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Research paper

Convergence of a generalized penalty and regularization method for quasi–variational–hemivariational inequalities

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ABSTRACT

In the paper an elliptic quasi–variational–hemivariational inequality with constraints in a Banach space is studied. First, we apply the Minty technique, the KKM principle and the theory of nonsmooth analysis to establish the solvability of the inequality problem. Then, we employ a generalized penalty and regularization method for the inequality and introduce a family of penalized and regularized problems with no constraints and with Gâteaux differentiable potentials. Through a limit procedure, we prove that the Kuratowski upper limit with respect to the weak topology of the solution sets to penalized and regularized problems, is a nonempty subset of the solution set to the original inequality problem. Next, if a set-valued operator in the inequality has $(S)_+$-property, then the Kuratowski upper limits with respect to the weak and strong topologies for the solution sets coincide. Finally, we illustrate our results by examining a nonlinear elliptic inclusion with the subgradient term of a locally Lipschitz function, mixed boundary conditions and an obstacle unilateral constraint which appears in a semipermeability problem.

1. Introduction

The theory of hemivariational inequalities has been introduced in early 1980s with the pioneering works of Panagiotopoulos, see [33–35], who initially used it to analyse several nonsmooth problems in mechanics involving set-valued nonmonotone operators. The theory is based on the notion of the generalized subgradient of Clarke [4] defined for a class of locally Lipschitz, non-differentiable and nonconvex energy functionals. The hemivariational inequalities has been extensively

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studied in many aspects and emerged as one of the most promising branches of pure and applied mathematics. Applications of hemivariational inequalities include models that describe various complex phenomena arising in physics, solid and fluid mechanics, economics, engineering sciences, optimization, control, etc (see [30–32,44,47,48]).

Nowadays, variational–hemivariational inequalities represent a natural generalization of both variational and hemivariational inequalities. They allow to enlarge significantly numerous applications involving the convex and nonconvex potentials. Recent advances in variational–hemivariational inequalities concern existence results [10,11,14,16,18,19,22,23,26–28,37,38,46], convergence, continuous dependence, and stability, see, e.g., [1,2,13,17,41,42,45], and numerical analysis, see, e.g., [9,12,13,29,43] and the references therein.

In this paper we explore a class of variational–hemivariational inequalities which serves as a variational formulation of a sempermeability model. The latter arises in several situations in heat conduction, fluid flow through porous media, hydraulic, and electrostatics, where the solution represents the temperature, the pressure and the electric potential [7]. The motivation to study such semipermeability problem comes from a prototype model in [7, Chapter I] (where the maximal monotone semipermeability relations were considered), and [33, Chapter 5.5.3] and [24,34,40] (where nonmonotone relations were studied).

In Section 4 we concentrate on an interesting real-world application. We study a semipermeability model for the stationary heat equation which involves two relations of subdifferential type. The first one is the multivalued nonmonotone law $-f \in \partial j(u)$ in the domain between the heat source f and the temperature u, where ∂j denotes the generalized subgradient of a locally Lipschitz function j. The second relation is the monotone law on the boundary of the domain between the heat flux and the temperature $-\frac{\partial u}{\partial \nu} \in \partial_c \Psi(u)$ with the convex subdifferential $\partial_c \Psi$ of a convex potential Ψ. The semipermeability model under consideration leads to the following formulation. Let V and X be reflexive Banach spaces with their dual spaces V^* and X^*, respectively, $\langle \cdot, \cdot \rangle$ stand for the duality brackets between V^* and V. Let K be a nonempty, closed, and convex subset of V. We shall study the elliptic quasi–variational–hemivariational inequality of the following form.

Problem 1. Find $u \in K$ such that $u^* \in Fu$ and

$$\langle u^*, v - u \rangle + J^0(\gamma u; \gamma v - \gamma u) + \varphi(v) - \varphi(u) \geq \langle f, v - u \rangle \quad \text{for all} \quad v \in K.$$

Here, the operators $F : V \to 2^{V^*}$ and $\gamma : V \to X$, and the functions $J : X \to \mathbb{R}$ and $\varphi : V \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, and the element $f \in V^*$ will be specified in Section 3. In a particular case, when F is a single-valued mapping, Problem 1 reduces to the one, which has been considered only recently in [42]. The novelty of the semipermeability model relies also on a unilateral obstacle effect in the domain which is modelled by the set K of constraints.

The main goal of this paper is threefold. The first contribution is to establish the existence of solution, as well as the weak compactness, convexity and boundedness of the solution set to Problem 1. In the proofs we apply the KKM principle, use the Minty approach, and the theory of nonsmooth analysis. The second aim is to investigate a generalized penalty and regularization method. To this end, we introduce a family of penalized and regularized problems without constraints and with the Gâteaux differentiable potentials. We shall establish the following assertions:

(i) the upper limit in the sense of Kuratowski with respect to the weak topology of the sequence of solution sets to the penalized and regularized problems, denoted by $w - \limsup_{n\to\infty} \mathcal{S}_n$, is nonempty, and it satisfies

$$w - \limsup_{n\to\infty} \mathcal{S}_n \subset \mathcal{S},$$

where the sets $\mathcal{S}_n$ and $\mathcal{S}$ are the sets of solutions to the penalized and regularized problem, Problem 8, and Problem 1, respectively.

(ii) further, if the operator F has (S_+)-property, then the identity holds

$$w - \limsup_{n\to\infty} \mathcal{S}_n = s - \limsup_{n\to\infty} \mathcal{S}_n \subset \mathcal{S}.$$

(iii) under the hypothesis that F has $(S)_+$-property, for each $u \in s - \limsup_{n\to\infty} \mathcal{S}_n$ and any sequence $\{\widetilde{u}_n\}$ with $\widetilde{u}_n \in \mathcal{S}_n$ and

$$\|\widetilde{u}_n - u\| \leq \|w - u\| \quad \text{for all} \quad w \in \mathcal{S}_n,$$

there exists a subsequence of $\{\widetilde{u}_n\}$ that converges strongly in V to u.

Our third goal is to illustrate the applicability of the aforementioned theoretical results and study an elliptic inclusion problem with mixed boundary conditions, which comes from the modeling of semipermeability phenomena.

The outline of the paper is as follows. Section 2 recalls the necessary notation and preliminary material. In Section 3, we establish the nonemptiness, closedness, convexity and boundedness of the set of solutions to Problem 1. Then, we employ a generalized penalty and regularization method to Problem 1 and deliver a convergence theorem, Theorem 9. Finally, Section 4 explores an elliptic problem involving a set-valued operator and the Clarke subgradient term.

2. Preliminaries

In this section we shall recall a preliminary material which will be used in the next sections, see [3–6,15,25,39] and the references therein.

Throughout the paper, we adopt the notation " $\xrightarrow{w}$ " and "$\to$" to denote the weak and the strong convergence, respectively. The set of all subsets of Y is denoted by 2^Y. Given topological spaces X and Y, we say that a set-valued mapping $F : X \to 2^Y$ is upper semicontinuous (u.s.c., for short) at $x \in X$ if, for every open set $\mathcal{O} \subset Y$ with $F(x) \subset \mathcal{O}$, there exists a neighborhood $N(x)$ of x such that

$$F(N(x)) := \bigcup_{y\in N(x)} F(y) \subset \mathcal{O}.$$

If this holds at every $x \in X$, then F is called upper semicontinuous.

We now recall the definition of the Kuratowski limits for a family of sets.

Definition 2. Let (X, τ) be a Hausdorff topological space and $\{A_n\}_{n\geq 1} \subset 2^X$. We define

$$\tau - \liminf_{n\to\infty} A_n := \left\{ x \in X \mid x = \tau - \lim_{n\to\infty} x_n,\ x_n \in A_n \text{ for all } \ n \geq 1 \right\},$$

and

$$\tau - \limsup_{n\to\infty} A_n := \left\{ x \in X \mid x = \tau - \lim_{k\to\infty} x_{n_k},\ x_{n_k} \in A_{n_k},\ n_1 < n_2 < \ldots < n_k < \ldots \right\}.$$

The sets $\tau - \liminf_{n\to\infty} A_n$ and $\tau - \limsup_{n\to\infty} A_n$ are called the τ–Kuratowski lower and upper limits of the sequence A_n, respectively. Further, if

$$A = \tau - \liminf_{n\to\infty} A_n = \tau - \limsup_{n\to\infty} A_n,$$

then the set A is called the τ–Kuratowski limit of the sequence A_n.

Let V be a reflexive Banach space with its dual space V^*, and $\langle \cdot, \cdot \rangle$ be the duality brackets between V^* and V. Let $A : V \to V^*$ be a single-valued operator. It is called bounded if it maps bounded sets of V into bounded subsets of V^*. A is called monotone if $\langle Av_1 - Av_2, v_1 - v_2 \rangle \geq 0$ for all $v_1, v_2 \in V$. An operator A is called hemicontinuous, if the functional $t \mapsto \langle A(u + tv), w \rangle$ is continuous on $[0, 1]$ for all $u, v, w \in V$. If for all $w \in V$ the functional $u \mapsto \langle Au, w \rangle$ is continuous, i.e., A is continuous as a map from V to V^* endowed with the weak topology, then A is called demicontinuous. It is known, see [6, Exercise I.9, Section 1.9] that for monotone operators, the notions of demicontinuity and hemicontinuity coincide. We say that a set-valued mapping $F : V \to 2^{V^*}$ is (S_+) (or F satisfies (S_+)-property, or has (S_+)-property), if for any sequences $\{u_n\} \subset V$ and $\{u_n^*\} \subset V^*$ with $u_n^* \in F(u_n)$ and $u_n \xrightarrow{w} u$ for some $u \in V$ such that $\limsup_{n\to\infty} \langle u_n^*, u_n - u \rangle \leq 0$, we have $u_n \to u$ as $n \to \infty$.

Finally, we recall the notion of the generalized gradient, see [4], and some its properties which are needed in what follows. A function $J : X \to \mathbb{R}$ defined on a Banach space $(X, \|\cdot\|_X)$ is locally Lipschitz continuous at $u \in X$, if there exist a neighborhood $N(u)$ of u in X and a constant $L_u > 0$ such that

$$|J(w) - J(v)| \leq L_u \|w - v\|_X \quad \text{for all} \quad w, v \in N(u).$$

We denote by $J^0(u; v)$ the generalized directional derivative of J at $u \in X$ in the direction $v \in X$ defined by

$$J^0(u; v) = \limsup_{\lambda\to 0^+,\, w\to u} \frac{J(w + \lambda v) - J(w)}{\lambda}.$$

The generalized (sub)gradient of $J : X \to \mathbb{R}$ at $u \in X$ is given by

$$\partial J(u) = \left\{ \xi \in X^* \mid J^0(u; v) \geq \langle \xi, v \rangle \quad \text{for all} \quad v \in X \right\}.$$

The generalized directional derivative and the generalized gradient share useful properties collected below, see [25, Proposition 3.23].

Proposition 3. *Assume that the function $J : X \to \mathbb{R}$ is locally Lipschitz. Then*

(i) *for all $x \in X$, the function $X \ni v \mapsto J^0(x; v) \in \mathbb{R}$ is positively homogeneous and subadditive, i.e., $J^0(x; \lambda v) = \lambda J^0(x; v)$ for all $\lambda \geq 0$, $v \in X$, and $J^0(x; v_1 + v_2) \leq J^0(x; v_1) + J^0(x; v_2)$ for all $v_1, v_2 \in X$.*
(ii) *for all $v \in X$, it holds $J^0(x; v) = \max \{ \langle \xi, v \rangle \mid \xi \in \partial J(x) \}$.*
(iii) *the function $X \times X \ni (u, v) \mapsto J^0(u; v) \in \mathbb{R}$ is upper semicontinuous.*
(iv) *the graph of the map $\partial J : X \to 2^{X^*}$ is closed in $X \times (w^* - X^*)$ topology, i.e., if $\{x_n\} \subset X$ and $\{\xi_n\} \subset X^*$ are sequences such that*

$$\xi_n \in \partial J(x_n), \quad x_n \to x \text{ in } X \quad \text{and} \quad \xi_n \to \xi \text{ weakly}^* \text{ in } X^*,$$

then $\xi \in \partial J(x)$, where $w^* - X^*$ denotes the space X^* equipped with weak* topology.

3. General existence and convergence results

In this section we study existence of solution to the quasi–variational–hemivariational inequality stated in Problem 1. We use the KKM principle, the Minty approach, and results from nonconvex and nonsmooth analysis. Then, we introduce a family of generalized penalty and regularization problems corresponding to Problem 1. Finally, we establish a convergence theorem on the Kuratowski upper limit for the generalized penalty and regularization problem.

Let $(V, \|\cdot\|)$ and $(X, \|\cdot\|_X)$ be reflexive Banach spaces with duals $(V^*, \|\cdot\|_{V^*})$ and $(X^*, \|\cdot\|_{X^*})$, respectively. To explore existence of solution to Problem 1, we need the following hypotheses.

$\underline{H(\varphi)}$: $\varphi : V \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function.

$\underline{H(\gamma)}$: $\gamma : V \to X$ is a bounded and linear operator.

$\underline{H(J)}$: $J : X \to \mathbb{R}$ is a locally Lipschitz function.

$\underline{H(h)}$: $h : V \to \mathbb{R}$ is a function such that $h(0_V) \le 0$ and

(i) $\limsup_{t\to 0^+} \frac{h(tv)}{t} \ge 0$ for all $v \in V$.

(ii) for all $\{v_n\} \subset V$ with $v_n \xrightarrow{w} v$ in V, it holds $h(v) \le \limsup_{n\to\infty} h(v_n)$.

$\underline{H(f)}$: $f \in V^*$.

$\underline{H(F)}$: $F : V \to 2^{V^*}$ satisfies the following conditions:

(i) F is u.s.c. and has compact values.

(ii) the set-valued map $u \mapsto Fu + \gamma^* \partial J(\gamma u) \subset V^*$ is relaxed h-monotone, namely, the inequality

$$\langle u^* - v^*, u - v\rangle + \langle \xi_u - \xi_v, \gamma(u - v)\rangle_{X^*\times X} \ge h(u - v)$$

holds for all $u^* \in Fu$, $v^* \in Fv$, $\xi_u \in \partial J(\gamma u)$, $\xi_v \in \partial J(\gamma v)$ and all $u, v \in V$.

(iii) there exists $v_0 \in K \cap dom(\varphi)$ such that

$$\liminf_{u\in V,\ \|u\|\to+\infty} \frac{\inf_{u^*\in F(u)} \langle u^*, u - v_0\rangle + \inf_{\xi_u\in\partial J(\gamma u)} \langle \xi_u, \gamma(u - v_0)\rangle_{X^*\times X}}{\|u\|} = +\infty. \tag{3.1}$$

$\underline{H(K)}$: K is a nonempty, closed, convex subset of V.

$\underline{H(T)}$: $T : V \to V^*$ is a bounded, hemicontinuous and monotone operator.

We now consider the following inequality problem.

Problem 4. Find $u \in K$ such that $u^* \in Fu$ and

$$\langle u^* + Tu, v - u\rangle + J^0(\gamma u; \gamma v - \gamma u) + \varphi(v) - \varphi(u) \ge \langle f, v - u\rangle \quad \text{for all} \quad v \in K.$$

The first main theorem of this section is the following result on existence and uniqueness of solution, and convexity and weak compactness of the solution set to Problem 4.

Theorem 5. *Assume that $H(\varphi)$, $H(\gamma)$, $H(J)$, $H(h)$, $H(f)$, $H(F)$, $H(K)$, and $H(T)$ are fulfilled. Then the following statements hold:*

(i) *$u \in K$ is a solution to Problem 4 if and only if, u solves the following problem: find $u \in K$ such that*

$$\langle v^* - f + Tv, v - u\rangle + \langle \xi_v, \gamma(v - u)\rangle_{X^*\times X} + \varphi(v) - \varphi(u) \ge h(v - u) \tag{3.2}$$

for all $v^* \in Fv$, $\xi_v \in \partial J(\gamma v)$ and $v \in K$.

(ii) *the set of solutions to Problem 4, denoted by $\mathcal{S}_T$, is nonempty and weakly compact in V.*

(iii) *if $h : V \to \mathbb{R}$ is convex, then $\mathcal{S}_T$ is convex as well.*

(iv) *if $h(v) > 0$ for all $v \in V \setminus \{0_V\}$, then Problem 4 has a unique solution.*

Proof. (i) We can argue as in the proof of [21, Lemma 3.3] and apply the Minty approach to conclude the desired assertion.

(ii) For the existence part, we first suppose that the constraint set K is additionally a bounded subset of V. Consider the set-valued mapping $\Xi : K \to 2^K$ defined by

$$\Xi(w) := \Big\{v \in K \mid \inf_{w^*\in F(w)} \langle w^*, w - v\rangle + \langle Tw - f, w - v\rangle + \inf_{\xi_w\in\partial J(\gamma w)} \langle \xi_w, \gamma(w - v)\rangle_{X^*\times X}$$

$$+\varphi(w) - \varphi(v) \ge h(w - v)\Big\} \quad \text{for} \quad w \in K.$$

Keeping in mind that $h(0_V) \le 0$ and $w \in \Xi(w)$ for every $w \in K$, it follows that the mapping Ξ is well-defined. Besides, we assert that Ξ has weakly closed values. Indeed, let $w \in K$ and $\{v_n\} \subset \Xi(w)$ be such that $v_n \xrightarrow{w} v$ in V as $n \to \infty$. Hence

$$\langle w^*, w - v_n\rangle + \langle Tw - f, w - v_n\rangle + \langle \xi_w, \gamma(w - v_n)\rangle_{X^*\times X} + \varphi(w) - \varphi(v_n) \ge h(w - v_n)$$

for all $w^* \in F(w)$ and $\xi_w \in \partial J(\gamma w)$. We pass to the upper limit in the last inequality and apply condition $H(h)$(ii) to get

$$\langle w^*, w - v\rangle + \langle Tw - f, w - v\rangle + \langle \xi_w, \gamma(w - v)\rangle_{X^*\times X} + \varphi(w) - \varphi(v)$$
$$\geq \lim_{n\to\infty} \left[\langle w^*, w - v_n\rangle + \langle Tw - f, w - v_n\rangle + \langle \xi_w, \gamma(w - v_n)\rangle_{X^*\times X}\right] + \varphi(w)$$
$$- \liminf_{n\to\infty} \varphi(v_n) \geq \limsup_{n\to\infty} h(w - v_n) \geq h(w - v),$$

where we have used the hypothesis that φ is l.s.c. Hence, we obtain

$$\inf_{w^*\in F(w)} \langle w^*, w - v\rangle + \langle Tw - f, w - v\rangle + \inf_{\xi_w\in\partial J(\gamma w)} \langle \xi_w, \gamma(w - v)\rangle_{X^*\times X}$$
$$+ \varphi(w) - \varphi(v) \geq h(w - v),$$

which means that the set $\Xi(w)$ is a weakly closed subset of V.

Further, if Ξ is a KKM mapping, then we are in a position to apply the KKM principle (see, e.g., [20, Lemma 2.6]) to conclude

$$\bigcap_{w\in K} \Xi(w) \neq \emptyset.$$

So, there exists an element $u \in K$ such that

$$\langle w^*, w - u\rangle + \langle Tw - f, w - u\rangle + \langle \xi_w, \gamma(w - u)\rangle_{X^*\times X} + \varphi(w) - \varphi(u) \geq h(w - u)$$

for all $w^* \in F(w)$, $\xi_w \in \partial J(\gamma w)$ and $w \in K$. By virtue of assertion (i), one has $u \in \mathcal{S}_T$.

On the other hand, consider the situation that Ξ is not a KKM mapping. This means that there are elements $\{w_1, w_2, \ldots, w_M\}$ and $w_0 = \sum_{k=1}^{M} t_k w_k \in K$ (since K is convex) such that

$$t_k \in [0, 1], \quad \sum_{k=1}^{M} t_k = 1 \quad \text{and} \quad w_0 \notin \bigcup_{k=1}^{M} \Xi(w_k).$$

Hence, for any $k = 1, 2, \ldots, M$, one has

$$\inf_{w_k^*\in F(w_k)} \langle w_k^*, w_k - w_0\rangle + \langle Tw_k - f, w_k - w_0\rangle + \varphi(w_k) - \varphi(w_0)$$
$$+ \inf_{\xi_k\in\partial J(\gamma w_k)} \langle \xi_k, \gamma(w_k - w_0)\rangle_{X^*\times X} < h(w_k - w_0). \tag{3.3}$$

Next, we claim that there exist a neighborhood U of w_0 such that for every $v \in U \cap K$, it holds

$$\inf_{w_k^*\in F(w_k)} \langle w_k^*, w_k - v\rangle + \langle Tw_k - f, w_k - v\rangle + \inf_{\xi_k\in\partial J(\gamma w_k)} \langle \xi_k, \gamma(w_k - v)\rangle_{X^*\times X}$$
$$+\varphi(w_k) - \varphi(v) < h(w_k - v) \tag{3.4}$$

for each $k = 1, 2, \ldots, M$. Suppose that this claim is not true. Thus, we are able to find a sequence $\{v_n\}$ such that

$$v_n \to w_0 \quad \text{in} \quad V, \quad \text{as } n \to \infty,$$

and

$$\inf_{w_{k_n}^*\in F(w_{k_n})} \langle w_{k_n}^*, w_{k_n} - v_n\rangle + \langle Tw_{k_n} - f, w_{k_n} - v_n\rangle + \varphi(w_{k_n}) - \varphi(v_n)$$
$$+ \inf_{\xi_{k_n}\in\partial J(\gamma w_{k_n})} \langle \xi_{k_n}, \gamma(w_{k_n} - v_n)\rangle_{X^*\times X} \geq h(w_{k_n} - v_n).$$

Hence, for n large enough, we can find $k_0 \in \{1, 2, \ldots, M\}$ such that

$$\langle w_{k_0}^*, w_{k_0} - v_n\rangle + \langle Tw_{k_0} - f, w_{k_0} - v_n\rangle + \langle \xi_{k_0}, \gamma(w_{k_0} - v_n)\rangle_{X^*\times X}$$
$$+\varphi(w_{k_0}) - \varphi(v_n) \geq h(w_{k_0} - v_n)$$

for all $w_{k_0}^* \in F(w_{k_0})$, $\xi_{k_0} \in \partial J(\gamma w_{k_0})$ and all $n \in \mathbb{N}$. We take the upper limit, as $n \to \infty$, in the inequality above and find

$$\langle w_{k_0}^*, w_{k_0} - w_0\rangle + \langle Tw_{k_0} - f, w_{k_0} - w_0\rangle + \langle \xi_{k_0}, \gamma(w_{k_0} - w_0)\rangle_{X^*\times X}$$
$$+\varphi(w_{k_0}) - \varphi(w_0)$$
$$\geq \lim_{n\to\infty} [\langle w_{k_0}^*, w_{k_0} - v_n\rangle + \langle Tw_{k_0} - f, w_{k_0} - v_n\rangle + \langle \xi_{k_0}, \gamma(w_{k_0} - v_n)\rangle_{X^*\times X}]$$
$$+\varphi(w_{k_0}) - \liminf_{n\to\infty} \varphi(v_n) \geq \limsup_{n\to\infty} h(w_{k_0} - v_n) \geq h(w_{k_0} - w_0)$$

for all $w^*_{k_0} \in F(w_{k_0})$ and $\xi_{k_0} \in \partial J(\gamma w_{k_0})$. The latter implies

$$\inf_{w^*_{k_0} \in F(w_{k_0})} \langle w^*_{k_0}, w_{k_0} - w_0 \rangle + \langle T w_{k_0} - f, w_{k_0} - w_0 \rangle + \varphi(w_{k_0}) - \varphi(w_0)$$

$$+ \inf_{\xi_{k_0} \in \partial J(\gamma w_{k_0})} \langle \xi_{k_0}, \gamma(w_{k_0} - w_0) \rangle_{X^* \times X} \geq h(w_{k_0} - w_0),$$

which contradicts the inequality (3.3). Therefore, the claim (3.4) is valid.

Let U be the neighborhood of w_0 such that (3.4) is satisfied. We shall verify that the element $w_0 \in K$ is a solution to Problem 4. Form the inequality (3.4), we obtain

$$\langle w^*_k, v - w_k \rangle + \langle T w_k - f, v - w_k \rangle + \langle \xi_k, \gamma(v - w_k) \rangle_{X^* \times X}$$

$$+ \varphi(v) - \varphi(w_k) > -h(w_k - v)$$

for some $w^*_k \in F(w_k)$ and $\xi_k \in \partial J(\gamma w_k)$. Next, due to the monotonicity of the operators $F(\cdot) + \gamma^* \partial J(\gamma \cdot)$ and T, we get

$$\langle v^*, v - w_k \rangle + \langle T v - f, v - w_k \rangle + \langle \xi_v, \gamma(v - w_k) \rangle_{X^* \times X} + \varphi(v) - \varphi(w_k) \geq 0$$

for all $v^* \in F(v)$, $\xi_v \in \partial J(\gamma v)$ and $k = 1, 2, \ldots, M$. Multiplying the above inequality by t_k and summing up the resulting inequalities from $k = 1$ to $k = M$, we have

$$\langle v^*, v - w_0 \rangle + \langle T v - f, v - w_0 \rangle + J^0(\gamma v; \gamma(v - w_0)) + \varphi(v) - \varphi(w_0)$$

$$\geq \langle v^*, v - w_0 \rangle + \langle T v - f, v - w_0 \rangle + \langle \xi_v, \gamma(v - w_0) \rangle_{X^* \times X} + \varphi(v) - \varphi(w_0) \geq 0 \tag{3.5}$$

for all $v^* \in F(v)$, $\xi_v \in \partial J(\gamma v)$ and all $v \in U \cap K$. Let $w \in K$ be arbitrary, and $t \in (0, 1)$ be small enough such that $v_t := tw + (1 - t) w_0 \in U \cap K$. So, letting $v = v_t$ in the inequality (3.5), it gives

$$\langle v^*_t, w - w_0 \rangle + \langle T v_t - f, w - w_0 \rangle + J^0(\gamma v_t; \gamma(w - w_0)) + \varphi(w) - \varphi(w_0) \geq 0 \tag{3.6}$$

for all $v^*_t \in F(v_t)$, where we have used the positive homogeneity of the map $v \mapsto J(u; v)$ and the convexity of φ. Recall that F is u.s.c. and has compact values, so for each sequence $\{v^*_t\}$ fixed, without any loss of generality, we may assume that

$$v^*_t \to w^*_0(w) \quad \text{in } V^*, \quad \text{as} \quad t \to 0^+$$

for some $w^*_0(w) \in F(w_0)$. We use the demicontinuity of T, the upper semicontinuity of the map $u \mapsto J^0(u; v)$, and pass to the upper limit, as $t \to 0^+$, in (3.6) to deduce

$$\langle w^*_0(w), w - w_0 \rangle + \langle T w_0 - f, w - w_0 \rangle + J^0(\gamma w_0; \gamma(w - w_0)) + \varphi(w) - \varphi(w_0) \geq 0.$$

In consequence, for each $w \in K$, there exists $w^*_0(w) \in F(w_0)$ such that

$$\langle w^*_0(w), w - w_0 \rangle + \langle T w_0 - f, w - w_0 \rangle + J^0(\gamma w_0; \gamma(w - w_0)) + \varphi(w) - \varphi(w_0) \geq 0.$$

Moreover, applying the same arguments as in the proof of [8, Proposition 3.3], we are able to find an element $w^*_0 \in F(w_0)$ such that

$$\langle w^*_0, w - w_0 \rangle + \langle T w_0 - f, w - w_0 \rangle + J^0(\gamma w_0; \gamma(w - w_0)) + \varphi(w) - \varphi(w_0) \geq 0$$

for all $w \in K$. Therefore $w_0 \in \mathcal{S}_T$.

Further, we consider the case when K is an unbounded set in V. For every $n \in \mathbb{N}$, let us introduce the set K_n defined by

$$K_n := \{ w \in K \mid \|w\| \leq n \},$$

and consider the following intermediate problem: find $w_n \in K_n$ and $w^*_n \in F(w_n)$ such that

$$\langle w^*_n, v - w_n \rangle + \langle T w_n - f, v - w_n \rangle + J^0(\gamma w_n; \gamma(v - w_n)) + \varphi(v) - \varphi(w_n) \geq 0 \tag{3.7}$$

for all $v \in K_n$. We now claim that there is a constant $N_0 \geq 1$ large enough such that for a solution to problem (3.7) with $n = N_0$, it holds

$$\|w_{N_0}\| < N_0. \tag{3.8}$$

Suppose, contrary to our claim, that for each $n \in \mathbb{N}$ and every solution w_n to problem (3.7), we have

$$\|w_n\| = n.$$

Let n be large enough such that $\|v_0\| < n$. Setting $v = v_0$ into (3.7) and using the monotonicity of operator T, we deduce

$$\langle w^*_n, v_0 - w_n \rangle + \langle T v_0 - f, v_0 - w_n \rangle + J^0(\gamma w_n; \gamma(v_0 - w_n)) + \varphi(v_0) - \varphi(w_n)$$

$$\geq \langle w^*_n, v_0 - w_n \rangle + \langle T w_n - f, v_0 - w_n \rangle + J^0(\gamma w_n; \gamma(v_0 - w_n)) + \varphi(v_0) - \varphi(w_n) \geq 0,$$

and hence

$$\|Tv_0 - f\|_{V^*}(\|w_n\| + \|v_0\|) + \varphi(v_0) \geq \langle Tv_0 - f, v_0 - w_n\rangle + \varphi(v_0)$$
$$\geq \langle w_n^*, w_n - v_0\rangle + \langle \xi_n, \gamma(w_n - v_0)\rangle_{X^*\times X} + \varphi(w_n)$$

for some $\xi_n \in \partial J(\gamma w_n)$ satisfying $\langle \xi_n, \gamma(v_0 - w_n)\rangle_{X^*\times X} = J^0(\gamma w_n; \gamma(v_0 - w_n))$. Since φ is a proper, convex and l.s.c. function, so, it follows from [5, Proposition 5.2.25] that there exist $\alpha_\varphi, \beta_\varphi > 0$ such that

$$\varphi(v) \geq -\alpha_\varphi\|v\| - \beta_\varphi \quad \text{for all} \quad v \in V. \tag{3.9}$$

We combine the last two inequalities to obtain

$$\|Tv_0 - f\|_{V^*}\left(1 + \frac{\|v_0\|}{\|w_n\|}\right) + \frac{\varphi(v_0) + \beta_\varphi}{\|w_n\|} + \alpha_\varphi$$
$$\geq \frac{\langle w_n^*, w_n - v_0\rangle + \inf_{\xi_n \in \partial J(\gamma w_n)}\langle \xi_n, \gamma(w_n - v_0)\rangle_{X^*\times X}}{\|w_n\|}.$$

Because of $\|w_n\| = n$, we pass to the limit as $n \to \infty$ in the inequality above and use the condition $H(F)$(iii) to get

$$\|Tv_0 - f\|_{V^*} + \alpha_\varphi \geq +\infty,$$

which is a contradiction. So, we deduce that there exists $N_0 \in \mathbb{N}$ such that (3.8) holds.

We now claim that w_{N_0} is a solution to Problem 4. For any $v \in K$, we take $t \in (0, 1)$ small enough such that $v_t = tv + (1-t)w_{N_0} \in K_{N_0}$. We consider (3.7) with $n = N_0$ and choose $v = v_t$ to get

$$\langle w_{N_0}^*, v - w_{N_0}\rangle + \langle Tw_{N_0} - f, v - w_{N_0}\rangle + J^0(\gamma w_{N_0}; \gamma(v - w_{N_0})) + \varphi(v) - \varphi(w_{N_0}) \geq 0$$

with $w_{N_0}^* \in F(w_{N_0})$. Since $v \in K$ is arbitrary, so, we conclude that w_{N_0} solves Problem 4.

Next, we shall prove the weak closedness of the set $\mathcal{S}_T$ of solutions to Problem 4. Let $\{w_n\} \subset \mathcal{S}_T$ be such that

$$w_n \xrightarrow{w} w \text{ in } V \text{ as } n \to \infty$$

for some $w \in K$ (since K is closed and convex). The assertion (i) guarantees that

$$\langle v^* - f + Tv, v - w_n\rangle + \langle \xi_v, \gamma(v - w_n)\rangle_{X^*\times X} + \varphi(v) - \varphi(w_n) \geq h(v - w_n)$$

for all $v^* \in F(v)$, $\xi_v \in \partial J(\gamma v)$ and all $v \in K$. We pass to the upper limit, as $n \to \infty$, in the inequality above and employ the weak lower semicontinuity of φ. We have

$$\langle v^* - f + Tv, v - w\rangle + \langle \xi_v, \gamma(v - w)\rangle_{X^*\times X} + \varphi(v) - \varphi(w) \geq h(v - w)$$

for all $v^* \in F(v)$, $\xi_v \in \partial J(\gamma v)$ and all $v \in K$. We use assertion (i) again and conclude that $w \in \mathcal{S}_T$. Hence, the set $\mathcal{S}_T$ is weakly closed.

For the boundedness of the solution set $\mathcal{S}_T$, we proceed by contradiction. Assume that there exists a sequence $\{w_n\}$ in $\mathcal{S}_T$ such that

$$\|w_n\| \to \infty \text{ as } n \to \infty.$$

Arguing as before, by a simple calculation, one finds

$$\|Tv_0 - f\|_{V^*}\left(1 + \frac{\|v_0\|}{\|w_n\|}\right) + \frac{\varphi(v_0) + \beta_\varphi}{\|w_n\|} + \alpha_\varphi$$
$$\geq \frac{\langle w_n^*, w_n - v_0\rangle + \inf_{\xi_n \in \partial J(\gamma w_n)}\langle \xi_n, \gamma(w_n - v_0)\rangle_{X^*\times X}}{\|w_n\|}.$$

Passing to the limit in the inequality above, we get a contradiction. Consequently, $\mathcal{S}_T$ is a bounded set. In conclusion, by the above analysis and the reflexivity of V, we infer that $\mathcal{S}_T$ is nonempty and weakly compact.

(iii) Let $u, w \in \mathcal{S}_T$, $t \in (0, 1)$, and $u_t = tu + (1-t)w$. From the assertion (i), one has

$$\langle v^* - f + Tv, v - u_t\rangle + \langle \xi_v, \gamma(v - u_t)\rangle_{X^*\times X} + \varphi(v) - \varphi(u_t)$$
$$\geq t\Big[\langle v^* - f + Tv, v - u\rangle + \langle \xi_v, \gamma(v - u)\rangle_{X^*\times X} + \varphi(v) - \varphi(u)\Big]$$
$$+(1-t)\Big[\langle v^* - f + Tv, v - w\rangle + \langle \xi_v, \gamma(v - w)\rangle_{X^*\times X} + \varphi(v) - \varphi(w)\Big]$$
$$\geq th(v - u) + (1-t)h(v - w) \geq h(v - u_t)$$

for all $v^* \in F(v)$, $\xi_v \in \partial J(\gamma v)$ and $v \in K$. So, by virtue of the assertion (i) again, we conclude that $u_t \in \mathcal{S}_T$. Thus, $\mathcal{S}_T$ is a convex set.

(iv) Let $w_1, w_2 \in \mathcal{S}_T$ be arbitrary. So, for $i = 1, 2$, there exists $w_i^* \in F(w_i)$ such that

$$\langle w_i^* - f + Tw_i, v - w_i\rangle + J^0(\gamma w_i; \gamma(v - w_i)) + \varphi(v) - \varphi(w_i) \geq 0$$

for all $v \in K$. Inserting $v = w_2$ and $v = w_1$, respectively, into the above inequality for $i = 1$ and $i = 2$, and summing up the resulting inequalities, we get

$$\langle w_1^* - w_2^* + \gamma^*\xi_1 - \gamma^*\xi_2 + Tw_1 - Tw_2, w_1 - w_2\rangle \le 0,$$

where $\xi_1 \in \partial J(\gamma w_1)$ and $\xi_2 \in \partial J(\gamma w_2)$ are such that

$$\langle \xi_1, \gamma(w_2 - w_1)\rangle_{X^*\times X} = J^0(\gamma w_1; \gamma(w_2 - w_1)),$$

$$\langle \xi_2, \gamma(w_1 - w_2)\rangle_{X^*\times X} = J^0(\gamma w_2; \gamma(w_1 - w_2)).$$

Hence and from the monotonicity of operators $T(\cdot)$ and $F(\cdot) + \gamma^* J(\gamma\cdot)$, we have

$$h(w_1 - w_2) \le \langle w_1^* - w_2^* + \gamma^*\xi_1 - \gamma^*\xi_2 + Tw_1 - Tw_2, w_1 - w_2\rangle \le 0.$$

Now the assumption $h(v) > 0$ for all $v \in V \setminus \{0_V\}$ yields $w_1 = w_2$. Therefore, Problem 4 admits a unique solution. This completes the proof of the theorem. □

If $T \equiv 0$, then we have the following consequence of Theorem 5.

Corollary 6. *Assume that $H(\varphi)$, $H(\gamma)$, $H(J)$, $H(h)$, $H(f)$, $H(F)$, and $H(K)$ are fulfilled. Then the following statements hold:*

(i) *$u \in K$ is a solution to Problem 1 if and only if, u solves the following problem: find $u \in K$ such that*

$$\langle v^* - f, v - u\rangle + \langle \xi_v, \gamma(v - u)\rangle_{X^*\times X} + \varphi(v) - \varphi(u) \ge h(v - u) \tag{3.10}$$

for all $v^* \in Fv$, $\xi_v \in \partial J(\gamma v)$ and $v \in K$.

(ii) *the set of solutions to Problem 1, denoted by $\mathcal{S}$, is nonempty and weakly compact in V.*

(iii) *if $h : V \to \mathbb{R}$ is convex, then $\mathcal{S}$ is convex too.*

(iv) *if $h(v) > 0$ for all $v \in V \setminus \{0_V\}$, then Problem 1 has a unique solution.*

Remark 7. Theorem 5(i) reveals the fact that Problem 4 is equivalent to problem (3.2). Further, arguing as in the proof of Theorem 5(i), we also can show that $u \in \mathcal{S}_T$, if and only if it solves the following Minty inequality: find $u \in K$ such that

$$\langle v^* - f + Tv, v - u\rangle + J^0(\gamma v; \gamma(v - u)) + \varphi(v) - \varphi(u) \ge h(v - u)$$

for all $v^* \in Fv$ and $v \in K$. In particular, when F is a single-valued mapping and $T \equiv 0$, then Theorem 5 reduces to a recent result of [42, Theorem 8].

Next, we draw our attention to the penalized and regularized inequality associated with Problem 4. In the study the generalized penalty and regularization approximation procedure, we need the following hypotheses on the data.

$\underline{H(\rho_n)}$: $\rho_n > 0$ for all $n \in \mathbb{N}$, and $\rho_n \to 0$ as $n \to \infty$.

$\underline{H(P_n)}$: $P_n : V \to V^*$ is a bounded, hemicontinuous and monotone operator for all $n \in \mathbb{N}$ such that

(i) for each $v \in K$, there is a sequence $\{v_n\} \subset V$ with the properties

$$P_n v_n = 0_{V^*} \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad v_n \to v \quad \text{in} \quad V \text{ as } n \to \infty.$$

(ii) there exists an operator $P : V \to V^*$ such that

$(\text{ii})_1$ $Pu = 0_{V^*}$ implies $u \in K$,

$(\text{ii})_2$ for any sequence $\{u_n\}$ with $u_n \xrightarrow{w} u$ in V and $\limsup_{n\to\infty} \langle P_n u_n, u_n - u\rangle \le 0$, we have

$$\liminf_{n\to\infty} \langle P_n u_n, u_n - v\rangle \ge \langle Pu, u - v\rangle \quad \text{for all} \quad v \in V.$$

$\underline{H(\varphi)'}$: $\varphi : V \to \mathbb{R}$ is a convex and l.s.c. function.

$\underline{H(\varphi_\rho)}$: $\varphi_\rho : V \to \mathbb{R}$ is such that

(i) φ_ρ is convex and Gâteaux differentiable, for each $\rho > 0$,

(ii) there exists $h_\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{cases} h_\varphi(\rho) \to 0 \text{ as } \rho \to 0^+, \\ |\varphi_\rho(v) - \varphi(v)| \le h_\varphi(\rho) \quad \text{for all} \quad v \in V, \quad \rho > 0. \end{cases}$$

$\underline{H(F)}(\text{iii})'$: There exists $v_0 \in K$ such that the coercivity condition holds

$$\liminf_{v\in B(v_0,1),\, u\in V,\, \|u\|\to+\infty} \frac{\inf_{u^*\in F(u)}\langle u^*, u - v\rangle + \inf_{\xi_u\in\partial J(\gamma u)}\langle \xi_u, \gamma(u - v)\rangle_{X^*\times X}}{\|u\|} = +\infty.$$

$\underline{H(\gamma)'}$: $\gamma : V \to X$ is a linear, continuous and compact operator.

In what follows, we set $\varphi_n := \varphi_{\rho_n}$. Consider the following penalized and regularized problem associated with Problem 4.

Problem 8. Find $u_n \in V$ such that $u_n^* \in Fu_n$ and

$$\langle u_n^* - f, v - u_n\rangle + \frac{1}{\rho_n}\langle P_n u_n, v - u_n\rangle + J^0(\gamma u_n; \gamma(v - u_n)) + \varphi_n(v) - \varphi_n(u_n) \geq 0$$

for all $v \in V$.

The second main result of this section concerns the convergence of the sets of solutions to the penalized and regularized problems, Problem 8.

Theorem 9. *Assume that $H(\varphi)'$, $H(J)$, $H(h)$, $H(f)$, $H(K)$, $H(F)(i) - (ii)$, $H(F)(iii)'$, $H(\gamma)'$, $H(\varphi_\rho)$, $H(\rho_n)$, and $H(P_n)$ hold, and $h : V \to \mathbb{R}$ is bounded. Then the following statements are true:*

(i) *for each $n \in \mathbb{N}$, the set of solutions to Problem 8, denoted by $\mathcal{S}_n$, is nonempty and weakly compact in V.*
(ii) $\emptyset \neq w - \limsup_{n\to\infty} \mathcal{S}_n \subset \mathcal{S}$.
(iii) *if F has (S_+)-property, then the equality holds*

$$w - \limsup_{n\to\infty} \mathcal{S}_n = s - \limsup_{n\to\infty} \mathcal{S}_n.$$

(iv) *if F satisfies $(S)_+$-property, then for each $u \in s - \limsup_{n\to\infty} \mathcal{S}_n$ and any sequence $\{\widetilde{u}_n\}$ with $\widetilde{u}_n \in \mathcal{S}_n$ and*

$$\|\widetilde{u}_n - u\| \leq \|w - u\| \quad \text{for all} \quad w \in \mathcal{S}_n$$

for each $n \in \mathbb{N}$, there exists a subsequence of $\{\widetilde{u}_n\}$ such that it converges strongly in V to u.
(v) *if Problem 1 has a unique solution $u \in K$, then Problem 8 has a unique solution u_n, and the whole sequence of solutions, $\{u_n\}$, of Problem 8 converges weakly to u. Moreover, if, in addition, F satisfies $(S)_+$-property, then the whole sequence $\{u_n\}$, of solutions to Problem 8 converges strongly in V to u.*

Proof. (i) For each $n \in \mathbb{N}$ fixed, hypotheses $H(P_n)$ and $H(\varphi_\rho)$ indicate that the operator $\frac{1}{\rho_n}P_n : V \to V^*$ is bounded, hemicontinuous and monotone, and $\varphi_n : V \to \mathbb{R}$ is a convex and l.s.c function. Then, the assertion follows directly from Theorem 5(ii).

(ii) The proof of this assertion is divided into the following three steps. □

Step 1. *The set $\cup_{n\in\mathbb{N}}\mathcal{S}_n$ is uniformly bounded in V.*
Assume by contractiction that we can find a sequence $\{u_n\}$ with $u_n \in \mathcal{S}_n$ satisfying

$$\|u_n\| \to \infty \quad \text{as} \quad n \to \infty.$$

Then, for each $n \in \mathbb{N}$, one has $u_n^* \in F(u_n)$ and

$$\langle u_n^* - f, v - u_n\rangle + \frac{1}{\rho_n}\langle P_n u_n, v - u_n\rangle + J^0(\gamma u_n; \gamma v - \gamma u_n) + \varphi_n(v) - \varphi_n(u_n) \geq 0 \tag{3.11}$$

for all $v \in V$. Condition $H(\varphi_\rho)$(ii) implies that

$$\varphi_n(v) \leq \varphi(v) + h_\varphi(\rho_n) \quad \text{and} \quad \varphi_n(v) \geq \varphi(v) - h_\varphi(\rho_n) \tag{3.12}$$

for all $v \in V$ and $n \in \mathbb{N}$. We take into account the last two inequalities to get

$$\langle u_n^* - f, v - u_n\rangle + \frac{1}{\rho_n}\langle P_n u_n, v - u_n\rangle + J^0(\gamma u_n; \gamma v - \gamma u_n) + \varphi(v) - \varphi(u_n) + 2h_\varphi(\rho_n) \geq 0 \tag{3.13}$$

for all $v \in V$. Keeping in mind that $v_0 \in K$ (see $H(F)$(iii)'), we use condition $H(P_n)$(i) to find a sequence $\{v_n\} \subset V$ such that

$$P_n v_n = 0_{V^*} \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad v_n \to v_0 \quad \text{in } V \quad \text{as} \quad n \to \infty. \tag{3.14}$$

Letting $v = v_n$ in the inequality (3.13), one has

$$\begin{aligned} &\langle u_n^* - f, u_n - v_n\rangle + \varphi(u_n) - \varphi(v_n) - J^0(\gamma u_n; \gamma v_n - \gamma u_n) \\ &\quad \leq 2h_\varphi(\rho_n) + \frac{1}{\rho_n}\langle P_n u_n - P_n v_n, v_n - u_n\rangle. \end{aligned}$$

Since operator P_n is monotone, we obtain

$$\begin{aligned} &2h_\varphi(\rho_n) \geq \langle u_n^* - f, u_n - v_n\rangle + \varphi(u_n) - \varphi(v_n) - J^0(\gamma u_n; \gamma v_n - \gamma u_n) \\ &= \langle u_n^* - f, u_n - v_n\rangle + \varphi(u_n) - \varphi(v_n) - \langle \widetilde{\xi}_n, \gamma(v_n - u_n)\rangle_{X^*\times X} \\ &\geq \langle u_n^*, u_n - v_n\rangle - \alpha_\varphi\|u_n\| - \beta_\varphi - \varphi(v_n) - \sup_{\xi_n \in \partial J(\gamma u_n)} \langle \xi_n, \gamma(v_n - u_n)\rangle_{X^*\times X} \\ &\quad - \|f\|_{V^*}\big(\|u_n\| + \|v_n\|\big), \end{aligned}$$

here the constants α_φ, $\beta_\varphi > 0$ are defined in (3.9), and $\widetilde{\xi}_n \in \partial J(\gamma u_n)$ satisfies

$$J^0(\gamma u_n; \gamma v_n - \gamma u_n) = \langle \widetilde{\xi}_n, \gamma(v_n - u_n)\rangle_{X^*\times X}.$$

Hence, we deduce

$$\begin{aligned}\frac{2h_\varphi(\rho_n)}{\|u_n\|} &\geq \frac{\langle u_n^* - f, u_n - v_n\rangle + \varphi(u_n) - \varphi(v_n) - J^0(\gamma u_n; \gamma v_n - \gamma u_n)}{\|u_n\|}\\ &\geq \frac{\inf_{w_n^*\in F(u_n)}\langle w_n^*, u_n - v_n\rangle + \inf_{\xi_n\in\partial J(\gamma u_n)}\langle \xi_n, \gamma(u_n - v_n)\rangle_{X^*\times X}}{\|u_n\|}\\ &\quad - \frac{\beta_\varphi + \varphi(v_n) + \|f\|_{V^*}M_0}{\|u_n\|} - \left(\alpha_\varphi + \|f\|_{V^*}\right)\end{aligned}$$

with $M_0 = \|v_0\|_V + 1$. Because of the convergence $\|u_n\| \to \infty$ as $n \to \infty$, we are now in a position to use hypothesis $H(F)(\text{iii})'$ to conclude that

$$0 \geq \frac{\inf_{w_n^*\in F(u_n)}\langle w_n^* - f, u_n - v_n\rangle + \varphi(u_n) - \varphi(v_n) - J^0(\gamma u_n; \gamma v_n - \gamma u_n)}{\|u_n\|} > 0,$$

where we have used the convergences $v_n \to v_0$ in V, $\rho_n \to 0^+$, and $h_\varphi(\rho_n) \to 0$ (see condition $H(\varphi_\rho)(\text{ii})$), as $n \to \infty$. This, obviously, leads to a contradiction. Hence, it follows that the set $\cup_{n\in\mathbb{N}}\mathcal{S}_n$ is uniformly bounded in V.

Step 2. $\emptyset \neq w - \limsup_{n\to\infty}\mathcal{S}_n$.

We take any sequence $\{u_n\} \subset V$ with $u_n \in \mathcal{S}_n$ for all $n \in \mathbb{N}$. By Step 1, this sequence is bounded in V. So, by passing to a subsequence if necessary, we can suppose that

$$u_n \xrightarrow{w} u \text{ in } V \text{ as } n \to \infty \tag{3.15}$$

with some $u \in V$, which implies $w - \limsup_{n\to\infty}\mathcal{S}_n \neq \emptyset$.

Step 3. $w - \limsup_{n\to\infty}\mathcal{S}_n \subset \mathcal{S}$.

Let us take any element $u \in w - \limsup_{n\to\infty}\mathcal{S}_n$. So, there is a sequence $\{u_n\} \subset V$ with $u_n \in \mathcal{S}_n$ for every $n \in \mathbb{N}$ satisfying (3.15). We are going to show that $u \in K$. For every $n \in \mathbb{N}$, it holds $u_n^* \in F(u_n)$ and

$$\frac{1}{\rho_n}\langle P_n u_n, u_n - v\rangle \leq \langle u_n^* - f, v - u_n\rangle + J^0(\gamma u_n; \gamma v - \gamma u_n) + \varphi(v) - \varphi(u_n) + 2h_\varphi(\rho_n)$$

for all $v \in V$, where we have used the inequalities (3.12). Invoking the monotonicity of operator $F(\cdot) + \gamma^*\partial J(\gamma\cdot)$, we obtain

$$\begin{aligned}&\frac{1}{\rho_n}\langle P_n u_n, u_n - v\rangle\\ &\leq \langle u_n^* - v^*, v - u_n\rangle + J^0(\gamma u_n; \gamma v - \gamma u_n) + J^0(\gamma v; \gamma u_n - \gamma v)\\ &\quad + \langle v^* - f, v - u_n\rangle - J^0(\gamma v; \gamma u_n - \gamma v) + \varphi(v) - \varphi(u_n) + 2h_\varphi(\rho_n)\\ &\leq \langle v^* - f, v - u_n\rangle - J^0(\gamma v; \gamma u_n - \gamma v) + \varphi(v) - \varphi(u_n) - h(u_n - v) + 2h_\varphi(\rho_n)\end{aligned}$$

for all $v \in V$ and $v^* \in F(v)$. A simple calculation shows that

$$\begin{aligned}\frac{1}{\rho_n}\langle P_n u_n, u_n - v\rangle &\leq \|v^* - f\|_{V^*}\|v - u_n\| + \|\partial J(\gamma v)\|_{X^*}\|\gamma\|\|v - u_n\|\\ &\quad + \varphi(v) - \varphi(u_n) - h(u_n - v) + 2h_\varphi(\rho_n)\end{aligned}$$

for all $v \in V$ and $v^* \in F(v)$. Note that h is a bounded function and the mapping F has compact values, so, for every $v \in V$, we are able to find a constant $c(v) > 0$, which depends on v but is independent of n, satisfying

$$\langle P_n u_n, u_n - v\rangle \leq \rho_n c(v).$$

Here, we have used the convergence $u_n \xrightarrow{w} u$ in V as $n \to \infty$, and hypotheses $H(\varphi)'$ as well as $H(\varphi_\rho)(\text{ii})$. We apply condition $H(\rho_n)$ and pass to the upper limit as $n \to \infty$ in the last inequality to obtain

$$\limsup_{n\to\infty}\langle P_n u_n, u_n - v\rangle \leq 0$$

for all $v \in V$. By choosing $v = u$, exploiting convergence (3.15) and the condition $H(P_n)(\text{ii})_2$, we have

$$\langle Pu, u - v\rangle \leq \liminf_{n\to\infty}\langle P_n u_n, u_n - v\rangle \leq \limsup_{n\to\infty}\langle P_n u_n, u_n - v\rangle \leq 0.$$

Since $v \in V$ is arbitrary, this shows that $\langle Pu, v\rangle = 0$ for all $v \in V$. Hence, $Pu = 0_{V^*}$. Thus, the hypothesis $H(P_n)(\text{ii})_1$ points out that $u \in K$.

Subsequently, we shall demonstrate that u is a solution to Problem 1. Let $v \in K$. The hypothesis $H(P_n)$(i) allows to find a sequence $\{v_n\} \subset V$ such that

$$P_n v_n = 0_{V^*} \quad \text{and} \quad v_n \to v \quad \text{in } V \text{ as } \quad n \to \infty. \tag{3.16}$$

For each $n \in \mathbb{N}$, it follows from Remark 7 that

$$\langle v_n^*, u_n - v_n \rangle + h(v_n - u_n) \le \langle f, u_n - v_n \rangle + \frac{1}{\rho_n} \langle P_n u_n, v_n - u_n \rangle + \varphi(v_n)$$
$$- \varphi(u_n) + 2h_\varphi(\rho_n) + J^0(\gamma v_n; \gamma v_n - \gamma u_n)$$

for all $v_n^* \in F(v_n)$. By the monotonicity of the operator P_n and the fact $P_n v_n = 0_{V^*}$, we note that

$$\begin{aligned} &\langle v_n^*, u_n - v_n \rangle + h(v_n - u_n) \\ &\le \langle f, u_n - v_n \rangle + \frac{1}{\rho_n} \langle P_n u_n - P_n v_n, v_n - u_n \rangle + \langle \xi_n, \gamma(v_n - u_n) \rangle_{X^* \times X} \\ &\quad + \varphi(v_n) - \varphi(u_n) + 2h_\varphi(\rho_n) \\ &\le \langle f, u_n - v_n \rangle + \langle \xi_n, \gamma(v_n - u_n) \rangle_{X^* \times X} + \varphi(v_n) - \varphi(u_n) + 2h_\varphi(\rho_n) \end{aligned} \tag{3.17}$$

for all $v_n^* \in F(v_n)$, where $\xi_n \in \partial J(\gamma v_n)$ is such that

$$J^0(\gamma v_n; \gamma v_n - \gamma u_n) = \langle \xi_n, \gamma(v_n - u_n) \rangle_{X^* \times X}.$$

Next, keeping in mind that $v_n \to v$ as $n \to \infty$ and the map $x \mapsto \partial J(x)$ is locally bounded, it follows that sequence $\{\xi_n\}$ is bounded in X^* too. We may assume that along a relabeled subsequence, we have

$$\xi_n \xrightarrow{w} \xi \quad \text{in} \quad X^* \text{ as } n \to \infty$$

for some $\xi \in X^*$. So, Proposition 3(iv) points out that $\xi \in \partial J(\gamma v)$. On the other hand, by the continuity and convexity of φ (see $H(\varphi)'$) as well as the compactness of γ, we deduce

$$\langle \xi_n, \gamma(v_n - u_n) \rangle_{X^* \times X} \to \langle \xi, \gamma(v - u) \rangle_{X^* \times X}, \tag{3.18}$$

$$[2mm] \quad \limsup_{n\to\infty} (\varphi(v_n) - \varphi(u_n)) \le \lim_{n\to\infty} \varphi(v_n) - \liminf_{n\to\infty} \varphi(u_n) \le \varphi(v) - \varphi(u). \tag{3.19}$$

Let us fix a sequence $\{v_n^*\}$. Since F is u.s.c. and has compact values, without any loss of generality, we may suppose that

$$v_n^* \to v^* \quad \text{in } V^* \text{ as } \quad n \to \infty \tag{3.20}$$

with some $v^* \in F(v)$. We pass to the upper limit, as $n \to \infty$, in (3.17) and take into account (3.18)–(3.20) to conclude

$$\begin{aligned} &\langle v^*, u - v \rangle + h(v - u) \\ &\le \langle f, u - v \rangle + \langle \xi, \gamma(v - u) \rangle_{X^* \times X} + \varphi(v) - \varphi(u) \\ &\le \langle f, u - v \rangle + J^0(\gamma v; \gamma v - \gamma u) + \varphi(v) - \varphi(u), \end{aligned}$$

where we have used the relation $\xi \in \partial J(\gamma v)$ and the convergences

$$\limsup_{n\to\infty} h(v_n - u_n) \ge h(v - u) \quad \text{and} \quad \lim_{n\to\infty} h_\varphi(\rho_n) = 0.$$

This means that for any $v \in K$, there exists an element $v^*(v) \in F(v)$ such that

$$\langle v^*(v) - f, v - u \rangle + J^0(\gamma v; \gamma v - \gamma u) + \varphi(v) - \varphi(u) \ge h(v - u).$$

We apply the Minty formulation and the same arguments as in the proof of [8, Proposition 3.3], and conclude that there exists $u^* \in F(u)$ such that

$$\langle u^* - f, v - u \rangle + J^0(\gamma u; \gamma v - \gamma u) + \varphi(v) - \varphi(u) \ge 0.$$

for all $v \in K$. Hence, $u \in K$ is a solution to Problem 1, i.e., $u \in \mathcal{S}$. Therefore, it holds $w - \limsup_{n\to\infty} \mathcal{S}_n \subset \mathcal{S}$.

(iii) Suppose now that F has (S_+)-property. It is trivial to see that $s - \limsup_{n\to\infty} \mathcal{S}_n \subset w - \limsup_{n\to\infty} \mathcal{S}_n$, so, we shall prove the opposite inclusion

$$w - \limsup_{n\to\infty} \mathcal{S}_n \subset s - \limsup_{n\to\infty} \mathcal{S}_n.$$

Let $u \in w - \limsup_{n\to\infty} \mathcal{S}_n$ be given. Passing to a subsequence if necessary, we are able to find a sequence $\{u_n\}$ with $u_n \in \mathcal{S}_n$ such that

$$u_n \xrightarrow{w} u \quad \text{in } V \text{ as } \quad n \to \infty.$$

We claim that $u_n \to u$ in V as $n \to \infty$. Since $u \in K$, it is clear that there exists a sequence $\{v_n\} \subset V$ satisfying $P_n v_n = 0_{V^*}$ and $v_n \to u$ in V as $n \to \infty$. Hence, there is $u_n^* \in F(u_n)$ such that

$$\langle u_n^* - f, v - u_n\rangle + \frac{1}{\rho_n}\langle P_n u_n, v - u_n\rangle + \varphi(v) - \varphi(u_n) + 2h_\varphi(\rho_n) + J^0(\gamma u_n; \gamma v - \gamma u_n) \geq 0 \tag{3.21}$$

for all $v \in V$. Inserting $v = v_n$ into the inequality above implies

$$\langle u_n^*, u_n - v_n\rangle \leq \langle f, u_n - v_n\rangle + \varphi(v_n) - \varphi(u_n) + 2h_\varphi(\rho_n) + J^0(\gamma u_n; \gamma v_n - \gamma u_n).$$

Passing to the upper limit as $n \to \infty$ in the above inequality, using the compactness of the operator γ, and the following result

$$\begin{aligned}\limsup_{n\to\infty}\langle u_n^*, u_n - v_n\rangle &= \limsup_{n\to\infty}\langle u_n^*, u_n - u + u - v_n\rangle \\ &= \limsup_{n\to\infty}\langle u_n^*, u_n - u\rangle + \lim_{n\to\infty}\langle u_n^*, u - v_n\rangle = \limsup_{n\to\infty}\langle u_n^*, u_n - u\rangle,\end{aligned}$$

we get

$$\limsup_{n\to\infty}\langle u_n^*, u_n - u\rangle \leq 0$$

We combine the latter with the $(S)_+$-property of F, and the convergence $u_n \xrightarrow{w} u$ in V as $n \to \infty$ to conclude

$$u_n \to u \quad \text{in } V \text{ as } \quad n \to \infty.$$

This means that $u \in s - \limsup_{n\to\infty} \mathcal{S}_n$. Therefore, it holds

$$s - \limsup_{n\to\infty} \mathcal{S}_n = w - \limsup_{n\to\infty} \mathcal{S}_n.$$

(iv) Let $u \in s - \limsup_{n\to\infty} \mathcal{S}_n$ be arbitrary. So, passing to a subsequence if necessary, we are able to find a sequence $\{u_n\} \subset V$ with $u_n \in \mathcal{S}_n$ such that

$$u_n \to u \quad \text{in } V \text{ as } n \to \infty.$$

Since $\mathcal{S}_n$ is nonempty and weakly compact, thus, for each $n \in \mathbb{N}$, there exists $\widetilde{u}_n \in \mathcal{S}_n$ such that

$$\|\widetilde{u}_n - u\| \leq \|w - u\| \quad \text{for all} \quad w \in \mathcal{S}_n.$$

Recall that $\cup_{n>0}\mathcal{S}_n$ is uniformly bounded (by Step 1), and hence, $\{\widetilde{u}_n\}$ is bounded as well. Then, without any loss of generality, one may suppoe that

$$\widetilde{u}_n \xrightarrow{w} \widetilde{u} \quad \text{in } V \text{ as } \quad n \to \infty$$

for some $\widetilde{u} \in V$. As before, we infer that $\widetilde{u} \in K$. On the other hand, hypothesis $H(P_n)$(i) allows one to find a sequence $\{v_n\} \subset V$ satisfying

$$v_n \to \widetilde{u} \quad \text{in } V \text{ as } \quad n \to \infty, \quad \text{and} \quad P_n v_n = 0.$$

By the monotonicity of P_n and condition $H(\varphi_\rho)$(ii), we have $u_n^* \in F(\widetilde{u}_n)$ and

$$\langle u_n^*, \widetilde{u}_n - v_n\rangle \leq \langle f, \widetilde{u}_n - v_n\rangle + \varphi(v_n) - \varphi(\widetilde{u}_n) + 2h_\varphi(\rho_n) + J^0(\gamma\widetilde{u}_n; \gamma v_n - \gamma\widetilde{u}_n).$$

Passing to the upper limit as $n \to \infty$ in the last inequality, using the compactness of γ, the continuity of φ, and the upper semicontinuity of $(u, v) \mapsto J^0(u; v)$, we obtain

$$\limsup_{n\to\infty}\langle u_n^*, \widetilde{u}_n - \widetilde{u}\rangle = \limsup_{n\to\infty}\langle u_n^*, \widetilde{u}_n - \widetilde{u}\rangle + \lim_{n\to\infty}\langle u_n^*, \widetilde{u} - v_n\rangle \leq 0.$$

Recalling that F has (S_+)-property, we conclude that $\widetilde{u}_n \to \widetilde{u}$ in V and there exists a subsequence of $\{u_n^*\}$, denoted still by $\{u_n^*\}$, such that $u_n^* \xrightarrow{w} u^*$ in V^* as $n \to \infty$ with $u^* \in F(\widetilde{u})$. Hence, $\widetilde{u} \in s - \limsup_{n\to\infty} \mathcal{S}_n \subset \mathcal{S}$.

Next, we are going to prove that $u = \widetilde{u}$. The definition of $\widetilde{u}_n$ implies

$$\|\widetilde{u}_n - u\| = d(u, \mathcal{S}_n) \leq \|u - u_n\|$$

with $u_n \in \mathcal{S}_n$. Since $u_n \to u$ in V as $n \to \infty$, the above results show $\|\widetilde{u}_n - u\| \to 0$ as $n \to \infty$. This together with the convergence $\widetilde{u}_n \to \widetilde{u}$ in V as $n \to \infty$ implies $u = \widetilde{u}$.

(v) Suppose that Problem 1 has a unique solution, i.e., $\mathcal{S} = \{\overline{u}\}$. It is not difficult to see that, for each $n \in \mathbb{N}$, Problem 8 also has a unique solution $u_n \in V$. Since $\emptyset \neq w - \limsup_{n\to\infty} \mathcal{S}_n \subset \mathcal{S} = \{\overline{u}\}$ (see assertion (ii)), we deduce that every converging subsequence of $\{u_n\}$ converges weakly to the same limit $\overline{u}$. Therefore, the whole sequence $\{u_n\}$ converges weakly in V to $\overline{u}$.

Moreover, if F has $(S)_+$-property, then we can see that every converging subsequence of $\{u_n\}$ converges strongly to the same limit $\overline{u}$, owing to $\emptyset \neq w - \limsup_{n\to\infty} \mathcal{S}_n = s - \limsup_{n\to\infty} \mathcal{S}_n = \mathcal{S} = \{\overline{u}\}$. Consequently, the whole sequence $\{u_n\}$ converges strongly in V to $\overline{u}$. $\square$

Remark 10. Given a closed and convex subset C of V, we denote by P_C the metric projection of V onto C. Then, under the hypotheses of Theorem 9(iv), we can see that for any $u \in s - \limsup_{n\to\infty} \mathcal{S}_n$, the sequence $\{P_{\mathcal{S}_n}(u)\}$ has a subsequence converging strongly in V to u.

We are now ready to complete this section with a series of colloraries.

Corollary 11. *Let $F : V \to V^*$ be a continuous function such that*

(F_1) the set-valued map $u \mapsto Fu + \gamma^* \partial J(\gamma u) \subset V^*$ is relaxed h-monotone, i.e., the inequality

$$\langle Fu - Fv, u - v\rangle + \langle \xi_u - \xi_v, \gamma(u - v)\rangle_{X^*\times X} \geq h(u - v)$$

holds for all $\xi_u \in \partial J(\gamma u)$, $\xi_v \in \partial J(\gamma v)$ and all $u, v \in V$,
(F_2) there exists $v_0 \in K$ such that the coercivity condition holds

$$\liminf_{v\in B(v_0,1),\, u\in V,\, \|u\|\to+\infty} \frac{\langle Fu, u - v\rangle + \inf_{\xi\in\partial J(\gamma u)}\langle \xi_u, \gamma(u - v)\rangle_{X^*\times X}}{\|u\|} = +\infty.$$

Assume that $H(\varphi)'$, $H(J)$, $H(h)$, $H(f)$, $H(K)$, $H(\gamma)'$, $H(\varphi_\rho)$, $H(\rho_n)$ and $H(P_n)$ hold, and $h : V \to \mathbb{R}$ is bounded. Then the following statements are true:

(i) *for each $n \in \mathbb{N}$, the set of solutions to Problem 8, denoted by $\mathcal{S}_n$, is nonempty and weakly compact.*
(ii) $\emptyset \neq w - \limsup_{n\to\infty} \mathcal{S}_n \subset \mathcal{S}$.
(iii) *if F has (S_+)-property, then the equality holds*

$$w - \limsup_{n\to\infty} \mathcal{S}_n = s - \limsup_{n\to\infty} \mathcal{S}_n.$$

(iv) *if F satisfies $(S)_+$-property, then for each $u \in s - \limsup_{n\to\infty} \mathcal{S}_n$ and any sequence $\{\widetilde{u}_n\}$ with $\widetilde{u}_n \in \mathcal{S}_n$ and*

$$\|\widetilde{u}_n - u\| \leq \|w - u\| \quad \text{for all} \quad w \in \mathcal{S}_n$$

for each $n \in \mathbb{N}$, there exists a subsequence of $\{\widetilde{u}_n\}$ strongly convergent in V to u.
(v) *if Problem 1 has a unique solution $u \in K$, then Problem 8 has a unique solution u_n, and the whole sequence of solutions $\{u_n\}$ of Problem 8 converges weakly in V to u. If, in addition, F satisfies $(S)_+$-property, then the whole sequence $\{u_n\}$ of solutions to Problem 8 converges strongly in V to u.*

Next, when we just use a generalized penalty method to Problem1, we have the following conclusion.

Corollary 12. *Assume that $H(\varphi)'$, $H(J)$, $H(h)$, $H(f)$, $H(K)$, $H(F)(i) - (ii)$, $H(F)(iii)'$, $H(\gamma)'$, $H(\varphi_\rho)$, $H(\rho_n)$ and $H(P_n)$ are satisfied, and $h : V \to \mathbb{R}$ is bounded. Then the following statements are true:*

(i) *for each $n \in \mathbb{N}$, the set of solutions (denoted by $\mathcal{S}_n$) to the following problem: find $u_n \in V$ and $u_n^* \in Fu_n$ such that*

$$\langle u_n^* - f, v - u_n\rangle + \frac{1}{\rho_n}\langle P_n u_n, v - u_n\rangle + J^0(\gamma u_n; \gamma(v - u_n)) + \varphi_n(v) - \varphi_n(u_n) \geq 0 \tag{3.22}$$

for all $v \in V$, is nonempty and weakly compact in V.
(ii) $\emptyset \neq w - \limsup_{n\to\infty} \mathcal{S}_n \subset \mathcal{S}$.
(iii) *if F has (S_+)-property, then the equality holds*

$$w - \limsup_{n\to\infty} \mathcal{S}_n = s - \limsup_{n\to\infty} \mathcal{S}_n.$$

(iv) *if F satisfies $(S)_+$-property, then for each $u \in s - \limsup_{n\to\infty} \mathcal{S}_n$ and any sequence $\{\widetilde{u}_n\}$ with $\widetilde{u}_n \in \mathcal{S}_n$ and*

$$\|\widetilde{u}_n - u\| \leq \|w - u\| \quad \text{for all} \quad w \in \mathcal{S}_n$$

for each $n \in \mathbb{N}$, there exists a subsequence of $\{\widetilde{u}_n\}$ such that it converges strongly in V to u.
(v) *if Problem 1 has a unique solution $u \in K$, then problem (3.22) has a unique solution u_n, and the whole sequence of solutions $\{u_n\}$ of problem (3.22) converges weakly in V to u. If, in addition, F satisfies $(S)_+$-property, then the whole sequence $\{u_n\}$ of soltions to problem (3.22) converges strongly in V to u.*

Remark 13. Note that if F is a single-valued mapping, which satisfies all conditions of Corollary 11 except $H(\varphi_\rho)$, then Corollary 12 reduces to a recent result of [42, Theorem 10].

Recall that an operator $P : V \to V^*$ is said to be a penalty operator of set K if P is bounded, hemicontinuous, monotone and $K = \{u \in V \mid Pu = 0\}$. Further, if P_n is specified as a penalty operator $P : V \to V^*$ of K, then we have the following corollary.

Corollary 14. *Let $P : V \to V^*$ be a penalty operator of K. Assume that $H(\varphi)'$, $H(J)$, $H(h)$, $H(f)$, $H(K)$, $H(F)(i) - (ii)$, $H(F)(iii)'$, $H(\gamma)'$, $H(\varphi_\rho)$ and $H(\rho_n)$ hold, and $h : V \to \mathbb{R}$ is bounded. Then the following statements are true:*

(i) *for each $n \in \mathbb{N}$, the set of solutions (denoted by $\mathcal{S}_n$) to the following problem: find $u_n \in V$ and $u_n^* \in Fu_n$ such that*

$$\langle u_n^* - f, v - u_n\rangle + \frac{1}{\rho_n}\langle Pu_n, v - u_n\rangle + J^0(\gamma u_n; \gamma(v - u_n)) + \varphi_n(v) - \varphi_n(u_n) \geq 0 \tag{3.23}$$

for all $v \in V$, is nonempty and weakly compact in V.

(ii) $\emptyset \neq w - \limsup_{n\to\infty} \mathcal{S}_n \subset \mathcal{S}$.

(iii) *if F has (S_+)-property, then one has*

$$w - \limsup_{n\to\infty} \mathcal{S}_n = s - \limsup_{n\to\infty} \mathcal{S}_n.$$

(iv) *if F satisfies $(S)_+$-property, then for each $u \in s - \limsup_{n\to\infty} \mathcal{S}_n$ and any sequence $\{\widetilde{u}_n\}$ with $\widetilde{u}_n \in \mathcal{S}_n$ and*

$$\|\widetilde{u}_n - u\| \leq \|w - u\| \quad \text{for all} \quad w \in \mathcal{S}_n$$

for each $n \in \mathbb{N}$, there exists a subsequence of $\{\widetilde{u}_n\}$ convergent strongly in V to u.

(v) *if Problem 1 has a unique solution $u \in K$, then problem (3.23) has a unique solution u_n, and the whole sequence of solutions $\{u_n\}$ of problem (3.23) converges weakly in V to u. Moreover, if, in addition, F satisfies $(S)_+$-property, then the whole sequence $\{u_n\}$ of solutions to problem (3.23) converges strongly in V to u.*

4. Elliptic problem with mixed boundary conditions and obstacle effect

To illustrate the applicability of the theoretical results established in Section 3, we now investigate a mixed boundary value problem involving the Clarke subgradient term of a locally Lipschitz function, and an obstacle effect. From the physical point of view, the elliptic inclusion problem under consideration originates from the modeling of semipermeability phenomena with nonconvex and nonsmooth potentials.

We assume that Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary such that

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \text{ and } \Gamma_i \cap \Gamma_j = \emptyset \text{ with } i \neq j \text{ for } i, j = 1, 2, 3$$

and $\text{meas}(\Gamma_1) > 0$. In what follows, we denote by $\boldsymbol{n}$ the outward unit normal to the boundary Γ. Let $j : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that $r \mapsto j(\boldsymbol{x}, r)$ is locally Lipschitz continuous for a.e. $\boldsymbol{x} \in \Omega$. For convenience, in the sequel, the symbol ∂j stands for the generalized subdifferential operator in the Clarke sense of the locally Lipschitz function $j : \Omega \times \mathbb{R} \to \mathbb{R}$ with respect to its last variable. The classical formulation of the elliptic inclusion problem is given as follows.

Problem 15. Find a function $u : \overline{\Omega} \to \mathbb{R}$ such that

$$-\text{div}(\alpha(\boldsymbol{x})\nabla u(\boldsymbol{x})) + \partial j(\boldsymbol{x}, u(\boldsymbol{x})) \ni f_0(\boldsymbol{x}) \text{ in } \Omega, \tag{4.1}$$

$$u(\boldsymbol{x}) \leq \Phi(\boldsymbol{x}) \text{ in } \Omega, \tag{4.2}$$

$$u(\boldsymbol{x}) = 0 \text{ on } \Gamma_1, \tag{4.3}$$

$$\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}_a} := \alpha(\boldsymbol{x})(\nabla u(\boldsymbol{x}), \boldsymbol{n})_{\mathbb{R}^N} = f_2(\boldsymbol{x}) \text{ on } \Gamma_2, \tag{4.4}$$

$$\begin{cases} \left|\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}_a}\right| \leq g(\boldsymbol{x}), \\ \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}_a} = -g(\boldsymbol{x})\frac{u(\boldsymbol{x})}{|u(\boldsymbol{x})|} \text{ if } u(\boldsymbol{x}) \neq 0 \end{cases} \quad \text{on} \quad \Gamma_3. \tag{4.5}$$

We note that Problem 15 is motivated by the study of semipermeability phenomena which may appear in the interior and on the boundary of the body Ω, and are met, for instance, in electrostatics, magnetostatics or stationary heat transfer. The problem serves as a mathematical model which describes the behavior of natural and artificial semipermeable membranes of finite thickness, temperature control problems, etc., see [33–35] and the references therein. The unknown function $u : \overline{\Omega} \to \mathbb{R}$ in the problem usually represents the electric potential, magnetic potential or temperature field, accordingly, while the function $\alpha : \Omega \to \mathbb{R}$ could be understood as the dielectric coefficient, the magnetic permeability or the thermal conductivity in the body occupied by a non-isotropic and heterogeneous material, and $f_0 = f_0(\boldsymbol{x})$ is a given source term. Moreover, condition (4.2) represents an additional unilateral constraint for the solution. It is not difficult to see that the relation (4.5) could be rewritten in the following inclusion form

$$-\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}_a} \in \partial_c \Psi(\boldsymbol{x}, u(\boldsymbol{x})) \quad \text{on} \quad \Gamma_3,$$

where $\partial_c \Psi$ denotes the convex subdifferential with respect to its second variable of the function $\Psi(\boldsymbol{x}, s) = g(\boldsymbol{x})|s|$ for $(\boldsymbol{x}, s) \in \Gamma_3 \times \mathbb{R}$.

In order to derive the weak variational formulation of Problem 15, we need the space

$$V = \left\{ v \in H^1(\Omega) \mid v(\boldsymbol{x}) = 0 \text{ for a.e. } \boldsymbol{x} \in \Gamma_1 \right\}.$$

Moreover, by the Poincaré inequality, we know that V endowed with the inner product

$$\langle u, v \rangle = \int_\Omega (\nabla u(\boldsymbol{x}), \nabla v(\boldsymbol{x}))\, d\boldsymbol{x} \quad \text{for all} \quad u, v \in V,$$

is a Hilbert space. Additionally, we denote by γ the embedding operator from $H^1(\Omega)$ to $X := L^2(\Gamma)$. Obviously, $\gamma : V \to X$ is linear, continuous, and compact. Let us introduce the constraint set K defined by

$$K := \{ v \in V \mid v(\boldsymbol{x}) \le \Phi(\boldsymbol{x}) \quad \text{for a.e.} \quad \boldsymbol{x} \in \Omega \}.$$

We also need the following assumptions for our problem.

$\underline{H(\alpha)}$: $\alpha \in L^\infty(\Omega)_+$.

$\underline{H(j)}$: $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that

(i) for a.e. $\boldsymbol{x} \in \Omega$, $s \mapsto j(\boldsymbol{x}, s)$ is locally Lipschitz,

(ii) for any $s \in \mathbb{R}$, $\boldsymbol{x} \mapsto j(\boldsymbol{x}, s)$ is measurable on Ω and there exists $e \in L^2(\Omega)$ satisfying $j(\cdot, e(\cdot)) \in L^1(\Omega)$,

(iii) there are constants $c_j \ge 0$, $\theta \in [1, 2)$ and a function $\beta_j \in L^1(\Omega)_+$ such that

$$j^0(\boldsymbol{x}, r; -r) \le \beta_j(\boldsymbol{x}) + c_j |r|^\theta \text{ for a.e. } \boldsymbol{x} \in \Omega \text{ and all } r \in \mathbb{R},$$

(iv) there exists $m_j \ge 0$ such that

$$j^0(\boldsymbol{x}, r_1; r_2 - r_1) + j^0(\boldsymbol{x}, r_2; r_1 - r_2) \le m_j |r_1 - r_2|^2$$

for a.e. $\boldsymbol{x} \in \Omega$ and all $r_1, r_2 \in \mathbb{R}$.

$\underline{H(0)}$: $f_0 \in L^2(\Omega)$, $f_2 \in L^2(\Gamma_2)$, $\Phi \in V$ with $\Phi(\boldsymbol{x}) \ge c_\Phi > 0$ for a.e. $\boldsymbol{x} \in \Omega$, $g \in L^\infty(\Gamma_3)$ with $g(\boldsymbol{x}) \ge 0$ for a.e. $\boldsymbol{x} \in \Gamma_3$ and $g \not\equiv 0$.

Assume now that $u \in K$ is a smooth function such that (4.1)–(4.5) hold. For any $v \in K$ fixed, it follows from Green's formula and the equality (4.1) that

$$\int_\Omega \alpha(\boldsymbol{x})(\nabla u(\boldsymbol{x}), \nabla v(\boldsymbol{x}) - \nabla u(\boldsymbol{x}))_{\mathbb{R}^N}\, d\boldsymbol{x} + \int_\Omega \xi(\boldsymbol{x})(v(\boldsymbol{x}) - u(\boldsymbol{x}))\, d\boldsymbol{x}$$
$$= \int_\Omega f_0(\boldsymbol{x})(v(\boldsymbol{x}) - u(\boldsymbol{x}))\, d\boldsymbol{x} + \int_\Gamma \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}_a}(v(\boldsymbol{x}) - u(\boldsymbol{x}))\, d\Gamma,$$

where $\xi : \Omega \to \mathbb{R}$ satisfies $\xi(\boldsymbol{x}) \in \partial j(\boldsymbol{x}, u(\boldsymbol{x}))$ for a.e. $\boldsymbol{x} \in \Omega$ and

$$-\mathrm{div}(\alpha(\boldsymbol{x})\nabla u(\boldsymbol{x})) + \xi(\boldsymbol{x}) = f_0(\boldsymbol{x}) \quad \text{for a.e.} \quad \boldsymbol{x} \in \Omega.$$

By Riesz's representation theorem, see, e.g., [25, Theorem 1.30] we define an element $f \in V^*$ which satisfies

$$\langle f, v \rangle = \int_\Omega f_0(\boldsymbol{x}) v(\boldsymbol{x})\, d\boldsymbol{x} + \int_{\Gamma_2} f_2(\boldsymbol{x}) v(\boldsymbol{x})\, d\Gamma \tag{4.6}$$

for all $v \in V$. From equality

$$\int_\Gamma \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}_a}(v(\boldsymbol{x}) - u(\boldsymbol{x}))\, d\Gamma = \int_{\Gamma_1} \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}_a}(v(\boldsymbol{x}) - u(\boldsymbol{x}))\, d\Gamma$$
$$+ \int_{\Gamma_2} \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}_a}(v(\boldsymbol{x}) - u(\boldsymbol{x}))\, d\Gamma + \int_{\Gamma_3} \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}_a}(v(\boldsymbol{x}) - u(\boldsymbol{x}))\, d\Gamma,$$

and boundary conditions (4.3) and (4.4), we obtain

$$\int_\Omega \alpha(\boldsymbol{x})(\nabla u(\boldsymbol{x}), \nabla v(\boldsymbol{x}) - \nabla u(\boldsymbol{x}))_{\mathbb{R}^N}\, d\boldsymbol{x} + \int_\Omega \xi(\boldsymbol{x})(v(\boldsymbol{x}) - u(\boldsymbol{x}))\, d\boldsymbol{x}$$
$$= \langle f, v - u \rangle + \int_{\Gamma_3} \frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}_a}(v(\boldsymbol{x}) - u(\boldsymbol{x}))\, d\Gamma.$$

From the definition of the Clarke subgradient, we get

$$\int_\Omega \alpha(\boldsymbol{x})\big(\nabla u(\boldsymbol{x}), \nabla v(\boldsymbol{x}) - \nabla u(\boldsymbol{x})\big)_{\mathbb{R}^N}\, d\boldsymbol{x} + \int_\Omega j^0(\boldsymbol{x}, u(\boldsymbol{x}); v(\boldsymbol{x}) - u(\boldsymbol{x}))\, d\boldsymbol{x}$$
$$+ \int_{\Gamma_3} g(\boldsymbol{x})\big(|v(\boldsymbol{x})| - |u(\boldsymbol{x})|\big)\, d\Gamma \ge \langle f, v - u \rangle.$$

We are now in a position to state the variational formulation of Problem 15 which reads as follows.

Problem 16. Find $u \in K$ such that

$$\int_\Omega \alpha(\boldsymbol{x})\big(\nabla u(\boldsymbol{x}), \nabla v(\boldsymbol{x}) - \nabla u(\boldsymbol{x})\big)_{\mathbb{R}^N}\, d\boldsymbol{x} + \int_\Omega j^0(\boldsymbol{x}, u(\boldsymbol{x}); v(\boldsymbol{x}) - u(\boldsymbol{x}))\, d\boldsymbol{x}$$
$$+ \int_{\Gamma_3} g(\boldsymbol{x})\big(|v(\boldsymbol{x})| - |u(\boldsymbol{x})|\big)\, d\Gamma \geq \langle f, v - u\rangle$$

for all $v \in K$.

The following result provides conditions on existence of a weak solution to Problem 16.

Theorem 17. *Assume that $H(\alpha)$, $H(j)$ and $H(0)$ hold. If, in addition, $\|\alpha\|_{L^\infty(\Omega)} \geq m_j\,\|\gamma\|^2$, then the set of solutions to Problem 16 is nonempty and weakly compact in V. Moreover, if $\|\alpha\|_{L^\infty(\Omega)} > m_j\,\|\gamma\|^2$, then Problem 16 has a unique solution.*

Proof. Let $X := L^2(\Omega)$. We introduce the operator $F : V \to V^*$, and functions $J : X \to \mathbb{R}$ and $\varphi : V \to \mathbb{R}$ defined by

$$\langle Fu, v\rangle := \int_\Omega \alpha(\boldsymbol{x})\big(\nabla u(\boldsymbol{x}), \nabla v(\boldsymbol{x})\big)_{\mathbb{R}^N}\, d\boldsymbol{x}, \tag{4.7}$$

$$J(w) := \int_\Omega j(\boldsymbol{x}, w(\boldsymbol{x}))\, d\boldsymbol{x}, \tag{4.8}$$

$$\varphi(u) := \int_{\Gamma_3} g(\boldsymbol{x})|u(\boldsymbol{x})|\, d\Gamma \tag{4.9}$$

for $u, v \in V$ and $w \in X$. We first consider the following intermediate problem: find $u \in K$ such that

$$\langle Fu - f, v - u\rangle + J^0(\gamma u; \gamma(v-u)) + \varphi(v) - \varphi(u) \geq 0 \quad \text{for all} \quad v \in K, \tag{4.10}$$

where $\gamma : V \to X$ is the embedding operator from $H^1(\Omega)$ to X. Employing [25, Theorem 3.47] we obtain that J is a locally Lipschitz function such that

$$\begin{cases} J^0(w; z) \leq \int_\Omega j^0(\boldsymbol{x}, w(\boldsymbol{x}); z(\boldsymbol{x}))\, d\boldsymbol{x} \\ \|\partial J(w)\|_{L^2(\Omega)} \leq d_j + \sqrt{2}c_j\|w\|_{L^2(\Omega)} \end{cases} \tag{4.11}$$

for all $w, z \in X$ with some $d_j \geq 0$. Observe that $u \in K$ is a solution to problem (4.10), then it solves Problem 16 as well.

To show that problem (4.10) admits a solution, we apply Corollary 6. We will prove that all conditions of this corollary are satisfied. It is easy to see that φ is a continuous and convex function, and F is a linear and bounded operator which is coercive in the following sense

$$\langle Fu, u\rangle \geq \|\alpha\|_{L^\infty(\Omega)}\|u\|^2 \quad \text{for all} \quad u \in V.$$

We combine the following estimate

$$\langle Fu, u\rangle + \inf_{\xi\in\partial J(\gamma u)} \langle \xi, \gamma u\rangle_{X^*\times X} = \langle Fu, u\rangle - \langle \overline{\xi}, -\gamma u\rangle_{X^*\times X}$$
$$\geq \|\alpha\|_{L^\infty(\Omega)}\|u\|^2 - J^0(\gamma u; -\gamma u),$$

the growth condition $H(j)$(iii) and (4.11) to deduce

$$\frac{\langle Fu, u\rangle + \inf_{\xi\in\partial J(\gamma u)}\langle \xi, \gamma u\rangle_{X^*\times X}}{\|u\|} \geq \|\alpha\|_{L^\infty(\Omega)}\|u\| - \frac{\int_\Omega \beta_j(\boldsymbol{x}) + c_j|u(\boldsymbol{x})|^\theta\, dx}{\|u\|},$$

where $\overline{\xi} \in \partial J(\gamma u)$ is such that $\inf_{\xi\in\partial J(\gamma u)}\langle \xi, \gamma u\rangle_{X^*\times X} = \langle \overline{\xi}, \gamma u\rangle_{X^*\times X}$. Keeping in mind that $\theta \in [1, 2)$, we use the Young inequality with $\varepsilon > 0$ to get

$$\frac{\langle Fu, u\rangle + \inf_{\xi\in\partial J(\gamma u)}\langle \xi, \gamma u\rangle_{X^*\times X}}{\|u\|} \geq (\|\alpha\|_{L^\infty(\Omega)} - \varepsilon)\|u\| - \frac{\int_\Omega \beta_j(\boldsymbol{x})\, dx + c(\varepsilon)}{\|u\|}$$

with some $c(\varepsilon) > 0$. Therefore, choosing $\varepsilon = \frac{1}{2}\|\alpha\|_{L^\infty(\Omega)}$ and passing to the limit as $\|u\| \to \infty$ with $u \in K$, we derive

$$\lim_{u\in K, \|u\|\to\infty} \frac{\langle Fu, u\rangle + \inf_{\xi\in\partial J(\gamma u)}\langle \xi, \gamma u\rangle_{X^*\times X}}{\|u\|} = +\infty.$$

Further, the monotonicity of F, the condition $H(j)$(iv) and the properties (4.11) imply the monotonicity of the operator $u \mapsto Fu + \gamma^*\partial J(\gamma u)$.

We are now in a position to apply Corollary 6 with $h \equiv 0$ to conclude that problem (4.10) has at least one solution. Hence, Problem 16 is solvable. Moreover, the boundedness and weak closedness of the solution set to Problem 16 can be obtained directly by using the same arguments as the proof of Theorem 5.

Finally, if $\|\alpha\|_{L^\infty(\Omega)} > m_j\|\gamma\|^2$, then it can observed that the function h can be specified as $h(u) = (\|\alpha\|_{L^\infty(\Omega)} - m_j\|\gamma\|^2)\|u\|^2$ for $u \in V$. Therefore, employing Corollary 6(iv), we conclude the uniqueness of solution to Problem 16, which completes the proof. □

Let us consider the sequence of functions $\{\Phi_n\}$ with

$$\Phi_n \to \Phi \quad \text{in } V \text{ as } n \to \infty. \tag{4.12}$$

From condition $H(0)$, without any loss of generality, we may assume that

$$\Phi_n(\boldsymbol{x}) \geq 0 \quad \text{for a.e. } \boldsymbol{x} \in \Omega \quad \text{and all } n \in \mathbb{N}.$$

For any $\rho_n > 0$, consider the penalized and regularized problem corresponding to Problem 15.

Problem 18. Find a function $u_n : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{aligned}
&-\mathrm{div}(\alpha(\boldsymbol{x})\nabla u_n(\boldsymbol{x})) + \partial j(\boldsymbol{x}, u_n(\boldsymbol{x})) + \frac{1}{\rho_n}(u_n(\boldsymbol{x}) - \Phi_n(\boldsymbol{x}))^+ \ni f_0(\boldsymbol{x}) && \text{in} \quad \Omega,\\
&u_n(\boldsymbol{x}) = 0 && \text{on} \quad \Gamma_1,\\
&\frac{\partial u_n(\boldsymbol{x})}{\partial \boldsymbol{n}_a} := \alpha(\boldsymbol{x})(\nabla u_n(\boldsymbol{x}), \boldsymbol{n})_{\mathbb{R}^N} = f_2(\boldsymbol{x}) && \text{on} \quad \Gamma_2,\\
&\frac{\partial u_n(\boldsymbol{x})}{\partial \boldsymbol{n}_a} = -g(\boldsymbol{x})\frac{u_n(\boldsymbol{x})}{\sqrt{|u_n(\boldsymbol{x})|^2 + \rho_n^2}} && \text{on} \quad \Gamma_3,
\end{aligned}$$

where $r^+ = \max\{0, r\}$ stands for the positive part of $r \in \mathbb{R}$.

Next, we define the operator $P_n : V \to V^*$ by

$$\langle P_n u, v\rangle = \int_\Omega (u(\boldsymbol{x}) - \Phi_n(\boldsymbol{x}))^+ v(\boldsymbol{x})\, d\boldsymbol{x} \quad \text{for} \quad u, v \in V \text{ and } n \in \mathbb{N}. \tag{4.13}$$

In fact, the family of operators $\{P_n\}$ satisfies condition $H(P_n)$ in Section 3, for its detailed proof, see [42, Lemma 20].

Lemma 19. *If the sequence $\{\Phi_n\}$ satisfies (4.12), then the family of operators $\{P_n\}$ defined by (4.13) satisfies condition $H(P_n)$ of Section 3.*

Let $\rho > 0$. We introduce the function $\varphi_\rho : V \to \mathbb{R}$ defined by

$$\varphi_\rho(u) := \int_{\Gamma_3} g(\boldsymbol{x})\sqrt{|u(\boldsymbol{x})|^2 + \rho^2}\, d\Gamma \tag{4.14}$$

for $u \in V$. The following result shows that for each $\rho > 0$, the function φ_ρ satisfies condition $H(\varphi_\rho)$.

Lemma 20. *Assume that $H(0)$ holds. Then, for each $\rho > 0$, φ_ρ satisfies condition $H(\varphi_\rho)$.*

Proof. It follows from the definition of φ_ρ and [36, Lemma 5.28] that for each $\rho > 0$, the function φ_ρ is Gâteaux differentiable and

$$\langle \nabla\varphi_\rho(u), w\rangle = \int_{\Gamma_3} g(\boldsymbol{x})\frac{u(\boldsymbol{x})w(\boldsymbol{x})}{\sqrt{|u(\boldsymbol{x})|^2 + \rho^2}}\, d\Gamma \tag{4.15}$$

for all $u, w \in V$. Additionally, arguing as in the proof of [36, Lemma 5.29], it is not difficult to see that the operator $\nabla\varphi_\rho : V \to V^*$ is Lipschitz continuous and monotone. Therefore, by virtue of [36, Proposition 1.32], we have that φ_ρ is convex. Hence, condition $H(\varphi_\rho)$(i) is verified.

Moreover, a simple calculation gives

$$|\varphi_\rho(w) - \varphi(w)| \leq \rho \int_{\Gamma_3} g(\boldsymbol{x})\, d\Gamma$$

for all $w \in V$ and $\rho > 0$. This means that condition $H(\varphi_\rho)$(ii) is also valid. □

Using the same procedure as we have done before, we derive the variational formulation of Problem 18.

Problem 21. Find $u_n \in V$ such that

$$\begin{aligned}
&\int_\Omega \alpha(\boldsymbol{x})\big(\nabla u_n(\boldsymbol{x}), \nabla v(\boldsymbol{x}) - \nabla u_n(\boldsymbol{x})\big)_{\mathbb{R}^N} d\boldsymbol{x} + \int_\Omega j^0(\boldsymbol{x}, u_n(\boldsymbol{x}); v(\boldsymbol{x}) - u_n(\boldsymbol{x}))\, d\boldsymbol{x}\\
&+\frac{1}{\rho_n}\int_\Omega (u_n(\boldsymbol{x}) - \Phi_n(\boldsymbol{x}))^+(v(\boldsymbol{x}) - u_n(\boldsymbol{x}))\, d\boldsymbol{x} + \int_{\Gamma_3} g(\boldsymbol{x})\sqrt{|v(\boldsymbol{x})|^2 + \rho_n^2}\, d\Gamma\\
&-\int_{\Gamma_3} g(\boldsymbol{x})\sqrt{|u_n(\boldsymbol{x})|^2 + \rho_n^2}\, d\Gamma \geq \langle f, v - u_n\rangle
\end{aligned}$$

for all $v \in V$.

We are now in a position to invoke Lemmata 19 and 20, and Corollary 11 to obtain the following convergence results.

Theorem 22. *Under the assumptions of Theorem 17, if, in addition, (4.12) holds and $\{\rho_n\}$ is a sequence such that*

$$\rho_n > 0 \quad \text{and} \quad \rho_n \to 0 \quad \text{as} \quad n \to \infty,$$

then we have

(i) *the set of solutions to Problem 21, denoted by $\mathcal{S}_n$, is nonempty and weakly compact,*
(ii) *it holds*

$$w - \limsup_{n\to\infty} \mathcal{S}_n = s - \limsup_{n\to\infty} \mathcal{S}_n,$$

(iii) *for any $u \in s - \limsup_{n\to\infty} \mathcal{S}_n$, let $\{\widetilde{u}_n\}$ be the sequence such that $\widetilde{u}_n \in \mathcal{S}_n$ and*

$$\|\widetilde{u}_n - u\| \le \|w - u\| \quad \text{for all} \quad w \in \mathcal{S}_n$$

for each $n \in \mathbb{N}$, then there exists a subsequence of $\{\widetilde{u}_n\}$ that converges strongly in V to u.
(iv) *if $\|\alpha\|_{L^\infty(\Omega)} > m_j \|\gamma\|^2$, then Problem 21 has a unique solution u_n, which converges strongly in V to the unqiue solution u of Problem 16.*

Credit Author Statement

The authors contributed equally in the paper.

Declaration of Competing Interest

The authors declared that they have no conflicts of interest to this work. We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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