \psi(v)$ for all $u, v \in X$. Then, it is easy to see that the sequence $\{\varphi_n\}$ satisfies conditions $H(\varphi_n)$ and (4.42). A similar example can be constructed for function j.

6. An elastic contact problem

The theoretical results presented in Sects. 3 and 4 can be applied in analysis of a large number of static contact problems with elastic materials. Details on modeling and construction of the corresponding boundary value problems can be found in [10,24]; for this reason, we will present the variational formulation of the contact model directly.

We denote by $\mathbb{S}^d$ $(d = 2, 3)$ the space of second-order symmetric tensors on $\mathbb{R}^d$, and by "$\cdot$", $\|\cdot\|$ and $\mathbf{0}$ the inner product, the norm and the zero element on the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$. Let Ω be a bounded domain in $\mathbb{R}^d$ with a Lipschitz continuous boundary Γ that is divided into three measurable disjoint parts Γ_1, Γ_2 and Γ_3 such that $meas\,\Gamma_1 > 0$. The unknown displacement will be sought in a subset of the space

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^d : \ \boldsymbol{v} = \mathbf{0} \ \text{ on } \Gamma_1 \,\}.$$

We denote by $\boldsymbol{\nu}$ the unit outward normal to Γ. For every $\boldsymbol{v} \in V$, we use the notation

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = \frac{1}{2}(\nabla \boldsymbol{v} + \nabla^T \boldsymbol{v}), \quad v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$$

for the symmetric part of the gradient of $\boldsymbol{v}$, the normal and tangential components of $\boldsymbol{v}$, respectively. It is well known that V is a real Hilbert space with the canonical inner product

$$(\boldsymbol{v}, \boldsymbol{u})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}x,$$

and the associated norm $\|\cdot\|_V$. We also use $\|\gamma\|$ for the norm of the trace operator $\gamma \colon V \to L^2(\Gamma_3)^d$.

Let $\mathcal{F}$, p_ν, p_τ, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, and θ be given data, satisfying the following conditions.

$$\left\{\begin{array}{l}
\mathcal{F} \colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d \text{ is such that}\\
\text{(a) there exists } L_{\mathcal{F}} > 0 \text{ such that}\\
\qquad \|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|\\
\qquad\quad \text{for all } \ \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega;\\
\text{(b) there exists } m_{\mathcal{F}} > 0 \text{ such that}\\
\qquad (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2\\
\qquad\quad \text{for all } \ \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega;\\
\text{(c) } \mathcal{F}(\cdot, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d,\\
\text{(d) } \mathcal{F}(\boldsymbol{x}, \mathbf{0}) = \mathbf{0} \ \text{ for a.e. } \boldsymbol{x} \in \Omega.
\end{array}\right. \tag{6.1}$$

$$\left\{\begin{array}{l}
\text{For } e = \nu, \tau, \ \text{the function } p_e \colon \mathbb{R} \to \mathbb{R}_+ \text{ is such that}\\
\text{(a) there exists } L_e > 0 \text{ such that}\\
\qquad |p_e(r_1) - p_e(r_2)| \leq L_e |r_1 - r_2| \quad \text{for all } r_1,\, r_2 \in \mathbb{R};\\
\text{(b) } p_e(r) = 0 \ \text{ if and only if } \ r \leq 0.
\end{array}\right. \tag{6.2}$$

$$(L_\nu + L_\tau)\,\|\gamma\|^2 < m_{\mathcal{F}}. \tag{6.3}$$

$$\boldsymbol{f}_0 \in L^2(\Omega)^d, \quad \boldsymbol{f}_2 \in L^2(\Gamma_2)^d. \tag{6.4}$$

$$\left\{\begin{array}{l}\text{There exist } G \in H^2(\Omega) \text{ and } M_0,\ M_1 \in \mathbb{R} \text{ such that}\\ \theta = \gamma(G) \text{ on } \Gamma_3 \text{ and } 0 < M_0 \le G(\boldsymbol{x}) \le M_1 \text{ for all } \boldsymbol{x} \in \Omega \cup \Gamma.\end{array}\right. \tag{6.5}$$

Introduce the following notation.

$$K = \{\, \boldsymbol{v} \in V :\ v_\nu \le \theta \quad \text{a.e. on } \Gamma_3 \,\}, \tag{6.6}$$

$$A : V \to V, \quad (A\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x, \tag{6.7}$$

$$\varphi\colon V \times V \to \mathbb{R}, \quad \varphi(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} p_\tau(u_\nu)\,\|\boldsymbol{v}_\tau\|\,\mathrm{d}s, \tag{6.8}$$

$$g\colon \mathbb{R} \to \mathbb{R}, \quad g_\nu(r) = \int_0^r p_\nu(s)\,\mathrm{d}s, \tag{6.9}$$

$$j\colon V \to \mathbb{R}, \quad j(\boldsymbol{v}) = \int_{\Gamma_3} g_\nu(v_\nu)\,\mathrm{d}s, \tag{6.10}$$

$$\boldsymbol{f} \in V, \quad (\boldsymbol{f}, \boldsymbol{v}) = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}\,\mathrm{d}s \tag{6.11}$$

for $\boldsymbol{u}, \boldsymbol{v} \in V$ and $r \in \mathbb{R}$. It can be proved that the function j is locally Lipschitz. Therefore, as usual, we shall use the notation $j^0(\boldsymbol{u}; \boldsymbol{v})$ for the generalized directional derivative of j at $\boldsymbol{u}$ in the direction $\boldsymbol{v}$. Moreover, a standard regularity result based on definition (6.9) and assumption (6.2) shows that

$$j^0(\boldsymbol{u}; \boldsymbol{v}) = \int_{\Gamma_3} p_\nu(u_\nu) v_\nu\,\mathrm{d}s \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V. \tag{6.12}$$

We now consider the following inequality problem.

Problem $\mathcal{Q}$. *Find a displacement field* $\boldsymbol{u} \in K$ *such that*

$$\langle A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle + \varphi(\boldsymbol{u}, \boldsymbol{v}) - \varphi(\boldsymbol{u}, \boldsymbol{u}) + j^0(\boldsymbol{u}; \boldsymbol{v} - \boldsymbol{u}) \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle \quad \forall\, \boldsymbol{v} \in K. \tag{6.13}$$

Problem $\mathcal{Q}$ represents the variational formulation of a mathematical model which describes the equilibrium of an elastic body in frictional contact with a foundation made of a rigid obstacle covered by a layer of asperities of thickness θ, under the action of body forces and surface tractions of densities $\boldsymbol{f}_0$ and $\boldsymbol{f}_2$, respectively. Here, $\mathcal{F}$ is a nonlinear constitutive function which describes the material's behavior and p_ν and p_τ are the so-called normal compliance functions.

Next, we consider a perturbed version of Problem $\mathcal{Q}$ in which, for simplicity, we assume that only the set K, the operator A and the function φ are perturbed. To this end, we consider sequences $\{\theta_n\}$, $\{\omega_n\}$ and $\{\varepsilon_n\}$ with the following properties.

$$\left\{\begin{array}{l}\text{For each } n \in \mathbb{N} \text{ there exist } G_n \in H^2(\Omega) \text{ such that}\\ \theta_n = \gamma(G_n) \text{ on } \Gamma_3 \text{ and } 0 < M_0 \le G_n(\boldsymbol{x}) \le M_1 \text{ for all } \boldsymbol{x} \in \Omega \cup \Gamma.\end{array}\right. \tag{6.14}$$

$$G_n \rightharpoonup G \quad \text{in } H^2(\Omega). \tag{6.15}$$

$$\omega_n \in L^\infty(\Omega), \quad \omega_n(\boldsymbol{x}) \ge 0 \text{ a.e. } \boldsymbol{x} \in \Omega,\ \forall\, n \in \mathbb{N},\ \|\omega_n\|_{L^\infty(\Omega)} \to 0. \tag{6.16}$$

$$\varepsilon_n \ge 0 \quad \forall\, n \in \mathbb{N},\ \varepsilon_n \to 0. \tag{6.17}$$

Define

$$K_n = \{\, \boldsymbol{v} \in V :\ v_\nu \le \theta_n \text{ a.e. on } \Gamma_3 \,\}, \tag{6.18}$$

$$A_n : V \to V, \quad (A_n \boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \big(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \omega_n \boldsymbol{\varepsilon}(\boldsymbol{u})\big) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}x, \tag{6.19}$$

$$\varphi_n \colon V \times V \to \mathbb{R}, \quad \varphi_n(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} p_\tau(u_\nu) \sqrt{\|\boldsymbol{v}_\tau\|^2 + \varepsilon_n^2}\, \mathrm{d}s \tag{6.20}$$

for $\boldsymbol{u}, \boldsymbol{v} \in V$, together with the following perturbed version of Problem $\mathcal{Q}$.

Problem $\mathcal{Q}_n$. *Find a displacement field* $\boldsymbol{u}_n \in K_n$ *such that*

$$\begin{aligned}(A_n \boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n)_X + \varphi_n(\boldsymbol{u}_n, \boldsymbol{v}) - \varphi_n(\boldsymbol{u}_n, \boldsymbol{u}_n) + j^0(\boldsymbol{u}_n; \boldsymbol{v} - \boldsymbol{u}_n)\\ \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_n)_X \quad \forall\, \boldsymbol{v} \in K_n.\end{aligned} \tag{6.21}$$

Note that Problem $\mathcal{Q}_n$ represents the variational formulation of a contact model similar to that associated with Problem $\mathcal{Q}$. The difference arises in the fact that the elasticity operator $\mathcal{F}$ and the thickness θ are perturbed and the friction law is regularized.

Regarding Problems $\mathcal{Q}$ and $\mathcal{Q}_n$, we have the following existence, uniqueness and convergence result.

Theorem 22. *Assume* (6.1)–(6.5), (6.14)–(6.17). *Then, the following statements hold.*

(a) Problem $\mathcal{Q}$ *admits a unique solution* $\boldsymbol{u} \in K$, *and for each* $n \in \mathbb{N}$, *Problem* $\mathcal{Q}_n$ *admits a unique solution* $\boldsymbol{u}_n \in K_n$.

(b) If, in addition, the smallness condition

$$(L_\nu + L_\tau)\, \|\gamma\|^2 \leq \frac{1}{2}\, m_{\mathcal{F}} \tag{6.22}$$

holds, then the solution $\boldsymbol{u}_n$ *of Problem* $\mathcal{Q}_n$ *converges to the solution* $\boldsymbol{u}$ *of Problem* $\mathcal{Q}$: $\boldsymbol{u}_n \to \boldsymbol{u}$ *in* V *as* $n \to \infty$.

Proof. We apply Theorems 11 and 15 on the space V. Since in our case the function j and the element f are not perturbed, we need to verify only the validity of conditions $H(K)$, $H(A)$, $H(\varphi)$, $H(j)$, $H(s)$, $H(f)$, $H(K_n)$, $H(A_n)$, $H(\varphi_n)$, $H(s_n)$, (4.13), (4.41), (4.42), (4.44) and (4.45). Some of the conditions are obviously satisfied, such as conditions $H(K)$, $H(A)$, $H(f)$, $H(K_n)$, $H(A_n)$ and (4.41). Therefore, we focus in what follows on the conditions $H(\varphi)$, $H(j)$ $H(s)$, $H(\varphi_n)$, $H(s_n)$ (4.13), (4.42), (4.44) and (4.45). Note that the constants m_A and L_A in $H(A)$ are given by $m_A = m_{\mathcal{F}}$ and $L_A = L_{\mathcal{F}}$.

Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V$. By the definition (6.8) and the assumption (6.2),

$$\begin{aligned}&\varphi(\boldsymbol{u}_1, \boldsymbol{v}_2) - \varphi(\boldsymbol{u}_1, \boldsymbol{v}_1) + \varphi(\boldsymbol{u}_2, \boldsymbol{v}_1) - \varphi(\boldsymbol{u}_2, \boldsymbol{v}_2)\\ &\quad\leq \int_{\Gamma_3} \big(p_\tau(u_{1\nu}) - p_\tau(u_{2\nu})\big)\big(\|\boldsymbol{v}_{2\tau}\| - \|\boldsymbol{v}_{1\tau}\|\big)\, \mathrm{d}s\\ &\quad\leq L_\tau \|\gamma\|^2 \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V.\end{aligned}$$

This shows that condition $H(\varphi)$ holds with $\alpha_\varphi = L_\tau \|\gamma\|^2$. Next, (6.12) and assumption (6.2) show that

$$\begin{aligned}j^0(\boldsymbol{v}_1, \boldsymbol{v}_2 - \boldsymbol{v}_1) + j^0(\boldsymbol{v}_1, \boldsymbol{v}_2 - \boldsymbol{v}_1) &= \int_{\Gamma_3} \big(p_\nu(v_{1\nu}) - p_\nu(v_{2\nu})\big)\big(\boldsymbol{v}_{1\nu} - \boldsymbol{v}_{2\nu}\big)\, \mathrm{d}s\\ &\leq L_\nu \|\gamma\|^2 \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V^2.\end{aligned}$$

This shows that condition $H(j)$ holds with $\alpha_j = L_\nu \|\gamma\|^2$. Therefore, using assumption (6.3) we deduce that condition $H(s)$ is satisfied. In addition, using the elementary inequality

$$|\sqrt{a^2 + \varepsilon^2} - \sqrt{b^2 + \varepsilon^2}| \leq |a - b| \quad \forall\, a, b, \varepsilon \in \mathbb{R},$$

we deduce that condition $H(\varphi_n)$ holds with $\alpha_{\varphi_n} = L_\tau \|\gamma\|^2$ and, therefore, condition $H(s_n)$ is satisfied, too.

On the other hand, it was proved in [22] that, under conditions (6.5), (6.14), (6.15) the set sequence $\{K_n\}$ defined by (6.18) converge to the set (6.6) in the sense of Mosco, in the space V. Therefore, condition (4.13) is satisfied.

Next, condition (4.42) follows from the compactness of the trace operator combined with inequality

$$|\sqrt{a^2+\varepsilon^2}-a| \le \varepsilon \qquad \forall\, a, \varepsilon \in \mathbb{R}_+,$$

while condition (4.44) is a consequence of the positivity of functions φ_n and j, guaranteed by assumptions (6.2).

Finally, to check condition (4.45) we assume that (6.22) holds. We remark that in our case $m_n = m_A$ and $L_n = L_A + \|\omega_n\|_{L^\infty(\Omega)}$ for all $n \in \mathbb{N}$. Therefore, we deduce that condition (4.45)(a) holds with $m = m_A = m_{\mathcal{F}}$ and, for n large enough, with any $L > L_{\mathcal{F}}$. In addition, (6.3) guarantees that (4.45)(b) holds with $\alpha = (L_\nu + L_\tau)\|\gamma\|^2$. Using the elementary inequality

$$x - \sqrt{x^2 - x} > \frac{1}{2} \quad \forall\, x \ge 1$$

with $x = L^2/m^2$, we deduce that

$$\frac{L^2}{m^2}\left(1 - \sqrt{1 - \frac{m^2}{L^2}}\right) > \frac{1}{2}$$

and, therefore,

$$\frac{1}{2}\, m < \frac{L^2}{m}\left(1 - \sqrt{1 - \frac{m^2}{L^2}}\right).$$

We conclude from here that if (6.22) holds then $\alpha \le m/2$ which implies that (4.45)(c) holds, too.

It follows from above that we are in a position to use Theorems 11 and 15 which concludes the proof. □

In addition to the mathematical interest in the convergence result in Theorem 22(b), it is important from the mechanical point of view, since it shows that the weak solution of the elastic frictional contact problem depends continuously on the perturbation of the constitutive operator, the thickness of the deformable layer and the friction law.

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Mircea Sofonea
Laboratoire de Mathématiques et Physique
University of Perpignan Via Domitia
52 Avenue Paul Alduy
66860 Perpignan
France
e-mail: sofonea@univ-perp.fr

Weimin Han
Department of Mathematics
University of Iowa
Iowa City IA52242
USA

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