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Research Article

# Strongly $\lambda -$ Convergence of Order $\alpha$ in Neutrosophic Normed Spaces

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**Abstract:** In this paper, we introduce the concept of strongly $\lambda -$convergence of order $\alpha$ in the neutrosophic normed spaces. We investigate some fundamental properties of this new concept.

**Keywords**: $\lambda -$convergence, Neutrosophic normed space, $\lambda$ sequence.

**AMS subject classification (2015):** Primary 40A05, 40C05; Secondary 46A45.

## I. Introduction

The concept neutrosophy implies impartial knowledge of thought and then neutral describes the basic difference between neutral, fuzzy, intuitive fuzzy set and logic. The neutrosophic set ($NS$) was investigated by Smarandache (Smarandache, 2003) who defined the degree of indeterminacy (i) as independent component. In (Smarandache, 2013), neutrosophic logic was firstly examined. It is a logic where each proposition is determined to have a degree of truth ($T$), falsity ($F$), and indeterminacy ($I$). A neutrosophic set ($NS$) is determined as a set where every component of the universe has a degree of $T$, $F$ and $I$. In intuitionistic fuzzy set ($IFSs$) the `degree of non-belongingness' is not independent but it is dependent on the "degree of belongingness". Fuzzy sets ($FSs$) can be thought as a remarkable case of an $IFS$ where the "degree of non-belongingness" of an element is absolutely equal to "1- degree of belongingness". Uncertainty is based on the belongingness degree in $IFSs$, whereas the uncertainty in $NS$ is considered independently from $T$ and $F$ values. Since no any limitations among the degree of $T$, $F$, $I$. Neutrosophic sets ($NSs$) are actually more general than $IFS$. Neutrosophic soft linear spaces ($NSLSs$) were considered by Bera and Mahapatra (Bera and Mahapatra, 2017). Subsequently, in (Bera and Mahapatra, 2018), the concept neutrosophic soft normed linear space ($NSNLS$) was defined and the features of ($NSNLS$) were examined.

Kirişci and Şimşek (Kirişci and Şimşek, 2020a) defined new a concept known as neutrosophic metric space ($NMS$) with continuous t-norms and continuous t-conorms. Some notable features of $NMS$ have been examined. Neutrosophic normed space ($NNS$) and statistical convergence in $NNS$ has been investigated by Kirişci and Şimşek (Kirişci and Şimşek, 2020b). Neutrosophic set and neutrosophic logic has used by applied sciences and theoretical science such as decision making, robotics, summability theory.

In (Kişi, 2020), lacunary statistical convergence of sequences in $NNS$ was examined. Also, lacunary statistically Cauchy sequence in $NNS$ was given and lacunary statistically completeness in connection with a neutrosophic normed space was presented. Kişi (Kişi, 2021a), defined lacunary ideal convergence and gave various results about lacunary ideal convergence in (Kişi, 2021a) and (Kişi, 2021b). Also, triple Lacunary Δ-statistical convergence and triple difference sequences of real numbers in neutrosophic normed space introduced by Kişi (Kişi, 2022a, 2022b).

***Definition 1.1 (Menger, 1942):*** Let $*: [0,1] \times [0,1] \to [0,1]$ be an operation. When $*$ satisfies following situations, it is called continuous $TN$ (Triangular norms (t-norms)). Take $p, q, r, s \in [0,1]$

(i) $p * 1 = p$

(ii) If $p \leq r$ and $q \leq s$, then $p * q \leq r * s$,

(iii) $*$ is continuous,

(iv) $*$ associative and commutative.

***Definition 1.2 (Menger, 1942):*** Let $\Diamond : [0,1] \times [0,1] \to [0,1]$ be an operation. When $\Diamond$ satisfies following situations, it is said to be continuous $TC$ (Triangular conorms (t-conorms)).

(i) $p \Diamond 0 = p$

(ii) If $p \leq r$ and $q \leq s$, then $p \Diamond q = r \Diamond s$,

(iii) $\Diamond$ is continuous,

(iv) $\Diamond$ associative and commutative.

***Definition 1.3 (Kirişci and Şimşek, 2020b):*** Let $F$ be a vector space, $N = \{\langle u, G(u),\ B(u), Y(u)\rangle : u \in F\}$ be a normed space ($NS$) such that $N: F \times \mathbb{R}^{+} \to [0,1]$. While following conditions hold, $V = (F, N,\ *, \Diamond\ )$ is called to be $NNS$. For each $u, v \in\ F$ and $\vartheta, \mu > 0$ and for all $\sigma \neq 0$,

(i) $0 \leq G(u,\vartheta) \leq 1, 0 \leq B(u,\vartheta) \leq 1, 0 \leq Y(u,\vartheta) \leq 1, \forall\, \vartheta \in \mathbb{R}^+$

(ii) $G(u,\vartheta) + B(u,\vartheta) + Y(u,\vartheta) \leq 3, \forall \vartheta \in \mathbb{R}^+$

(iii) $G(u,\vartheta) = 1$ (for $\vartheta > 0$) iff $u = 0$,

(iv) $G(\sigma u,\vartheta) = G\left(u,\frac{\vartheta}{|\sigma|}\right)$,

(v) $G(u,\mu) * G(v,\vartheta) \leq G(u+v,\mu+\vartheta)$

(vi) $G(u,.)$ is non-decreasing continuous function,

(vii) $\lim_{\vartheta\to\infty} G(u,\vartheta) = 1$,

(viii) $B(u,\vartheta) = 0$ (for $\vartheta > 0$) iff $u = 0$,

(ix) $B(\sigma u,\vartheta) = B\left(u,\frac{\vartheta}{|\sigma|}\right)$,

(x) $B(u,\mu) \lozenge B(v,\vartheta) \geq B(u+v,\mu+\vartheta)$,

(xi) $B(u,.)$ is non-increasing continuous function,

(xii) $\lim_{\vartheta\to\infty} B(u,\vartheta) = 0$,

(xiii) $Y(u,\vartheta) = 0$ (for $\vartheta > 0$) iff $u = 0$,

(xiv) $Y(\sigma u,\vartheta) = Y\left(u,\frac{\vartheta}{|\sigma|}\right)$,

(xv) $Y(u,\mu) \lozenge Y(v,\vartheta) \geq Y(u+v,\mu+\vartheta)$,

(xvi) $Y(u,.)$ is non-increasing continuous function,

(xvii) $\lim_{\vartheta\to\infty} Y(u,\vartheta) = 0$,

(xviii) If $\vartheta \leq 0$, then $G(u,\vartheta) = 0, B(u,\vartheta) = 1$ and $Y(u,\vartheta) = 1$.

Then $N = (G,B,Y)$ is called Neutrosophic norm $(NN)$.

***Definition 1.4 (Kirişci and Şimşek, 2020b):*** Let $V$ be an $NNS$, the sequence $(x_k)$ in $V$, $\varepsilon \in (0,1)$ and $\vartheta > 0$. Then, the squence $(x_k)$ is converges to $\ell$ iff there is $N \in \mathbb{N}$ such that $G(x_k-\ell,\vartheta) > 1-\varepsilon$, $B(x_k-\ell,\vartheta) < \varepsilon$, $Y(x_k-\ell,\vartheta) < \varepsilon$. That is, $\lim_{k\to\infty} G(x_k-\ell,\vartheta) = 1$, $\lim_{k\to\infty} B(x_k-\ell,\vartheta) = 0$ and $\lim_{k\to\infty} Y(x_k-\ell,\vartheta) = 0$ as $\vartheta > 0$. In that case, the sequence $(x_k)$ is named a convergent sequence in $V$. The convergent in $NNS$ is indicated by $N-lim(x_k) = \ell$.

***Definition 1.5 (Kirişci and Şimşek, 2020b):*** Let $V$ be an $NNS$. For $\vartheta > 0$, $w \in F$ and $\varepsilon \in (0,1)$,

$$OB(w,\varepsilon,\vartheta) = \{u \in F : G(w-u,\vartheta) > 1-\varepsilon, B(w-u,\vartheta) < \varepsilon, Y(w-u,\vartheta) < \varepsilon\}$$

is called open ball with center $w$, radius $\varepsilon$.

***Definition 1.6 (Kirişci and Şimşek, 2020b):*** The set $A \subset F$ is called neutrosophic-bounded $NB$ in $NNS\, V$, if there exist $\lambda > 0$ and $\varepsilon \in (0,1)$ such that $G(u,\vartheta) > 1-\varepsilon, B(u,\vartheta) < \varepsilon$ and $Y(u,\vartheta) < \varepsilon$ for each $u \in A$.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to $\infty$ such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. The generalized de la Vallée-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k \ ,$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1,2,\cdots$. A sequence $(x_k)$ is said to $(V,\lambda) -$summable to a number $L$ if $t_n(x) \to L$ as $n \to \infty$. If $(\lambda_n) = n$, then $(V,\lambda) -$summability is reduced to Cesàro summability. By $\Lambda$ we denote the class of all non-decreasing sequence of positive real numbers tending to $\infty$ such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. $\lambda -$convergence was studied by Mursaleen (Mursaleen, 2000), Şengül et. al. (Şengül, 2019, 2018), Başarır et. al. (Başarır, 2019) and some investigations concerning this concept refer to (Aral, 2022) and (Çakallı, 2019).

Statistical convergence of order $\alpha$ is more general than statistical convergence for sequences. This has motivated us to study the strongly $\lambda -$convergence of order $\alpha$ in NNS. In this paper, we introduce the concept of strongly $\lambda -$convergence of order $\alpha$ in $NNS$ by using the $\lambda -$summable and we obtain some inclusion results.

## II. Main Results

***Definition 2.1:*** Take an $NNS\ V$. For $0 < \alpha \leq 1$, a sequence $x = (x_k)$ is named to be strongly $\lambda -$convergent to $\ell \in F$ of order $\alpha$ with regards to $NN\ (LC - NN)$, if for every $\vartheta > 0$ and $\varepsilon \in (0,1)$, there is $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} G(x_k - \ell, \vartheta) > 1 - \varepsilon \text{ and } \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} B(x_k - \ell, \vartheta) < \varepsilon, \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} Y(x_k - \ell, \vartheta) < \varepsilon$$

for all $n \geq n_0$. We indicate $(G,B,Y)_\lambda^\alpha - \lim x = \ell$. For $(\lambda_n) = n$, we shall write $(G,B,Y)^\alpha$ instead of $(G,B,Y)_\lambda^\alpha$ and in the special case $\alpha = 1$ and $(\lambda_n) = n$, we shall write $(G,B,Y)$ instead of $(G,B,Y)_\lambda^\alpha$.

***Theorem 2.2:*** Let $V$ be an $NNS$. If $x$ is strongly $\lambda -$convergent of order $\alpha$ with regards to $NN$, then $(G,B,Y)_\lambda^\alpha - \lim x = \ell$ is unique.

***Proof:*** Suppose that $(G,B,Y)_\lambda^\alpha - \lim x = \ell_1$, $(G,B,Y)_\lambda^\alpha - \lim x = \ell_2$ and $\ell_1 \neq \ell_2$. Given $\varepsilon > 0$, select $\rho \in (0,1)$ such that $(1-\rho) * (1-\rho) > 1 - \varepsilon$ and $p \lozenge p < \varepsilon$. For each $\vartheta > 0$, there is $n_1 \in \mathbb{N}$ such that

$$\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} G(x_k - \ell_1, \vartheta) > 1 - \rho \ \text{ and } \ \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} B(x_k - \ell_1, \vartheta) < \rho, \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} Y(x_k - \ell_1, \vartheta) < \rho$$

for all $n \geq n_1$. Also, there is $n_2 \in \mathbb{N}$ such that

$$\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} G(x_k - \ell_2, \vartheta) > 1 - \rho \ \text{ and } \ \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} B(x_k - \ell_2, \vartheta) < \rho, \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} Y(x_k - \ell_2, \vartheta) < \rho$$

for all $n \geq n_2$. Assume that $n_0 = max\{n_1, n_2\}$. Then for $n \geq n_0$, we can find a positive integer $m \in \mathbb{N}$ such that

$$G(\ell_1-\ell_2,\vartheta)\geq\ G\left(x_m-\ell_1,\frac{\vartheta}{2}\right)*G\left(x_m-\ell_2,\frac{\vartheta}{2}\right)>(1-\rho)*(1-\rho)>1-\varepsilon,$$

$$B(\ell_1-\ell_2,\vartheta)\leq\ B\left(x_m-\ell_1,\frac{\vartheta}{2}\right)\Diamond B\left(x_m-\ell_2,\frac{\vartheta}{2}\right)<\rho\Diamond\rho<\varepsilon$$

and

$$Y(\ell_1-\ell_2,\vartheta)\leq\ Y\left(x_m-\ell_1,\frac{\vartheta}{2}\right)\Diamond Y\left(x_m-\ell_2,\frac{\vartheta}{2}\right)<\rho\Diamond\rho<\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we get $G(\ell_1-\ell_2,\vartheta)=1, B(\ell_1-\ell_2,\vartheta)=0$ and $Y(\ell_1-\ell_2,\vartheta)=0$ for all $\vartheta>0$, which gives that $\ell_1=\ell_2$.

Now, we give an example to denote the sequence strongly $\lambda-$convergence of order $\alpha$ in an $NNS$.

***Example 2.3:*** Let $(F,\|.\|)$ be an $NNS$. For all $u,v,\alpha\in[0,1]$, define $u*v=uv$ and $u\Diamond v=min\{u+v;1\}$. For all $x\in F$ and every $\vartheta>0$, we take $G(x,\vartheta)=\frac{\vartheta}{\vartheta+\|x\|}$, $B(x,\vartheta)=\frac{\|x\|}{\vartheta+\|x\|}$ and $Y(x,\vartheta)=\frac{\|x\|}{\vartheta}$. Then $V$ is an $NNS$. We define a sequence $(x_k)$ by

$$x_k=\begin{cases}1, & if\ k=t^2(t\in\mathbb{N})\\ 0, & otherwise\end{cases}.$$

Consider

$$A=\{k\in I_n: G(x,\vartheta)>1-\varepsilon\ and\ B(x,\vartheta)<\varepsilon, Y(x,\vartheta)<\varepsilon\}.$$

Then, for any $\vartheta>0$ and $\varepsilon\in(0,1)$, the following set

$$A=\left\{k\in I_n:\frac{\vartheta}{\vartheta+\|x\|}>1-\varepsilon\ and\ \frac{\|x_k\|}{\vartheta+\|x_k\|}<\varepsilon,\frac{\|x_k\|}{\vartheta}<\varepsilon\right\}$$

$$=\left\{k\in I_n:\|x_k\|\leq\frac{\vartheta\varepsilon}{1-\varepsilon}\ and\ \|x_k\|<\vartheta\varepsilon\right\}$$

$$\subset\{k\in I_n:\|x_k\|=1\}=\{k\in I_n: k=t^2\}$$

i.e.,

$$A_n(\varepsilon,\vartheta)=\left\{n\in\mathbb{N}:\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}G(x_k,\vartheta)>1-\varepsilon\ and\ \frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}B(x_k,\vartheta)<\varepsilon,\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}Y(x_k,\vartheta)<\varepsilon\right\}$$

will be a finite set.

***Theorem 2.4:*** If $(G,B,Y)_\lambda^\alpha-\lim x=\ell$, then there is a subsequence $(x_{\rho_k})$ of $x$ such that $(G,B,Y)_\lambda^\alpha-\lim x_{\rho_k}=\ell$.

***Proof:*** Take $(G,B,Y)_\lambda^\alpha-\lim x=\ell$. Then, for every $\vartheta>0$ and $\varepsilon\in(0,1)$, there is $n_0\in\mathbb{N}$ such that

$$\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}G(x_k-\ell,\vartheta)>1-\varepsilon \text{ and } \frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}B(x_k-\ell,\vartheta)<\varepsilon,\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}Y(x_k-\ell,\vartheta)<\varepsilon$$

for all $n \geq n_0$. Obviously, for each $n \geq n_0$, we choose $\rho_k \in I_n$ such that

$$G\left(x_{\rho_k} - \ell, \vartheta\right) > \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} G(x_k - \ell, \vartheta) > 1 - \varepsilon$$

$$B\left(x_{\rho_k} - \ell, \vartheta\right) < \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} B(x_k - \ell, \vartheta) < \varepsilon$$

$$Y\left(x_{\rho_k} - \ell, \vartheta\right) < \tfrac{1}{\lambda_n^\alpha} \textstyle\sum_{k \in I_n} Y(x_k - \ell, \vartheta) < \varepsilon.$$

It follows that $(G, B, Y)_\lambda^\alpha - \lim x_{\rho_k} = \ell$.

***Theorem 2.5:*** Let $0 < \alpha \leq 1$. Then $(G, B, Y)^\alpha \subset (G, B, Y)_\lambda^\alpha$.

***Proof:*** Take $(G, B, Y)^\alpha - \lim x = \ell$. We can write

$$\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} G(x_k - \ell, \vartheta) = \frac{1}{\lambda_n^\alpha} \sum_{k=1}^{n} G(x_k - \ell, \vartheta) - \frac{1}{\lambda_n^\alpha} \sum_{k=1}^{n-\lambda_n} G(x_k - \ell, \vartheta)$$

$$= \frac{(n)^\alpha}{\lambda_n^\alpha} \left( \frac{1}{(n)^\alpha} \sum_{k=1}^{n} G(x_k - \ell, \vartheta) \right) - \frac{(n-\lambda_n)^\alpha}{\lambda_n^\alpha} \left( \frac{1}{(n-\lambda_n)^\alpha} \sum_{k=1}^{n-\lambda_n} G(x_k - \ell, \vartheta) \right)$$

$$> \frac{(n)^\alpha}{\lambda_n^\alpha} \left( \frac{1}{(n)^\alpha} \sum_{k=1}^{n} G(x_k - \ell, \vartheta) \right) - \frac{(n)^\alpha}{\lambda_n^\alpha} \left( \frac{1}{(n-\lambda_n)^\alpha} \sum_{k=1}^{n-\lambda_n} G(x_k - \ell, \vartheta) \right).$$

From here, $\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} B(x_k - \ell, \vartheta) < \varepsilon$ and $\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} Y(x_k - \ell, \vartheta) < \varepsilon$ are obtained. Thus, $(G, B, Y)_\lambda^\alpha - \lim x = \ell$.

***Theorem 2.6:*** Let $\lambda = (\lambda_n)$, $\mu = (\mu_n) \in \Lambda$ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$, $\alpha$ and $\beta$ be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$. If

$$\lim_{n\to\infty} \inf \frac{(\mu_n)^\beta}{(\lambda_n)^\alpha} > 0 \text{ and } \lim_{n\to\infty} \frac{\mu_n}{(\lambda_n)^\beta} = 1 \tag{1}$$

holds and $A \subset F$ is neutrosophic-bounded $(NB)$ in $NNS\, V$, then $(G, B, Y)_\lambda^\alpha \subset (G, B, Y)_\mu^\beta$.

***Proof:*** Let $x \in (G, B, Y)_\lambda^\alpha$ and assume that (1) holds. Since $A \subset F$ is neutrosophic-bounded $(NB)$ in $NNS\, V$, then there exists some $\vartheta > 0$ such that

$\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} G(x_k - \ell, \vartheta) > 1 - \varepsilon$ and $\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} B(x_k - \ell, \vartheta) < \varepsilon$, $\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} Y(x_k - \ell, \vartheta) < \varepsilon$ for each $(x_k - \ell) \in A$. Now, since $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$, we may write

$$\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} G(x_k - \ell, \vartheta) \leq \frac{1}{\lambda_n^\alpha} \sum_{k \in J_n} G(x_k - \ell, \vartheta)$$

$$=\frac{(\mu_n)^\beta}{(\lambda_n)^\alpha}\frac{1}{(\mu_n)^\beta}\sum_{k\in J_n}G(x_k-\ell,\vartheta)$$

for all $n\in\mathbb{N}$. Therefore, we obtain

$$\frac{1}{(\mu_n)^\beta}\sum_{k\in J_n}B(x_k-\ell,\vartheta)=\frac{1}{(\mu_n)^\beta}\sum_{k\in J_n-I_n}B(x_k-\ell,\vartheta)+\frac{1}{(\mu_n)^\beta}\sum_{k\in I_n}B(x_k-\ell,\vartheta)$$

$$\leq\frac{\mu_n-\lambda_n}{(\mu_n)^\beta}\varepsilon+\frac{1}{(\mu_n)^\beta}\sum_{k\in I_n}B(x_k-\ell,\vartheta)$$

$$\leq\frac{\mu_n-(\lambda_n)^\beta}{(\lambda_n)^\beta}\varepsilon+\frac{1}{(\lambda_n)^\alpha}\sum_{k\in I_n}B(x_k-\ell,\vartheta)$$

$$\leq\left(\frac{\mu_n}{(\lambda_n)^\beta}-1\right)\varepsilon+\frac{1}{(\lambda_n)^\alpha}\sum_{k\in I_n}B(x_k-\ell,\vartheta)$$

for every $n\in\mathbb{N}$. Therefore $\frac{1}{(\mu_n)^\beta}\sum_{k\in J_n}G(x_k-\ell,\vartheta)>1-\varepsilon$ and $\frac{1}{(\mu_n)^\beta}\sum_{k\in J_n}B(x_k-\ell,\vartheta)<\varepsilon$. It can be shown to be $\frac{1}{(\mu_n)^\beta}\sum_{k\in J_n}Y(x_k-\ell,\vartheta)<\varepsilon$ by similar operations. $(G,B,Y)_\lambda^\alpha\subset(G,B,Y)_\mu^\beta$ is obtained as the result.

Thus in the light of Theorem 2.6, we have the following result:

***Corollary 2.7:*** Let $\lambda=(\lambda_n)$, $\mu=(\mu_n)\in\Lambda$ such that $\lambda_n\leq\mu_n$ for all $n\in\mathbb{N}$. If (1) holds and $NB$ then,

(i) $(G,B,Y)_\lambda^\alpha\subset(G,B,Y)_\mu$ for $0<\alpha\leq 1$,

(ii) $(G,B,Y)_\lambda\subset(G,B,Y)_\mu$.

***Definition 2.8:*** Take an $NNS\ V$. A sequence $x=(x_k)$ is named to be strongly $\lambda-$Cauchy of order $\alpha$ with regards to $NN\ N$ $(LC\alpha-NN)$, if for every $\varepsilon\in(0,1)$ and $\vartheta>0$, there are $n_0,p\in\mathbb{N}$ is satisfying

$$\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}G(x_k-x_p,\vartheta)>1-\varepsilon \text{ and } \frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}B(x_k-x_p,\vartheta)<\varepsilon,\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}Y(x_k-x_p,\vartheta)<\varepsilon$$

for all $n\geq n_0$.

***Theorem 2.9:*** If a sequence $x=(x_k)$ in $NN\ S$ is strongly $\lambda-$convergent of order $\alpha$ with regards to $NN\ N$, then it is strongly Cauchy of order $\alpha$ with regards to $NN\ N$.

***Proof:*** Let $(G,B,Y)_\lambda^\alpha-\lim x=\ell$. Select $\varepsilon>0$. Then, for a given $\rho\in(0,1)$, $(1-\rho)*(1-\rho)>1-\varepsilon$ and $p\lozenge p<\varepsilon$. Then, we have

$\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}G\left(x_k-\ell,\frac{\vartheta}{2}\right)>1-\rho$ and $\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}B\left(x_k-\ell,\frac{\vartheta}{2}\right)<\rho,\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}Y\left(x_k-\ell,\frac{\vartheta}{2}\right)<\rho.$

We have to show that

$$\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}G(x_k-x_m,\vartheta)>1-\varepsilon \text{ and } \frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}B(x_k-x_m,\vartheta)<\varepsilon,\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}Y(x_k-x_m,\vartheta)<\varepsilon.$$

We have three possible cases

Case (i) we get for $\vartheta > 0$

$$G(x_k - x_m, \vartheta) \geq G\left(x_k - \ell, \frac{\vartheta}{2}\right) * G\left(x_m - \ell, \frac{\vartheta}{2}\right) > (1-\rho) * (1-\rho) > 1 - \varepsilon.$$

Case (ii) we obtain

$$B(x_k - x_m, \vartheta) \leq B\left(x_k - \ell, \frac{\vartheta}{2}\right) \lozenge B\left(x_m - \ell, \frac{\vartheta}{2}\right) < p \lozenge p < \varepsilon.$$

Case (iii) we have

$$Y(x_k - x_m, \vartheta) \leq Y\left(x_k - \ell, \frac{\vartheta}{2}\right) \lozenge Y\left(x_m - \ell, \frac{\vartheta}{2}\right) < p \lozenge p < \varepsilon.$$

This shows that $(x_k)$ is strongly Cauchy of order $\alpha$ with regards to $NN\,N$.

## III. Conclusion and Recommendations

In this paper, we define strongly $\lambda -$convergence of order $\alpha$ of sequences with in $NNS$. Some fundamental theorems and example are given. The notation of strongly $\lambda -$Cauchy of order $\alpha$ for sequences in $NNS$ is examined and relations between this new concepts is established.

The results of the paper are expected to be a source for sequences and applications in $NNS$. In future studies on this topic, it is also possible $\lambda -$ ideal convergence in $NNS$.

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