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Well-Posedness of History/State-Dependent Implicit Sweeping Processes

Shengda Zeng[1,2] · **Emilio Vilches**[3]

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Abstract
This paper is devoted to the study of a new class of implicit state-dependent sweeping processes with history-dependent operators. Based on the methods of convex analysis, we prove the equivalence of the history/state dependent implicit sweeping process and a nonlinear differential equation, which, through a fixed point argument for history-dependent operators, enables us to prove the existence, uniqueness, and continuous dependence of the solution in a very general framework. Moreover, we present some new convergence results with respect to perturbations in the data, including perturbations of the associated moving sets. Finally, the theoretical results are applied to prove the well-posedness of a history-dependent quasi-static contact problem.

Keywords Implicit sweeping process · History-dependent operator · Frictional contact problem · Viscoelastic material · Unilateral constraints · Moreau's sweeping process · Evolution variational inequality

Mathematics Subject Classification 49J40 · 47J20 · 47J22 · 35L15 · 35L86 · 35L87 · 74HXX · 74M10

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✉ Emilio Vilches
emilio.vilches@uoh.cl

Shengda Zeng
zengshengda@163.com

[1] Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, P.R. China

[2] Jagiellonian University in Krakow, Chair of Optimization and Control, ul. Lojasiewicza 6, 30348 Krakow, Poland

[3] Universidad de O'Higgins, Rancagua, Chile

1 Introduction

The Moreau's sweeping process is a first-order differential inclusion, involving the normal cone to a moving set depending on time. Roughly speaking, a point is swept by a moving closed set. The sweeping process was introduced and deeply studied by Moreau in a series of papers (see [1–3]) to model an elasto-plastic mechanical system. Since then, many other applications have been given, such as applications in switched electrical circuits [4], nonsmooth mechanics [5,6], crowd motion [7], hysteresis in elasto-plastic models [8], among others. Moreover, due to the development of new techniques to deal with differential inclusions involving normal cones, new variants of the sweeping process have been introduced. We can mention the state-dependent sweeping process, the second-order sweeping process, the implicit sweeping process [9–12], and some others variants. For more details, we refer to [13–17] and the references therein. The aim of this paper is to study the existence, uniqueness, and stability for a class of history/state-dependent implicit sweeping processes. The latter was introduced in [18] (for the special, case where the moving sets are state-independent) to model a history-dependent viscoelastic contact problem (see [18, Sect. 4]). Moreover, the history-dependent implicit sweeping process includes several others models as studied in [9–11].

In this paper, we aim at extending the results from [11,18] to a more general framework, by using tools from differential equations and convex analysis. Specially, under general assumptions, we show the equivalence of the history/state-dependent implicit sweeping process and a nonlinear differential equation for which the existence can be obtained through a fixed point theorem for history-dependent operators. This equivalence result will play a crucial role in this work, because it allows, on the one hand, to apply classical existence theorems from differential equations to study the implicit sweeping processes and, on the other hand, to investigate its parametric stability.

The paper is organized as follows. In Sect. 2, we recall some preliminary material and gather the hypotheses needed in the study of history/state-dependent implicit sweeping processes. Then, Sect. 3 establishes the equivalence of the history/state-dependent sweeping process and a nonlinear differential equation, as well as proves the existence, uniqueness and continuous dependence of the solution for the history/state-dependent sweeping process. In Sect. 4, we give a stability theorem and a convergence result for history/state-dependent sweeping process. In Sect. 5, as an illustrative application, we study a quasi-static contact problem for viscoelastic materials in which, the constitutive law is given by a history-dependent process. The paper ends with some concluding remarks.

2 Mathematical Background and Hypotheses

Let H be a separable Hilbert space endowed with a scalar product $\langle \cdot, \cdot \rangle$ and unit ball $\mathbb{B}$. The weak convergence of a sequence (x_n) to x is denoted by $x_n \rightharpoonup x$. $B(H)$ denotes the set of linear and continuous operators over H.

Given a closed and convex set $S \subset H$ we define, the convex normal cone to S at $x \in S$ as $N_S(x) := \{\zeta \in H \colon \langle \zeta, y - x \rangle \leq 0 \text{ for all } y \in S\}$. For a closed and convex set $S \subset H$, we consider the distance function $d_S : H \to \mathbb{R}$ and the projection $\operatorname{proj}_S : H \to S$ over S as the maps

$$d_S(x) := \inf_{y \in S} \|x - y\|, \quad \operatorname{proj}_S(x) := \{y \in H : d_S(x) = \|x - y\|\}.$$

Clearly, for a closed and convex set S, the map $x \mapsto d_S^2(x)$ is Fréchet differentiable with $\nabla d_S^2(x) = 2\left(x - \operatorname{proj}_S(x)\right)$ for all $x \in H$ (see, e.g., [19, Corollary 12.31]). Moreover, the following inclusion holds

$$x - \operatorname{proj}_S(x) \in N_S\left(\operatorname{proj}_S(x)\right) \quad \text{for all } x \in H. \tag{1}$$

The last inclusion indicates that the map $x \mapsto (I + N_S(\cdot))^{-1}(x)$, where I stands for the identity operator in H, is single-valued and coincides with the projection onto the closed convex set S, that is,

$$(I + N_S(\cdot))^{-1}(x) = \operatorname{proj}_S(x) \quad \text{for all } x \in H. \tag{2}$$

Given two nonempty closed sets $A, B \subset H$, we recall the Hausdorff distance between A and B is defined by $\mathcal{H}(A, B) = \max\{\sup_{x \in A} d_B(x), \sup_{x \in B} d_A(x)\}$. Moreover, for two closed and convex sets $A, B \subset H$ the following inequality holds (see [3, Formula 2.17]):

$$\left\|\operatorname{proj}_A(x) - \operatorname{proj}_B(y)\right\|^2 \leq \|x - y\|^2 + 2\left[d_A(x) + d_B(y)\right]\mathcal{H}(A, B). \tag{3}$$

The following result states continuity properties for the projection onto state-dependent closed and convex sets.

Lemma 2.1 *Let Q be a symmetric and invertible operator in $B(H)$. Let $C : [0, T] \times H \rightrightarrows H$ be a set-valued map with nonempty, closed and convex values. Assume that the following conditions hold.*

(a) *For all $x \in H$, the set-valued map $t \mapsto C(t, x)$ is measurable (that is,* graph $C(\cdot, x)$ *is measurable (see, e.g.,* [20, Chapter 3]) *such that there exists a non negative constant $\mu \geq 0$ such that for a.e. $t \in [0, T]$ and all $x \in H$*

$$d_{C(t,x)}(0) \leq \mu \cdot (\|x\| + 1).$$

(b) *The normal cone of C is hypomonotone-like, in the sense that for a given $r > 0$, there exists $\tilde{L}_r \geq 0$ (independent of t) such that if*

$$a_i \in N_{C(t,u_i)}(b_i) \text{ for } a_i \in H, u_i, b_i \in r\mathbb{B}, i = 1, 2 \text{ and } t \in [0, T],$$

then

$$\langle a_1 - a_2, b_1 - b_2 \rangle \geq -\tilde{L}_r \|a_1 - a_2\| \, \|u_1 - u_2\|. \tag{4}$$

Then the following assertions hold.

(i) *For all $x, y \in H$, the map $t \mapsto \operatorname{proj}_{QC(t,x)}(y)$ is measurable.*

(ii) *For all $y \in H$ and a.e. $t \in [0, T]$, the map $x \mapsto \operatorname{proj}_{QC(t,x)}(y)$ is $\|Q\| \cdot \tilde{L}_\rho$-Lipschitz continuous on $r\mathbb{B}$, where $\rho := \max\{\|y\| + \mu\|Q\|(r+1), r\}$.*

Proof Since the sets $C(t, x)$ are closed and convex and the operator Q is invertible, then the sets $QC(t, x)$ are closed and convex. Therefore, the projection is single-valued and well defined. The first assertion follows from [21, Theorem III.41]. To prove (ii), fix $y \in H$ and consider $x_1, x_2 \in r\mathbb{B}$. Define $z_i = \operatorname{proj}_{QC(t,x_i)}(y)$ for $i = 1, 2$. Then, for $i = 1, 2$,

$$\begin{aligned}\|z_i\| &\le \|\operatorname{proj}_{QC(t,x_i)}(y) - \operatorname{proj}_{QC(t,x_i)}(0)\| + \|\operatorname{proj}_{QC(t,x_i)}(0)\| \\ &\le \|y\| + \|Q\|\|\operatorname{proj}_{C(t,x_i)}(0)\| \\ &\le \|y\| + \mu\|Q\|\,(\|x_i\| + 1) \le \|y\| + \mu\|Q\|(r+1).\end{aligned}$$

Moreover, by virtue of formula (1), we obtain that $y - z_i \in N_{QC(t,x_i)}(z_i)$ for $i = 1, 2$, which is equivalent to $Q(y - z_i) \in N_{C(t,x_i)}\left(Q^{-1}z_i\right)$ for $i = 1, 2$. Thus, by property (4), we get that $\|z_1 - z_2\|^2 \le \tilde{L}_\rho\|Q\|\|z_1 - z_2\|\,\|x_1 - x_2\|$, which implies (ii). □

Remark 2.1 The hypothesis (4) is a weakening of [11, Assumption 2], where similar conditions were introduced to deal with state-dependent implicit sweeping processes.

Definition 2.1 An operator $\mathcal{R}\colon C([0, T]; H) \to C([0, T]; H)$ is called history-dependent if there exists a constant $L_{\mathcal{R}} \ge 0$ such that

$$\|(\mathcal{R}x)(t) - (\mathcal{R}y)(t)\| \le L_{\mathcal{R}} \int_0^t \|x(s) - y(s)\|\,\mathrm{d}s,\; x, y \in C([0, T]; H), t \in [0, T].$$

An important property of history-dependent operators is provided by the following fixed point principle (see [22, Theorem 25]).

Lemma 2.2 *If $\mathcal{R}\colon C([0, T]; H) \to C([0, T]; H)$ is a history-dependent operator, then there exists a unique function $x^* \in C([0, T]; H)$ such that $\mathcal{R}x^* = x^*$.*

Now, to study the stability of history/state-dependent implicit sweeping processes, we recall the concept of Mosco convergence (see, e.g., [23]).

Definition 2.2 (*Mosco convergence*) Let (C_n) be a sequence of closed subsets of H and $C \subset H$. We say that the sequence $(C_n)_n$ *Mosco converges* to C, if $\text{w-}\limsup C_n \subset \liminf C$. In that case, we write

$$\text{M-}\lim C_n := \liminf C_n = \limsup C_n = \text{w-}\liminf C_n = \text{w-}\limsup C_n.$$

Clearly, in finite dimension Mosco convergence is equivalent to convergence in the sense of Painlevé–Kuratowski.

The following result points out the relationship between Mosco convergence and convergence of projections (see, e.g., [23, Proposition 3.33]).

Proposition 2.1 *Let $(C_n)_n$ be a sequence of nonempty, closed, and convex subsets of H. Then the following statements are equivalent:*

(i) $C = \text{M-}\lim C_n$.
(ii) *For all $x \in H$, $(\text{proj}_{C_n}(x))_n$ converges strongly to $\text{proj}_C(x)$ as $n \to +\infty$.*
(iii) *For all $x \in H$, $d_{C_n}(x)$ converges to $d_C(x)$ as $n \to +\infty$.*

As a product of Proposition 2.1, we obtain the following convergence result.

Lemma 2.3 *Let $C \subset H$ be a nonempty, closed and convex set with $\text{proj}_C(0)$ bounded. If $(Q_n) \subset B(H)$ is a sequence of invertible operators converging to an invertible operator $Q \in B(H)$, then $QC = \text{M-}\, Q_n C$.*

Proof By virtue of Proposition 2.1, it is enough to prove that for all $x \in H$, $d_{Q_nC}(x) \to d_{QC}(x)$ as $n \to +\infty$.
On the one hand, taking $z := Q^{-1}\,\text{proj}_{QC}(y) \in C$, we obtain

$$d_{Q_nC}(x) \leq \|x - Q_n z\| \leq \|x - Qz\| + \|Qz - Q_n z\| = d_{QC}(x) + \|Q_n - Q\|\,\|z\|,$$

which implies that $\limsup_{n\to+\infty} d_{Q_nC}(x) \leq d_{QC}(x)$.
On the other hand, taking $z_n := Q_n^{-1}\,\text{proj}_{Q_nC}(x) \in C$, we obtain

$$\begin{aligned} d_{QC}(x) &\leq \|x - Qz_n\| \leq \|x - Q_n z_n\| + \|Q_n z_n - Qz_n\| \\ &= d_{Q_nC}(x) + \|\,\text{proj}_{Q_nC}(x) - QQ_n^{-1}\,\text{proj}_{Q_nC}(y)\| \\ &\leq d_{Q_nC}(x) + \|\,\text{proj}_{Q_nC}(x)\| \cdot \|I - QQ_n^{-1}\| \\ &\leq d_{Q_nC}(x) + \|x\| \cdot \|I - QQ_n^{-1}\| + \|\,\text{proj}_{Q_nC}(0)\| \cdot \|I - QQ_n^{-1}\| \\ &\leq d_{Q_nC}(x) + \|x\| \cdot \|I - QQ_n^{-1}\| + \|Q_n\|\,\|\,\text{proj}_C(0)\| \cdot \|I - QQ_n^{-1}\|, \end{aligned}$$

which, since $\text{proj}_C(0)$ is bounded, implies that $d_{QC}(x) \leq \liminf_{n\to+\infty} d_{Q_nC}(x)$. □

For the sake of readability, furthermore, we collect the hypotheses used along with the paper.

Hypotheses on the operator $A\colon H \to H$:

($\mathcal{H}_A$) $A\colon H \to H$ is linear, bounded, symmetric, and coercive with constant $\alpha > 0$, i.e., $\langle Ax, x\rangle \geq \alpha\|x\|^2$ for all $x \in H$.

Remark 2.2 Under ($\mathcal{H}_A$), it is well known that there exists an invertible operator $P \in B(H)$ such that $A = PP$. In what follows, we denote $Q := (P)^{-1}$.

Hypotheses on the map $B\colon H \to H$:

($\mathcal{H}_B$) $B\colon H \to H$ is a Lipschitz continuous mapping, that is, there exists $L_B \geq 0$ such that $\|Bx - By\| \leq L_B\|x - y\|$ for all $x, y \in H$.

Hypotheses on the map $\mathcal{R}\colon C([0, T]; H) \to C([0, T]; H)$:

$(\mathcal{H}_{\mathcal{R}})$ $\mathcal{R}\colon C([0,T];H) \to C([0,T];H)$ is a history-dependent operator, that is, there exists $L_{\mathcal{R}} \geq 0$ such that for all $x, y \in C([0,T];H)$, and $t \in [0,T]$

$$\|(\mathcal{R}x)(t) - (\mathcal{R}y)(t)\| \leq L_{\mathcal{R}} \int_0^t \|x(s) - y(s)\|\,\mathrm{d}s.$$

Hypotheses on the set-valued map $C\colon [0,T] \times H \rightrightarrows H$:

$(\mathcal{H}_{Cx})$

(a) For each $t \in [0,T]$, $x \in H$ the set $C(t,x)$ is nonempty, closed and convex. For all $x \in H$, the set-valued map $t \mapsto C(t,x)$ is measurable and there exists a constant μ non-negative such that for a.e. $t \in [0,T]$ and all $x \in H$, $d_{C(t,x)}(0) \leq \mu(\|x\| + 1)$.

(b) The normal cone of C is hypomonotone-like, i.e., for all $r > 0$, there exists $\tilde{L}_r \geq 0$ (independent of t) such that if

$$a_i \in N_{C(t,u_i)}(b_i) \text{ for } a_i \in H, u_i, b_i \in r\mathbb{B}, i = 1,2 \text{ and } t \in [0,T],$$

then $\langle a_1 - a_2, b_1 - b_2\rangle \geq -\tilde{L}_r \|a_1 - a_2\|\,\|u_1 - u_2\|$.

3 Well-Posedness Results for History/State-Dependent Implicit Sweeping Processes

This section is devoted to study the well-posedness of history/state-dependent implicit sweeping process, including the existence, uniqueness, and continuous dependence with respect to the initial data of the problem: find $x\colon [0,T] \to H$ such that $x(0) = x_0$ and

$$-\dot{x}(t) \in N_{C(t,x(t))}(A\dot{x}(t) + Bx(t) + (\mathcal{R}x)(t)) \quad \text{a.e. } t \in [0,T]. \tag{5}$$

Here, $C\colon [0,T] \times H \rightrightarrows H$, $A\colon H \to H$, $B\colon H \to H$, and $\mathcal{R}$ are assumed to satisfy hypotheses $(\mathcal{H}_{Cx})$, $(\mathcal{H}_A)$, $(\mathcal{H}_B)$ and $(\mathcal{H}_R)$, respectively.

We follow the ideas developed in [10], to transform the differential inclusion (5) into a nonlinear differential equation. Particularly, we shall demonstrate that the differential inclusion (5) is equivalent to the following nonlinear differential equation: find $x\colon [0,T] \to H$ such that $x(0) = x_0$ and

$$\begin{aligned} \dot{x}(t) &= -P^{-1}(QBx(t) + Q(\mathcal{R}x)(t)) \\ &\quad + P^{-1}\operatorname{proj}_{QC(t,x(t))}(QBx(t) + Q(\mathcal{R}x)(t)) \quad \text{a.e. } t \in [0,T], \end{aligned} \tag{6}$$

where $A = PP$, $Q := P^{-1}$ and $QC(t,x) := \{Qy\colon y \in C(t,x)\}$.

Proposition 3.1 *Assume, in addition to $(\mathcal{H}_A)$, that $C(t,y)$ is closed and convex for all $t \in [0,T]$ and all $y \in H$. Then x is a solution of the differential inclusion (5) if and only if it is a solution of the Cauchy problem (6).*

Proof Before starting the proof, we observe that under hypothesis $(\mathcal{H}_A)$, the following formula holds:

$$\begin{aligned} &\zeta \in N_{C(t,x(t))}\left(A\dot{x}(t) + Bx(t) + (\mathcal{R}x)(t)\right) \\ &\quad \Leftrightarrow P\zeta \in N_{QC(t,x(t))}\left(P\dot{x}(t) + QBx(t) + Q(\mathcal{R}x)(t)\right), \end{aligned} \tag{7}$$

where $Q := P^{-1}$ is well defined. Let x be a solution of (5). Then, according to (7), one has for a.e. $t \in [0, T]$

$$QBx(t) + Q(\mathcal{R}x)(t) \in \left(I + N_{QC(t,x(t))}(\cdot)\right)\left(P\dot{x}(t) + QBx(t) + Q(\mathcal{R}x)(t)\right).$$

Moreover, we have that $\left(I + N_{QC(t,x(t))}(\cdot)\right)^{-1} = \text{proj}_{QC(t,x(t))}$ (see (2)). Hence, for a.e. $t \in [0, T]$

$$P\dot{x}(t) + QBx(t) + Q(\mathcal{R}x)(t) = \text{proj}_{QC(t,x(t))}\left(QBx(t) + Q(\mathcal{R}x)(t)\right),$$

which shows that x is a solution of (6). Reciprocally, let x be a solution of (6). Then, for a.e. $t \in [0, T]$,

$$\begin{aligned} P\dot{x}(t) &= -QBx(t) - Q(\mathcal{R}x)(t) + \text{proj}_{QC(t,x(t))}\left(QBx(t) + Q(\mathcal{R}x)(t)\right) \\ &\in -N_{QC(t,x(t))}\left(\text{proj}_{QC(t,x(t))}\left(QBx(t) + Q(\mathcal{R}x)(t)\right)\right) \\ &\in -N_{QC(t,x(t))}\left(P\dot{x}(t) + QBx(t) + Q(\mathcal{R}x)(t)\right), \end{aligned}$$

where we have used the inclusion (1). Therefore,

$$-P\dot{x}(t) \in N_{QC(t,x(t))}\left(P\dot{x}(t) + QBx(t) + Q(\mathcal{R}x)(t)\right) \quad \text{a.e. } t \in [0.T],$$

which proves, according to formula (7), that x is a solution of (5). □

From Proposition 3.1, we are able to state the main result of this section, concerning the well-posedness result for problem (5). It is worth noting that, contrary to the existence results for the sweeping process, we do not require that the variation of the moving sets is absolutely continuous or Lipschitz continuous in the sense of Hausdorff distance. Additionally, the following result generalizes the recent one [18, Theorem 12], by considering state-dependent moving sets without absolute continuity conditions on time. Moreover, it extends the results from [11, Theorem 3.1] and [9, Theorem 3.1], by considering history-dependent operators.

Theorem 3.1 *Assume that* $(\mathcal{H}_A)$, $(\mathcal{H}_B)$, $(\mathcal{H}_R)$ *and* $(\mathcal{H}_{Cx})$ *hold. Then for any* $x_0 \in H$ *there exists a unique* $x \in \text{AC}([0, T]; H)$ *satisfying* (5). *Moreover, the map* $x_0 \mapsto x(x_0)\colon H \to \text{AC}([0, T]; H)$ *is locally Lipschitz continuous.*

Remark 3.1 We emphasize that the method to prove Theorem 3.1 follows the ideas developed in [10], which is of a different nature to the methods used in [9,11,18].

Proof The proof is divided into four steps.
Step 1: Given $w \in C([0,T];H)$, there exists a unique solution $x\colon [0,T] \to H$ such that $x(0) = x_0$ and

$$\dot{x}(t) \in -N_{C(t,x(t))}\left(A\dot{x}(t) + Bw(t) + (\mathcal{R}w)(t)\right) \quad \text{a.e. } t \in [0,T]. \tag{8}$$

□

Proof of Step 1 Fix $R > 0$ such that $\|w\|_{C([0,T];H)} \le R$ and let us consider the map $F\colon [0,T] \times H \to H$ defined by

$$F(t,x) := -P^{-1}(z(t)) + P^{-1}\operatorname{proj}_{QC(t,x)}(z(t)),$$

where $z(t) := QBw(t) + Q(\mathcal{R}w)(t)$. Firstly, as a consequence of $(\mathcal{H}_B)$, $(\mathcal{H}_{Cx})$ and Lemma 2.1, for all $x \in H$, the map $t \mapsto F(t,x)$ is measurable on $[0,T]$. We now claim that there exist $c, d \in L^1(0,T)$ with $c(t) \ge 0$ and $d(t) \ge 0$ for all $t \in [0,T]$ (depending on w) such that for a.e. $t \in [0,T]$ and all $x \in H$

$$\|F(t,x)\| \le c(t)\|x\| + d(t).$$

In fact, by assumptions $(\mathcal{H}_B)$ and $(\mathcal{H}_{Cx})$, we find

$$\begin{aligned}
\|F(t,x)\| &\le \|P^{-1}\| d_{QC(t,x(t))}\left(QBw(t) + Q(\mathcal{R}w)(t)\right)\\
&\le \|P^{-1}\| \cdot \|QBw(t) + Q(\mathcal{R}w)(t)\| + \|P^{-1}\| d_{QC(t,x)}(0)\\
&\le \|P^{-1}\| \cdot \|QBw(t) + Q(\mathcal{R}w)(t)\| + \|P^{-1}\|\,\|Q\| d_{C(t,x)}(0)\\
&\le \|P^{-1}\| \cdot \|QBw(t) + Q(\mathcal{R}w)(t)\| + \|P^{-1}\|\,\|Q\|\,\mu \cdot (\|x\| + 1)\\
&\le c(t)\|x\| + d(t),
\end{aligned}$$

where c and d are the integrable functions defined by:

$$\begin{aligned}
c(t) &:= \|P^{-1}\|\,\|Q\|\,\mu,\\
d(t) &:= \|P^{-1}\| \cdot \|QBw(t) + Q(\mathcal{R}w)(t)\| + \|P^{-1}\|\,\|Q\|\,\mu.
\end{aligned}$$

Third, by virtue of Lemma 2.1, for all $r > 0$, a.e. $t \in [0,T]$ and all $x, y \in r\mathbb{B}$

$$\|F(t,x) - F(t,y)\| \le \|P^{-1}\|\,\|Q\|\tilde{L}_\rho\|x - y\|,$$

where $\rho := \max\{R + \mu\|Q\|(1+r), r\}$ and $\tilde{L}_\rho$ is the constant given by $(\mathcal{H}_{Cx})$. Therefore, according to [24, Theorem 10.5], the differential Eq. (6) has a unique solution $x(w)$ defined on $[0,T]$, which is absolutely continuous. Finally, the existence and uniqueness of solution for the differential inclusion (8) are obtained directly from Proposition 3.1, which ends the proof of Step 1. □

According to Step 1, for any $w \in C([0,T];H)$ there exists a unique absolutely continuous solution $x = x(w)$ of problem (8). Now, we introduce the solution mapping $\mathcal{S}\colon C([0,T];H) \to \mathrm{AC}([0,T];H) \subset C([0,T];H)$ defined by $\mathcal{S}w = x(w)$. Let $\operatorname{Fix}\mathcal{S}$ be the set of fixed points of $\mathcal{S}$, i.e.,

$$\operatorname{Fix}\mathcal{S} := \{x \in C([0,T];H) \mid x = \mathcal{S}x\}.$$

Step 2: There exists a constant $M_0 > 0$ such that for all $x \in \operatorname{Fix}\mathcal{S}$ it holds

$$\|x\|_{C([0,T];H)} \leq M_0. \tag{9}$$

Proof of Step 2 Indeed, let $x \in \operatorname{Fix}\mathcal{S}$, i.e., $x(0) = x_0$ and for a.e. $t \in [0,T]$

$$-\dot{x}(t) = P^{-1}\left(QBx(t) + Q(\mathcal{R}x)(t)\right) - P^{-1}\operatorname{proj}_{QC(t,x(t))}\left(QBx(t) + Q(\mathcal{R}x)(t)\right).$$

Hence, for all $t \in [0,T]$, it has

$$\begin{aligned} x(t) = x_0 &- P^{-1}\int_0^t \left(QBx(s) + Q(\mathcal{R}x)(s)\right) \mathrm{d}s \\ &+ P^{-1}\int_0^t \operatorname{proj}_{QC(s,x(s))}\left(QBx(s) + Q(\mathcal{R}x)(s)\right) \mathrm{d}s. \end{aligned}$$

Therefore, for all $t \in [0,T]$, the following estimates hold

$$\begin{aligned} \|x(t)\| &\leq \|x_0\| + \|P^{-1}\| \int_0^t d_{QC(s,x(s))}\left(QBx(s) + Q(\mathcal{R}x)(s)\right) \mathrm{d}s \\ &\leq \|x_0\| + \|P^{-1}\| \int_0^t \|QBx(s) + Q(\mathcal{R}x)(s)\| \mathrm{d}s + \|P^{-1}\| \int_0^t d_{QC(s,x(s))}(0)\, \mathrm{d}s \\ &\leq \|x_0\| + \|P^{-1}\|\, \|Q\| \int_0^t \left(\|Bx(s) - B0\| + \|(\mathcal{R}x)(s) - (\mathcal{R}0)(s)\|\right) \mathrm{d}s \\ &\quad + \|P^{-1}\|\, \|Q\| T \left(\|B0\| + \|\mathcal{R}0\|_{C([0,T];H)}\right) + \|P^{-1}\|\, \|Q\| \int_0^t \mu\left(\|x(s)\| + 1\right) \mathrm{d}s \\ &\leq \|P^{-1}\|\, \|Q\| \left(L_B + L_{\mathcal{R}}T\right) \int_0^t \|x(s)\| \mathrm{d}s + \|P^{-1}\|\, \|Q\| \int_0^t \mu\|x(s)\| \mathrm{d}s \\ &\quad + \|x_0\| + \|P^{-1}\|\, \|Q\| T \left(\|\mathcal{R}0\|_{C([0,T];H)} + \|B0\|\right) + \|P^{-1}\|\, \|Q\| \mu T. \end{aligned}$$

Thus, from Gronwall's inequality, we obtain that for all $t \in [0,T]$

$$\|x(t)\| \leq k_0 \exp\left(\|P^{-1}\|\, \|Q\| \int_0^t \left(L_B + L_{\mathcal{R}}T + \mu\right) \mathrm{d}s\right),$$

where k_0 is defined by

$$k_0 = \|x_0\| + \|P^{-1}\|\, \|Q\| T \left(\|\mathcal{R}0\|_{C([0,T];H)} + \|B0\|\right) + \|P^{-1}\|\, \|Q\| \mu T.$$

Therefore, inequality (9) holds for $M_0 > 0$ given by

$$M_0 := k_0 \exp\Big(\|P^{-1}\|\,\|Q\|(L_B + L_{\mathcal{R}}T)T + \|P^{-1}\|\,\|Q\|\mu T\Big),$$

which ends the proof of Step 2. □

We introduce the M_0-radial retraction, $P_{M_0}\colon C([0,T];H) \to C([0,T];H)$, defined by

$$P_{M_0}(x) := \begin{cases} x & \text{if } \|x\|_{C([0,T];H)} \le M_0, \\ \dfrac{M_0 x}{\|x\|_{C([0,T];H)}} & \text{if } \|x\|_{C([0,T];H)} > M_0. \end{cases}$$

It is obvious that P_{M_0} is uniformly bounded and Lipschitz continuous. Further, consider the mapping $\mathcal{S}_{M_0}\colon C([0,T];H) \to C([0,T];H)$ given by

$$\mathcal{S}_{M_0}(w) := \mathcal{S}\left(P_{M_0}(w)\right) \quad \text{for all } w \in C([0,T];H).$$

It is clear from Step 2 and the definition of P_{M_0} that $\operatorname{Fix}\mathcal{S} \subset \operatorname{Fix}\mathcal{S}_{M_0}$.
Step 3: The operator $\mathcal{S}_{M_0}$ is history-dependent.

Proof of Step 3 Let $w_i \in C([0,T];H)$ and denote $x_i = \mathcal{S}_{M_0}(w_i)$ for $i = 1, 2$. We observe that $\|P_{M_0}(w)\|_{C([0,T];H)} \le M_0$ for all $w \in C([0,T];H)$, and thus, it is possible to find $r > 0$ such that for all $w \in C([0,T];H)$

$$\max\{\|\mathcal{S}_{M_0}(w)\|_{C([0,T];H)}, \|A\dot{x}_i + B(P_{M_0}(w)) + \mathcal{R}(P_{M_0}(w))\|_\infty\} \le r \quad i = 1, 2.$$

Next, due to $(\mathcal{H}_{Cx})$, it follows that for a.e. $t \in [0,T]$

$$\begin{aligned} &\langle -\dot{x}_1(t) + \dot{x}_2(t), \big(A\dot{x}_1(t) + B(P_{M_0}(w_1))(t) + \mathcal{R}(P_{M_0}(w_1))(t)\big)\rangle \\ &\quad - \langle -\dot{x}_1(t) + \dot{x}_2(t), \big(A\dot{x}_2(t) + B(P_{M_0}(w_2))(t) + \mathcal{R}(P_{M_0}(w_2))(t)\big)\rangle \\ &\ge -\tilde{L}_r\|\dot{x}_1(t) - \dot{x}_2(t)\|\,\|x_1(t) - x_2(t)\|, \end{aligned}$$

which implies that for a.e. $t \in [0,T]$

$$\begin{aligned} &\langle A\dot{x}_1(t) - A\dot{x}_2(t), \dot{x}_1(t) - \dot{x}_2(t)\rangle \\ &\quad \le \tilde{L}_r\|\dot{x}_1(t) - \dot{x}_2(t)\|\,\|x_1(t) - x_2(t)\| \\ &\qquad + \|\dot{x}_1(t) - \dot{x}_2(t)\|\,\|B(P_{M_0}(w_1))(t) - B(P_{M_0}(w_2))(t)\| \\ &\qquad + \|\dot{x}_1(t) - \dot{x}_2(t)\|\,\|(\mathcal{R}(P_{M_0}(w_1))(t) - (\mathcal{R}(P_{M_0}(w_2))(t)\|. \end{aligned} \tag{10}$$

Moreover, according to $(\mathcal{H}_A)$, for a.e. $t \in [0,T]$, it reads

$$\alpha\|\dot{x}_1(t) - \dot{x}_2(t)\|^2 \le \langle A\dot{x}_1(t) - A\dot{x}_2(t), \dot{x}_1(t) - \dot{x}_2(t)\rangle. \tag{11}$$

Thus, by combining inequalities (10) and (11), we obtain that for a.e. $t \in [0, T]$

$$\begin{aligned} \|\dot{x}_1(t) - \dot{x}_2(t)\| &\le \frac{\tilde{L}_r}{\alpha}\|x_1(t) - x_2(t)\| + \frac{1}{\alpha}\|B(P_{M_0}(w_1))(t) - B(P_{M_0}(w_2))(t)\| \\ &\quad + \frac{1}{\alpha}\|(\mathcal{R}(P_{M_0}(w_1))(t) - (\mathcal{R}(P_{M_0}(w_2))(t)\| \\ &\le \frac{\tilde{L}_r}{\alpha}\|x_1(t) - x_2(t)\| + \frac{L_B L_{P_{M_0}}}{\alpha}\|w_1(t) - w_2(t)\| \\ &\quad + \frac{L_{\mathcal{R}} L_{P_{M_0}}}{\alpha}\int_0^t \|w_1(s) - w_2(s)\|\,\mathrm{d}s, \end{aligned} \tag{12}$$

where $L_{P_{M_0}} > 0$ is the Lipschitz constant of P_{M_0}. Notice that for all $t \in [0, T]$

$$\|x_1(t) - x_2(t)\| = \left\| \int_0^t (\dot{x}_1(s) - \dot{x}_2(s))\mathrm{d}s \right\| \le \int_0^t \|\dot{x}_1(s) - \dot{x}_2(s)\|\mathrm{d}s,$$

integrating (12) over $[0, t]$ implies for all $t \in [0, T]$

$$\begin{aligned} \|x_1(t) - x_2(t)\| &\le \frac{\tilde{L}_r}{\alpha}\int_0^t \|x_1(s) - x_2(s)\|\mathrm{d}s + \frac{L_B L_{P_{M_0}}}{\alpha}\int_0^t \|w_1(s) - w_2(s)\|\mathrm{d}s \\ &\quad + \frac{L_{\mathcal{R}} T L_{P_{M_0}}}{\alpha}\int_0^t \|w_1(s) - w_2(s)\|\mathrm{d}s \end{aligned}$$

Now, we are in position to use to Gronwall's inequality (see, e.g., [25, Proposition 2.4.1]), to obtain that for all $t \in [0, T]$

$$\|x_1(t) - x_2(t)\| \le \frac{1}{\alpha}\exp\left(\frac{\tilde{L}_r T}{\alpha}\right)\left(L_B L_{P_{M_0}} + L_{\mathcal{R}} T L_{P_{M_0}}\right)\int_0^t \|w_1(s) - w_2(s)\|\mathrm{d}s,$$

which is equivalent to

$$\begin{aligned} &\|\mathcal{S}_{M_0}(w_1)(t) - \mathcal{S}_{M_0}(w_2)(t)\| \\ &\quad \le \frac{1}{\alpha}\exp\left(\frac{\tilde{L}_r T}{\alpha}\right)\left(L_B L_{P_{M_0}} + L_{\mathcal{R}} T L_{P_{M_0}}\right)\int_0^t \|w_1(s) - w_2(s)\|\mathrm{d}s \end{aligned}$$

for all $t \in [0, T]$. This means $\mathcal{S}_{M_0}$ is a history-dependent operator. □

Step 4: The operator $\mathcal{S}_{M_0}$ has a unique fixed point.

Proof of Step 4 By using Lemma 2.2, the operator $\mathcal{S}_{M_0}$ has a unique fixed point $x^* \in C([0, T]; H)$, i.e., $x^* = \mathcal{S}_{M_0} x^*$. However, since the operator $\mathcal{S}_{M_0}$ takes values in the space $\mathrm{AC}([0, T]; H)$, we deduce that $x^* \in \mathrm{AC}([0, T]; H)$ too. A simple calculation shows that $\|x^*\|_{C([0,T];H)} \le M_0$ (see Step 2). Therefore, $x^* = \mathcal{S}_{M_0}(x^*) = \mathcal{S}(x^*)$. We conclude that x^* is also the unique fixed point of $\mathcal{S}$. Hence $x^* = \mathcal{S}x^*$ is a solution to problem (5), which proves the existence part in Theorem 3.1. Whereas, the uniqueness part follows directly from the uniqueness of the fixed point operator $\mathcal{S}$, guaranteed by Lemma 2.2. □

Step 5: The operator $x_0 \mapsto x(x_0)\colon H \to \mathrm{AC}\,([0,T];H)$ is locally Lipschitz continuous.

Proof of Step 5 Let $x_0^1, x_2^2 \in H$, and for $i = 1, 2$ denote by $x_i \in \mathrm{AC}\,([0,T];H)$ the solution of problem (5) corresponding to the initial data x_0^i. By the same arguments used in Step 2, it is not difficult to find $r > 0$ and $\tilde{L}_r > 0$ (depending on x_0^1 and x_0^2) such that for all $t \in [0,T]$ and $i = 1, 2$,

$$\max\{\|x\|_{C([0,T];H)}, \|A\dot{x} + B(x) + \mathcal{R}(x)\|_\infty\} \leq r,$$
$$\|x_1(t) - x_2(t)\| \leq \|x_0^1 - x_0^2\| + \frac{1}{\alpha}\mathrm{e}^{\tilde{L}_r T/\alpha}\,(L_B + T L_{\mathcal{R}}) \int_0^t \|x_1(s) - x_2(s)\|.$$

Thus, by applying Gronwall's inequality (see, e.g., [25, Proposition 2.4.1]), we find that

$$\|x_1(t) - x_2(t)\| \leq L_0^r \|x_0^1 - x_0^2\| \quad \text{for all } t \in [0,T], \tag{13}$$

for some constant $L_0^r > 0$. Moreover, from (13), we have (see (12))

$$\|\dot{x}_1(t) - \dot{x}_2(t)\| \leq L_1^r \|x_0^1 - x_0^2\| \quad \text{for a.e. } t \in [0,T], \tag{14}$$

for some $L_1^r > 0$. Finally, combining the inequalities (13) and (14), we obtain

$$\|x_1 - x_2\|_{\mathrm{AC}([0,T];H)} \leq (L_0^r + L_1^r)\|x_0 - x_1\|,$$

which completes the proof of the theorem. □

Remark 3.2 A detailed analysis of the previous proof reveals that Theorem 3.1 is still valid if in $(\mathcal{H}_{Cx})$ we request that $\mu \in L^1(0,T)$ and $\tilde{L}_r \equiv L$. We denote these hypotheses as $(\mathcal{H}'_{Cx})$.

The following result, consequence of Theorem 3.1, Proposition 3.1 and (3), reveals that the unique solution of (5) is Lipschitz provided the variation of the moving sets is continuous, which generalizes the main result of [18].

Corollary 3.1 *Assume, in addition to the hypotheses of Theorem 3.1, that exist $L_C \geq 0$ and a continuous function $v\colon [0,T] \to \mathbb{R}_+$ such that*

$$\sup_{z\in H} |d_{C(t,x)}(z) - d_{C(s,y)}(z)| \leq |v(t) - v(s)| + L_C\|x - y\| \quad x, y \in H, s, t \in [0,T].$$

Then, the unique solution given by Theorem 3.1 belongs to $W^{1,\infty}([0,T];H)$. Moreover, the operator $x_0 \mapsto x(x_0)\colon H \to W^{1,\infty}([0,T];H)$ is Lipschitz.

We end this section, by considering the case $\mathcal{R} \equiv 0$. The following result generalizes the main result of [11], by relaxing the hypotheses on the continuity on the moving sets (see [11, Theorem 3.1]).

Corollary 3.2 *Assume that hypotheses* $(\mathcal{H}_A)$, $(\mathcal{H}_B)$ *and* $(\mathcal{H}_{Cx})$ *hold. Then for any* $x_0 \in H$ *there exists a unique* $x \in \mathrm{AC}([0,T];H)$ *satisfying* $x(0) = x_0$ *and*

$$\dot{x}(t) \in -N_{C(t,x(t))}\left(A\dot{x}(t) + Bx(t)\right) \quad a.e.\ t \in [0,T].$$

Moreover, the map $x_0 \mapsto x(x_0): H \to \mathrm{AC}([0,T];H)$ *is Lipschitz continuous.*

4 Stability of History/State-Dependent Implicit Sweeping Processes

In this section, we are concerned with the study of some new convergence results with respect to perturbations in the data, including perturbation of the associated moving sets, for the history/state-dependent implicit sweeping process (5). These results, in some sense, can be seen as stability results for the solution map of the problem (5). We refer to [26,27] for similar results about the sweeping process and some relevant differential variational inequalities.

In what follows, we assume that $(\mathcal{H}_A)$, $(\mathcal{H}_B)$, $(\mathcal{H}_R)$ and $(\mathcal{H}_{Cx})$ hold always. For each $n \in \mathbb{N}$ fixed, let us consider the following perturbed problem:

$$\begin{cases} -\dot{x}(t) \in N_{C_n(t,x(t))}\left(A_n\dot{x}(t) + B_nx(t) + (\mathcal{R}_nx)(t)\right) & \text{a.e. } t \in [0,T], \\ x(0) = x_0^n, \end{cases} \tag{15}$$

where $(A_n, B_n, \mathcal{R}_n, C_n)$ satisfy the following conditions:

$(\mathcal{H}_A^n)$ Let $A_n: H \to H$ be a linear, bounded, symmetric operator satisfying:

(a) There exists $\alpha > 0$ such that $\langle A_nx, x\rangle \geq \alpha\|x\|^2$ for all $x \in H$.
(b) $A_n = P_nP_n$ with $P_n \to P$.

$(\mathcal{H}_B^n)$ Let $B_n: H \to H$ be Lipschitz continuous mapping satisfying:

(a) There exists $L_B \geq 0$ such that

$$\|B_nx - B_ny\| \leq L_B\|x - y\| \quad \text{for all } x, y \in H.$$

(b) For all $x \in H$, the sequence (B_nx) strongly converges to Bx.

$(\mathcal{H}_{\mathcal{R}}^n)$ Let $\mathcal{R}_n: C([0,T];H) \to C([0,T];H)$ be such that

(a) There exists $L_{\mathcal{R}} \geq 0$ satisfying

$$\|(\mathcal{R}_nx)(t) - (\mathcal{R}_ny)(t)\| \leq L_{\mathcal{R}}\int_0^t \|x(s) - y(s)\|\,\mathrm{d}s$$

for all $x, y \in C([0,T];H)$ and $t \in [0,T]$.

(b) For all $x \in C([0,T];H)$, $\mathcal{R}_n x$ converges to $\mathcal{R}x$ in $C([0,T];H)$.

$(\mathcal{H}_{Cx}^n)$ Let $C_n\colon [0,T]\times H \rightrightarrows H$ be a set-valued map satisfying:

(a) C_n satisfies $(\mathcal{H}_{Cx})$.
(b) For all $t \in [0,T]$ and every sequence $(x_n)_n$ converging to x, the sequence $\{C_n(t,x_n)\}$ Mosco-converges to $C(t,x)$ (see Definition 2.2).

Theorem 4.1 *Assume that $(\mathcal{H}_A^n)$, $(\mathcal{H}_B^n)$, $(\mathcal{H}_R^n)$ and $(\mathcal{H}_{Cx}^n)$ hold. Then, the following assertions hold.*

(i) *For each $n \in \mathbb{N}$, the problem (15) has a unique solution $x_n \in \mathrm{AC}([0,T];H)$.*
(ii) *If $x_0^n \to x_0$ in H and $x_n \to x$ in $C([0,T];H)$, then $x \in \mathrm{AC}([0,T];H)$ and x is a solution of (5) with $x(0) := x_0$.*

Proof Assertion (i) follows directly from Theorem 3.1. We now prove (ii). In fact, Proposition 3.1 indicates that x_n satisfies $x_n(0) = x_0$ and

$$\begin{aligned}\dot{x}_n(t) &= -P_n^{-1}\left(Q_nB_nx_n(t) + Q_n(\mathcal{R}_nx_n)(t)\right)\\ &\quad + P_n^{-1}\operatorname{proj}_{Q_nC_n(t,x_n(t))}\left(Q_nB_nx_n(t) + Q_n(\mathcal{R}_nx_n)(t)\right) \quad \text{a.e. } t \in [0,T],\end{aligned}$$

where $Q_n = (P_n)^{-1}$ is well defined. Thus, for each $n \in \mathbb{N}$ and all $t \in [0,T]$

$$\begin{aligned}x_n(t) &= x_0^n - P_n^{-1}\int_0^t \left(Q_nB_nx_n(s) + Q_n(\mathcal{R}_nx_n)(s)\right)\,\mathrm{d}s\\ &\quad + P_n^{-1}\int_0^t \operatorname{proj}_{Q_nC_n(s,x_n(s))}\left(Q_nB_nx_n(s) + Q_n(\mathcal{R}_nx_n)(s)\right)\,\mathrm{d}s.\end{aligned} \tag{16}$$

However, the convergence $x_n \to x$ in $C([0,T];H)$ and assumptions $(\mathcal{H}_A^n)$, $(\mathcal{H}_B^n)$ and $(\mathcal{H}_R^n)$, guarantee that, as $n \to +\infty$, for all $s \in [0,T]$

$$\begin{cases} x_n(0) = x_0^n \to x(0) =: x_0\\ z_n(s) := Q_nB_nx_n(s) + Q_n(\mathcal{R}_nx_n)(s) \to z(s) := QBx(s) + Q(\mathcal{R}x)(s).\end{cases} \tag{17}$$

Moreover, according to $(\mathcal{H}_{Cx}^n)$ and Proposition 2.1, it follows that for all $y \in H$

$$\operatorname{proj}_{Q_nC_n(t,x_n(t))}(y) \to \operatorname{proj}_{QC(t,x(t))}(y) \text{ as } n \to +\infty. \tag{18}$$

Consequently, combining (17) and (18) with the nonexpansiveness of the distance, we get that $\operatorname{proj}_{Q_nC_n(t,x_n(t))}(z_n(t)) \to \operatorname{proj}_{QC(t,x(t))}(z(t))$ as $n \to +\infty$. Hence, by using this convergence and the dominated convergence theorem, we can pass to the limit in (16) to obtain that x is a solution of (6). Finally, by Proposition 3.1 and Theorem 3.1, $x \in \mathrm{AC}([0,T];H)$ solves (5), which completes the proof. □

Now, we present a novel convergence result with respect to perturbations in the data for the differential Eq. (5). Unlike Theorem 4.1, the moving sets are not affected by the perturbations. Thus, we are interested in the stability of history/state-dependent implicit sweeping process: Find $x_n : [0, T] \to H$ such that $x_n(0) = x_0^n$ and

$$-\dot{x}_n(t) \in N_{C(t,x_n(t))}\left(A_n\dot{x}_n(t) + B_n x_n(t) + (\mathcal{R}_n x_n)(t)\right) \text{ a.e. } t \in [0, T]. \quad (19)$$

Now, we present the main result of this section.

Theorem 4.2 *Assume, in addition to* $(\mathcal{H}_A^n)$, $(\mathcal{H}_B^n)$, $(\mathcal{H}_R^n)$ *and* $(\mathcal{H}_{Cx})$, *that there exists* $L_C \geq 0$ *such that for all* $t \in [0, T]$

$$|d_{C(t,x)}(z) - d_{C(t,y)}(z)| \leq L_C \|x - y\| \text{ for all } x, y, z \in H. \quad (20)$$

Then, the following assertions hold.

(i) *For each* $n \in \mathbb{N}$, *the problem* (19) *has a unique solution* $x_n \in \mathrm{AC}([0, T]; H)$.
(ii) *If* $x_0^n \to x_0$ *strongly in* H, *then* $x_n \to x$ *in* $C([0, T]; H)$ *as* $n \to +\infty$, *where* x *is the unique solution of* (5) *with* $x(0) = x_0$.

Proof The assertion (i) is a consequence of Theorem 3.1. To prove the conclusion (ii), let x be the unique solution of (5) with $x(0) = x_0$ and let us consider the intermediate problem: find $\tilde{x}_n : [0, T] \to H$ such that $\tilde{x}_n(0) = x_0^n$ and

$$-\dot{\tilde{x}}_n(t) \in N_{C(t,x(t))}\left(A_n\dot{\tilde{x}}_n(t) + B_n x(t) + (\mathcal{R}_n x)(t)\right) \text{ a.e. } t \in [0, T]. \quad (21)$$

Assume that $\tilde{x}_n \in C([0, T]; H)$ is the unique solution to problem (21). Then, by similar arguments to be given in the proof of Proposition 3.1, it follows that for all $t \in [0, T]$

$$\begin{aligned}\tilde{x}_n(t) = x_0^n + P_n^{-1} \int_0^t \mathrm{proj}_{Q_n C(s,x(s))}\left(Q_n B_n x(s) + Q_n(\mathcal{R}_n x)(s)\right) \mathrm{d}s \\ - P_n^{-1} \int_0^t \left(Q_n B_n x(s) + Q_n(\mathcal{R}_n x)(s)\right) \mathrm{d}s.\end{aligned} \quad (22)$$

Let $x \in C([0, T]; H)$ be the unique solution to the problem (5) associated to x_0. Define $z_n(t) := Q_n B_n x(t) + Q_n(\mathcal{R}_n x)(t)$ and $z(t) := QBx(t) + Q(\mathcal{R}x)(t)$. Then, for all $t \in [0, T]$, we have

$$x(t) = x_0 + P^{-1} \int_0^t \mathrm{proj}_{QC(s,x(s))}(z(s))\, \mathrm{d}s - P^{-1} \int_0^t (z(s))\, \mathrm{d}s. \quad (23)$$

Combining (22) and (23), we get that for all $t \in [0, T]$

$$\begin{aligned}\tilde{x}_n(t) - x(t) = x_0^n - x_0 - (P_n^{-1} - P^{-1}) \int_0^t (z(s))\, \mathrm{d}s \\ - P_n^{-1} \int_0^t (z_n(s) - z(s))\, \mathrm{d}s \\ + P_n^{-1} \int_0^t \left[\mathrm{proj}_{Q_n C(s,x(s))}(z_n(s)) - \mathrm{proj}_{Q_n C(s,x(s))}(z(s))\right] \mathrm{d}s \\ + P_n^{-1} \int_0^t \left[\mathrm{proj}_{Q_n C(s,x(s))}(z(s)) - \mathrm{proj}_{QC(s,x(s))}(z(s))\right] \mathrm{d}s \\ + (P_n^{-1} - P^{-1}) \int_0^t \mathrm{proj}_{QC(s,x(s))}(z(s))\, \mathrm{d}s.\end{aligned}$$

Now, we recall that $P_n^{-1} \to P^{-1}$, $P_n \to P$ and $Q_n = (P_n)^{-1} \to (P)^{-1} = Q$. Thus, by a careful calculation, we get for all $s \in [0, T]$

$$\begin{aligned}\|z_n(s) - z(s)\| &\le \|Q_n B_n x(s) + Q_n(\mathcal{R}_n x)(s) - Q_n Bx(s) - Q_n(\mathcal{R}x)(s)\| \\ &\quad + \|Q_n Bx(s) + Q_n(\mathcal{R}x)(s) - QBx(s) - Q(\mathcal{R}x)(s)\| \\ &\le \|Q_n\| \left(\|B_n x(s) - Bx(s)\| + \|(\mathcal{R}_n x)(s) - (\mathcal{R}x)(s)\|\right) \\ &\quad + \|Q_n - Q\| \left(\|Bx(s)\| + \|(\mathcal{R}x)(s)\|\right).\end{aligned}$$

Moreover, by virtue of Lemma 2.3, $(\mathcal{H}_{Cx})$(a) and Proposition 2.1, for a.e. $s \in [0, T]$, $\mathrm{proj}_{Q_n C(s,x(s))}(z(s)) \to \mathrm{proj}_{QC(s,x(s))}(z(s))$ as $n \to +\infty$.

Besides, hypotheses $(\mathcal{H}_B^n)$ and $(\mathcal{H}_R^n)$ guarantee that for all $s \in [0, T]$, $B_n x(s) \to Bx(s)$ and $(\mathcal{R}_n x)(s) \to (\mathcal{R}x)(s)$. Passing to the limit in (22), we conclude that $\tilde{x}_n \to x$ in $C([0, T]; H)$ as $n \to +\infty$.

Let x_n be the unique solution to problem (15). Then, for all $t \in [0, T]$, it has

$$\begin{aligned}x_n(t) = x_0^n + P_n^{-1} \int_0^t \mathrm{proj}_{Q_n C(s,x_n(s))} \left(Q_n B_n x_n(s) + Q_n(\mathcal{R}_n x_n)(s)\right) \mathrm{d}s \\ - P_n^{-1} \int_0^t \left(Q_n B_n x_n(s) + Q_n(\mathcal{R}_n x_n)(s)\right) \mathrm{d}s.\end{aligned} \tag{24}$$

Combining (22) and (24), we have that for all $t \in [0, T]$

$$\begin{aligned}\|x_n(t) - \tilde{x}_n(t)\| \leq \|P_n^{-1}\| \, \|Q_n\|(L_B + L_{\mathcal{R}}T) \int_0^t \|x_n(s) - x(s)\| \mathrm{d}s \\ + 2\|P_n^{-1}\| \int_0^t d_{Q_nC(s,x_n(s))}(z_n(s))\, \mathcal{H}(Q_nC(s, x_n(s)), Q_nC(s, x(s)))\; \mathrm{d}s \\ + 2\|P_n^{-1}\| \int_0^t d_{Q_nC(s,x(s))}(z_n(s))\, \mathcal{H}(Q_nC(s, x_n(s)), Q_nC(s, x(s)))\; \mathrm{d}s \\ \leq \|P_n^{-1}\| \, \|Q_n\|(L_B + L_{\mathcal{R}}T) \int_0^t \|x_n(s) - x(s)\| \, \mathrm{d}s \\ + 2\|P_n^{-1}\| \int_0^t m_n(s) \mathcal{H}(Q_nC(s, x_n(s)), Q_nC(s, x(s))\; \mathrm{d}s,\end{aligned}$$

where m_n is defined by

$$m_n(s) := \|Q_nB_nx(s)\| + \|Q_n(\mathcal{R}_nx)(s)\| + \|Q_n\|\mu\left(\|x_n(s)\| + \|x(s)\| + 2\right).$$

The latter inequality combined with (20) implies

$$\begin{aligned}\|x_n(t) - \tilde{x}_n(t)\| \leq \|P_n^{-1}\| \, \|Q_n\|(L_B + L_{\mathcal{R}}T) \int_0^t \|x_n(s) - x(s)\| \, \mathrm{d}s \\ + 2\|P_n^{-1}\| \, \|Q_n\| L_C \int_0^t m_n(s)\|x_n(s) - x(s)\| \, \mathrm{d}s.\end{aligned}$$

Moreover, by employing the same arguments from the proof of Theorem 3.1, we are able to find a constant $M_2 > 0$ such that $\|x_n\|_{C([0,T];H)} \leq M_2$ for all $n \in \mathbb{N}$. From the two previous inequalities, there exists a non-negative integrable function k^*, which is independent of n, such that for all $t \in [0, T]$

$$\|\tilde{x}_n(t) - \tilde{x}_n(t)\| \leq \int_0^t k^*(s)\|x_n(s) - x(s)\| \, \mathrm{d}s.$$

Therefore, we conclude that for all $t \in [0, T]$

$$\begin{aligned}\|x_n(t) - x(t)\| &\leq \|x_n(t) - \tilde{x}_n(t)\| + \|\tilde{x}_n(t) - x(t)\| \\ &\leq \int_0^t k^*(s)\|x_n(s) - x(s)\| \, \mathrm{d}s + \|\tilde{x}_n(t) - x(t)\|.\end{aligned}$$

Whereas, Gronwall's inequality indicates that

$$\|x_n(t) - x(t)\| \leq \|\tilde{x}_n - x\|_{C([0,T];H)} \exp\left(\int_0^t k^*(s)\mathrm{d}s\right) \quad \text{for all } t \in [0, T].$$

Finally, due to the convergence $\tilde{x}_n \to x$ in $C([0,T];H)$, we obtain the convergence that the sequence (x_n) converges to x in $C([0,T];H)$, as $n \to \infty$, which ends the proof. □

5 A History-Dependent Quasi-static Contact Problem

In this section, we apply our theoretical results to study the well-posedness of a quasi-static contact problem for viscoelastic materials in which the constitutive law for the viscoelastic material is given by a history-dependent process. We refer to [18,20,22,28] for more details on quasi-static contact problems.

The physical setting of the contact model is described as follows. We assume that a viscoelastic body occupies a bounded domain Ω in $\mathbb{R}^d$, $d = 2, 3$, with a Lipschitz continuous boundary $\Gamma := \partial\Omega$ such that the boundary is decomposed into four mutually disjoint and measurable parts Γ_D, Γ_N, Γ_{C_1} and Γ_{C_2} with $\mathrm{meas}(\Gamma_D) > 0$.

For the sake of convenience, we shall adopt the following standard notation. Denote by $\boldsymbol{\nu} = (\nu_i)$ and $\boldsymbol{x} \in \overline{\Omega} = \Omega \cup \partial\Omega$ the unit outward normal vector on boundary and the position vector in the body, respectively. In what follows, the indices i, j, k, l run from 1 to d, and the summation convention over repeated indices is used. For simplicity, we often will not explicitly indicate the dependence on the variable $\boldsymbol{x}$. Let $(\mathbb{S}^d, \|\cdot\|_{\mathbb{S}^d})$ be the space of second order symmetric tensors on $\mathbb{R}^d$. The inner products and norms in $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by $\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i$, $\|\boldsymbol{v}\|_{\mathbb{R}^d} = (\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}}$ for all $\boldsymbol{u} = (u_i)$, $\boldsymbol{v} = (v_i) \in \mathbb{R}^d$, $\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}$, $\|\boldsymbol{\tau}\|_{\mathbb{S}^d} = (\boldsymbol{\tau} : \boldsymbol{\tau})^{\frac{1}{2}}$ for $\boldsymbol{\sigma} = (\sigma_{ij})$, $\boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d$, respectively. Besides, we use the notation $\boldsymbol{u} = (u_i)$, $\boldsymbol{\sigma} = (\sigma_{ij})$, and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$, $\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\big(u_{i,j} + u_{j,i}\big)$, $i, j = 1, \ldots, d$, to stand for the displacement vector, the stress tensor, and the linearized strain tensor, respectively. For a vector $\boldsymbol{w}$ on the boundary, its normal and tangential components are formulated by $w_\nu = \boldsymbol{w}\cdot\boldsymbol{\nu}$ and $\boldsymbol{w}_\tau = \boldsymbol{w} - w_\nu\boldsymbol{\nu}$, accordingly. For the stress tensor $\boldsymbol{\sigma}$, its normal and tangential components on the boundary are denoted by $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$, respectively. Set $\mathcal{Q} = \Omega\times]0,T[$, $\Sigma = \Gamma\times]0,T[$, $\Sigma_D = \Gamma_D\times]0,T[$, $\Sigma_N = \Gamma_N\times]0,T[$, $\Sigma_{C_1} = \Gamma_{C_1}\times]0,T[$, and $\Sigma_{C_2} = \Gamma_{C_2}\times]0,T[$.

The classical formulation of the contact model reads as follows.

Problem 5.1 *Find a displacement field* $\boldsymbol{u} : \mathcal{Q} \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \mathcal{Q} \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}'(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,\mathrm{d}s \quad \text{in } \mathcal{Q}, \tag{25}$$

$$\mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in } \mathcal{Q}, \tag{26}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Sigma_D, \tag{27}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_N(t) \quad \text{on } \Sigma_N, \tag{28}$$

$$\begin{cases} -\sigma_\nu(t) = p_\nu(u_\nu(t) - g) \\ \boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \end{cases} \quad \text{on } \Sigma_{C_1}, \tag{29}$$

$$\begin{cases} -\sigma_\nu(t) = F, \\ \|\boldsymbol{\sigma}_\tau(t)\|_{\mathbb{S}^d} \le \mu|\sigma_\nu(t)|, \\ -\boldsymbol{\sigma}_\tau(t) = \mu|\sigma_\nu(t)| \frac{\boldsymbol{u}'_\tau(t)}{\|\boldsymbol{u}'_\tau(t)\|_{\mathbb{R}^d}} \quad \textit{if}\ \boldsymbol{u}'_\tau(t) \neq \boldsymbol{0} \end{cases} \quad \text{on } \Sigma_{C_2}, \tag{30}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{31}$$

To understand the contact model, we now give a short description of its equations and relations. First, Eq. (25) represents a general constitutive law for viscoelastic materials with long memory in which $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$ are the linear viscosity operator, nonlinear elasticity operator, and the relaxation tensor, respectively. In fact, this kind of constitutive law has been considered and studied in the literature, for example, [20,22]. The equality (26) is called the equation of equilibrium, where "Div" denotes the divergence operator

$$\text{Div}\boldsymbol{\sigma} = (\sigma_{ij,j}) = \left(\frac{\partial \sigma_{ij}}{\partial x_j}\right),$$

and $\boldsymbol{f}_0$ denotes the density of volume forces. The boundary conditions (27) and (28) characterize the physical phenomena that the viscoelastic body is clamped on Γ_D and it is subjected to the density $\boldsymbol{f}_N$ of surface tractions on Γ_N. On boundary Γ_{C_1}, the contact is described by the normal compliance condition (29) and frictionless (see, for instance [29,30]). However, the relation (30) illustrates a version of Tresca's law of dry friction, in which the normal stress on the contact boundary is assumed to be given (see, for instance [31] and the references therein). Here, F is a positive function, $\mu \ge 0$ denotes the coefficient of friction and, therefore, μF represents the friction bound. Finally, condition (31) is the initial condition in which $\boldsymbol{u}_0$ stands for the initial displacement field.

To obtain the weak formulation of Problem 5.1, let us introduce the following function spaces

$$H = \{\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D\}, \quad \mathcal{V} = L^2(\Omega; \mathbb{S}^d).$$

By virtue of Korn's inequality, since meas$(\Gamma_D) > 0$, it is possible to prove that H is a Hilbert space endowed with the inner product

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_H = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) : \boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x} \quad \text{for } \boldsymbol{u}, \boldsymbol{v} \in H,$$

and the associated norm $\|\cdot\|_H$. Moreover, $\mathcal{V}$ is also a Hilbert space equipped with the inner product

$$\langle \boldsymbol{\sigma}, \boldsymbol{\tau}\rangle_{\mathcal{V}} = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{x}) : \boldsymbol{\tau}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \text{for all } \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{V},$$

and the associated norm $\|\cdot\|_{\mathcal{V}}$.

Consider the trace operator $\gamma\colon H \to L^2(\Gamma_C;\mathbb{R}^d)$, with $\Gamma_C = \Gamma_{C_1}\cup\Gamma_{C_2}$, it follows from Korn's inequality that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_C;\mathbb{R}^d)} \le \|\gamma\|\cdot\|\boldsymbol{v}\|_H \quad \text{for all } \boldsymbol{v}\in H.$$

On the other hand, let us introduce a function space

$$\mathcal{Q}_\infty = \{\,\mathcal{E} = (\mathcal{E}_{ijkl}) \mid \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega),\ 1\le i,j,k,l\le \mathrm{d}\,\},$$

which endowed with the norm $\|\mathcal{E}\|_{\mathcal{Q}_\infty} = \max_{1\le i,j,k,l\le d}\|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}$ is a real Banach space.

Further, we impose the following hypotheses.

<u>$H(\mathcal{A})$</u>: $\mathcal{A}\in\mathcal{Q}_\infty$ and it satisfies $\mathcal{A}\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} \ge L_{\mathcal{A}}\|\boldsymbol{\varepsilon}\|^2_{\mathbb{S}^d}$ for all $\boldsymbol{\varepsilon}\in\mathbb{S}^d$ for some $L_{\mathcal{A}}>0$.

<u>$H(\mathcal{B})$</u>: $\mathcal{B}\colon \Omega\times\mathbb{S}^d\to\mathbb{S}^d$ is such that

(a) there exists $L_{\mathcal{B}}>0$ such that $\|\mathcal{B}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{B}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\|_{\mathbb{S}^d} \le L_{\mathcal{B}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d}$ for all $\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d$, a.e. $\boldsymbol{x}\in\Omega$.
(b) the mapping $\boldsymbol{x}\mapsto\mathcal{B}(\boldsymbol{x},\boldsymbol{\varepsilon})$ is measurable on Ω for all $\boldsymbol{\varepsilon}\in\mathbb{S}^d$.
(c) the mapping $\boldsymbol{x}\mapsto\mathcal{B}(\boldsymbol{x},\boldsymbol{0})$ belongs to $\mathcal{V}$.

<u>$H(\mathcal{R})$</u>: $\mathcal{R}\in C([0,T];\mathcal{Q}_\infty)$.

<u>$H(p_\nu)$</u>: $p_\nu\colon\Gamma_{C_1}\times\mathbb{R}\to\mathbb{R}_+$ is such that

(a) there exists $L_\nu>0$ such that $|p_\nu(\boldsymbol{x},r_1)-p_\nu(\boldsymbol{x},r_2)|\le L_\nu|r_1-r_2|$ for all $r_1,r_2\in\mathbb{R}$ a.e. $\boldsymbol{x}\in\Gamma_{C_1}$.
(b) $\boldsymbol{x}\mapsto p_\nu(\boldsymbol{x},r)$ is measurable on Γ_{C_1} for all $r\in\mathbb{R}$.
(c) the mapping $p_\nu(\boldsymbol{x},r)=0$ for all $r\le 0$ and a.e. $\boldsymbol{x}\in\Gamma_{C_1}$.

<u>$H(f)$</u>: $\boldsymbol{f}_0\in L^2(0,T;L^2(\Omega;\mathbb{R}^d))$, $\boldsymbol{f}_N\in L^2(0,T;L^2(\Gamma_N;\mathbb{R}^d))$, $F\in L^2(\Gamma_{C_2})$, $F(\boldsymbol{x})\ge 0$ a.e. $\boldsymbol{x}\in\Gamma_{C_2}$, $\mu\in L^\infty(\Gamma_{C_2})$, $\mu(\boldsymbol{x})\ge 0$ a.e. $\boldsymbol{x}\in\Gamma_{C_2}$, $g\in L^2(\Gamma_{C_1})$, $g(\boldsymbol{x})\ge 0$ for a.e. $\boldsymbol{x}\in\Gamma_{C_1}$.

By a standard procedure, it is not difficult to obtain the following weak formulation of Problem 5.1 as folllows.

Problem 5.2 *Find a displacement field $\boldsymbol{u}\colon[0,T]\to H$ such that*

$$\begin{aligned}
&\langle\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}'(t))+\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))+\int_0^t\mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,\mathrm{d}s,\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}'(t))\rangle_{\mathcal{V}}\\
&\quad+\int_{\Gamma_{C_2}}F(v_\nu-u'_\nu(t))\,\mathrm{d}\Gamma+\int_{\Gamma_{C_2}}\mu F\|\boldsymbol{v}_\tau\|_{\mathbb{R}^d}\,\mathrm{d}\Gamma-\int_{\Gamma_{C_2}}\mu F\|\boldsymbol{u}'_\tau(t)\|_{\mathbb{R}^d}\,\mathrm{d}\Gamma\\
&\quad+\int_{\Gamma_{C_1}}p_\nu(u_\nu(t)-g(x))(v_\nu-u'_\nu(t))\,\mathrm{d}\Gamma\ge\langle\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}'(t)\rangle_H\\
&\qquad\textit{for all }\ \boldsymbol{v}\in H\quad\textit{and a.e. }\ t\in[0,T],\\
&\quad\boldsymbol{u}(0)=\boldsymbol{u}_0\quad\textit{in}\ \ \Omega,
\end{aligned}$$

where $\boldsymbol{f}\in L^2(0,T;H)$ is such that for all $\boldsymbol{v}\in H$ and $t\in[0,T]$

$$\langle\boldsymbol{f}(t),\boldsymbol{v}\rangle_H=\langle\boldsymbol{f}_0(t),\boldsymbol{v}\rangle_{L^2(\Omega;\mathbb{R}^d)}+\langle\boldsymbol{f}_N(t),\boldsymbol{v}\rangle_{L^2(\Gamma_N;\mathbb{R}^d)}.$$

Our main result for Problem 5.2 reads as follows.

Theorem 5.3 *Under the hypotheses $H(\mathcal{A})$, $H(\mathcal{B})$, $H(\mathcal{R})$, $H(p_\nu)$ and $H(f)$, then, for any initial displacement $\boldsymbol{u}_0 \in H$, Problem 5.2 has a unique solution $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{u}_0) \in$ AC $([0,T]; H)$. Moreover, the map $\boldsymbol{u}_0 \mapsto \boldsymbol{u}(\boldsymbol{u}_0)$ defined from H into AC $([0,T]; H)$ is locally Lipschitz continuous.*

Proof We shall apply Theorem 3.1 to prove the desired conclusion. To this end, first, we show that Problem 5.2 is equivalent to a history/state-dependent sweeping process, i.e., problem (5).

Let $A, B\colon H \to H$ and $\mathcal{R}\colon C([0,T]; H) \to C([0,T]; H)$ defined by

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle_H = \langle \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathcal{V}}, \tag{32}$$

$$\langle B\boldsymbol{u}, \boldsymbol{v}\rangle_H = \langle \mathcal{B}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathcal{V}}, \tag{33}$$

$$\langle (\mathcal{R}\boldsymbol{w})(t), \boldsymbol{v}\rangle_H = \Big\langle \int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{w}(s))\,\mathrm{d}s, \boldsymbol{\varepsilon}(\boldsymbol{v})\Big\rangle_{\mathcal{V}} \tag{34}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in H$, all $\boldsymbol{w} \in C([0,T]; H)$ and all $t \in [0,T]$. Also, we consider the functions $\varphi\colon H \to \mathbb{R}$ and $P\colon H \to H$ defined by

$$\varphi(\boldsymbol{v}) = \int_{\Gamma_{C_2}} \mu F\big(v_\nu + \|\boldsymbol{v}_\tau\|_{\mathbb{R}^d}\big)\,\mathrm{d}\Gamma \quad \text{for all } \boldsymbol{v} \in H,$$

$$\langle P(\boldsymbol{u}), \boldsymbol{v}\rangle = \int_{\Gamma_{C_1}} p_\nu(\boldsymbol{x}, u_\nu(\boldsymbol{x}) - g(\boldsymbol{x}))v_\nu(\boldsymbol{x})\,\mathrm{d}\Gamma \quad \text{for all } \boldsymbol{v} \in H.$$

Under the above definitions, it is easy to see that Problem 5.2 can be rewritten equivalently as the problem: find $\boldsymbol{u}\colon [0,T] \to H$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and

$$\begin{aligned}&\langle A\boldsymbol{u}'(t) + B\boldsymbol{u}(t) + (\mathcal{R}\boldsymbol{u})(t) - \boldsymbol{f}(t) + P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}'(t)\rangle_H \\ &\quad + \varphi(\boldsymbol{v}) - \varphi(\boldsymbol{u}'(t)) \geq 0 \qquad \text{for all } \boldsymbol{v} \in H \quad \text{and a.e. } t \in [0,T],\end{aligned}$$

From the definition of φ and hypotheses $H(f)$, it can observe that φ is a convex and continuous function. So, we can directly reformulate the above inequality to the following differential inclusion problem: find a displacement field $\boldsymbol{u}\colon [0,T] \to H$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and

$$\boldsymbol{f}(t) - A\boldsymbol{u}'(t) - B\boldsymbol{u}(t) - (\mathcal{R}\boldsymbol{u})(t) - P\boldsymbol{u}(t) \in \partial\varphi(\boldsymbol{u}'(t)) \quad \text{a.e. } t \in [0,T].$$

Also φ is convex and positively homogeneous of degree 1 (i.e., $\varphi(\lambda\boldsymbol{u}) = \lambda\varphi(\boldsymbol{u})$ for all $\lambda > 0$ and $\boldsymbol{u} \in H$). A simple calculating gives

$$\partial\varphi(\boldsymbol{0}_H) = \{\boldsymbol{\xi} \in H \mid \varphi(\boldsymbol{v}) \geq \langle \boldsymbol{\xi}, \boldsymbol{v}\rangle_H \quad \text{for all } \boldsymbol{v} \in H\}.$$

For simplicity, we denote $C_0 := \partial\varphi(\boldsymbol{0}_H)$. Obviously, it holds

$$\varphi(\boldsymbol{v}) = \sup_{\boldsymbol{\xi} \in C_0} \langle \boldsymbol{\xi}, \boldsymbol{v}\rangle_H = \sigma_{C_0}(\boldsymbol{v}) = I^*_{C_0}(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in H,$$

where $I^*_{C_0}$ is the Legendre–Fenchel conjugate of the indicator function I_{C_0}. The latter combined with the facts

$$\partial\varphi(\boldsymbol{v}) = \partial I^*_{C_0}(\boldsymbol{v}), \quad \varphi^*(\boldsymbol{v}) = I^{**}_{C_0}(\boldsymbol{v}) = I_{C_0}(\boldsymbol{v}),$$

and $\boldsymbol{\xi} \in \partial\varphi(\boldsymbol{v}) \Leftrightarrow \boldsymbol{v} \in \partial I_{C_0}(\boldsymbol{\xi}) \Leftrightarrow \boldsymbol{v} \in N_{C_0}(\boldsymbol{\xi})$, implies that for a.e. $t \in [0, T]$

$$\begin{aligned}
&\boldsymbol{f}(t) - A\boldsymbol{u}'(t) - B\boldsymbol{u}(t) - (\mathcal{R}\boldsymbol{u})(t) - P\boldsymbol{u}(t) \in \partial\varphi(\boldsymbol{u}'(t)) \\
&\quad \Longleftrightarrow -\boldsymbol{u}'(t) \in N_{C_0}\big(\boldsymbol{f}(t) - A\boldsymbol{u}'(t) - B\boldsymbol{u}(t) - (\mathcal{R}\boldsymbol{u})(t) - P\boldsymbol{u}(t)\big) \\
&\quad \Longleftrightarrow -\boldsymbol{u}'(t) \in N_{C_0 + P\boldsymbol{u}(t) - \boldsymbol{f}(t)}\big(- A\boldsymbol{u}'(t) - B\boldsymbol{u}(t) - (\mathcal{R}\boldsymbol{u})(t)\big) \\
&\quad \Longleftrightarrow -\boldsymbol{u}'(t) \in N_{C(t,\boldsymbol{u}(t))}\big(A\boldsymbol{u}'(t) + B\boldsymbol{u}(t) + (\mathcal{R}\boldsymbol{u})(t)\big),
\end{aligned}$$

where $C(t, \boldsymbol{u}(t)) := \boldsymbol{f}(t) - C_0 - P\boldsymbol{u}(t) = \boldsymbol{f}(t) - \partial\varphi(\boldsymbol{0}_H) - P\boldsymbol{u}(t)$. To conclude, we can see that Problem 5.2 is equivalent to the following history/state-dependent sweeping process: find $\boldsymbol{u}\colon [0, T] \to H$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and

$$-\boldsymbol{u}'(t) \in N_{C(t,\boldsymbol{u}(t))}\big(A\boldsymbol{u}'(t) + B\boldsymbol{u}(t) + (\mathcal{R}\boldsymbol{u})(t)\big) \quad \text{for a.e. } t \in [0, T].$$

It remains us to verify that all conditions of Theorem 3.1 are valid. Indeed, hypothesis $H(\mathcal{A})$ implies that operator A defined in (32) is linear, bounded, symmetric and coercive with constant $\alpha = L_{\mathcal{A}}$, i.e., $(\mathcal{H}_A)$ holds. From hypotheses $H(\mathcal{B})$ and $H(\mathcal{R})$, it is not difficult to corroborate that operators B and R defined in (33) and (34) satisfy conditions $(\mathcal{H}_B)$ and $(\mathcal{H}_R)$, respectively. Further, we shall demonstrate that the set-valued map $C\colon [0, T] \times H \rightrightarrows H$ enjoys hypothesis $(\mathcal{H}'_{Cx})$ (see Remark 3.2). It is clear that for all $\boldsymbol{u} \in H$ the mapping $t \mapsto C(t, \boldsymbol{u}) = \boldsymbol{f}(t) - \partial\varphi(\boldsymbol{0}) - P\boldsymbol{u}$ is measurable and

$$d_{C(t,\boldsymbol{u})}(\boldsymbol{0}_H) = \inf_{\boldsymbol{v}\in C(t,\boldsymbol{u})} \|\boldsymbol{v}\|_H \le \|P\boldsymbol{u}\|_H + \|\boldsymbol{f}(t)\|_H + d_{C_0}(\boldsymbol{0}_H). \tag{35}$$

Moreover, through the definition of P and Hölder's inequality, we obtain

$$\begin{aligned}
\|P\boldsymbol{u}\|_H &\le \sup_{\boldsymbol{w}\in H, \|\boldsymbol{w}\|=1} \int_{\Gamma_{C_1}} p_\nu(\boldsymbol{x}, u_\nu(x) - g(x)) w_\nu(x)\, \mathrm{d}\Gamma \\
&\le \|\gamma\| \left(\int_{\Gamma_{C_1}} |p_\nu(\boldsymbol{x}, u_\nu(x) - g(x))|^2 \,\mathrm{d}\Gamma \right)^{\frac{1}{2}} \\
&\le \|\gamma\| L_\nu \left(\int_{\Gamma_{C_1}} |u_\nu(x) - g(x)|^2 \,\mathrm{d}\Gamma \right)^{\frac{1}{2}} \\
&\le \|\gamma\| L_\nu \sqrt{2}\big(\|\gamma\| \cdot \|\boldsymbol{u}\|_H + \|g\|_{L^2(\Gamma_{C_1})}\big).
\end{aligned}$$

Thus, due to the last inequality and (35), we get

$$d_{C(t,\boldsymbol{u})}(\boldsymbol{0}_H) \le \mu(t)(1 + \|\boldsymbol{u}\|_H) \quad \text{for all } \boldsymbol{u} \in H \text{ and a.e. } t \in [0, T],$$

where $\mu \in L^1(0, T)$ is the function defined by

$$\mu(t) := \|\gamma\| L_\nu \sqrt{2} \max\{\|\gamma\|, \|g\|_{L^2(\Gamma_{C_1})}\} + \|\boldsymbol{f}(t)\|_H + d_{C_0}(\mathbf{0}_H).$$

Hence, $(\mathcal{H}'_{Cx})$(a) holds. Next, fix $r \in \mathbb{R}$ and for $i = 1, 2$, let $\boldsymbol{w}_i \in N_{C(t,\boldsymbol{u}_i)}(\boldsymbol{v}_i)$ with $\boldsymbol{u}_1, \boldsymbol{v}_i \in r\mathbb{B}$. Then $\boldsymbol{w}_i \in N_{C(t,\boldsymbol{u}_i)}(\boldsymbol{v}_i) = N_{\boldsymbol{f}(t)-\partial\varphi(\mathbf{0}_H)}(\boldsymbol{v}_i + P\boldsymbol{u}_i)$. By virtue of the monotonicity of the normal cone, it follows that

$$\langle \boldsymbol{w}_1 - \boldsymbol{w}_2, \boldsymbol{v}_1 + P\boldsymbol{u}_1 - \boldsymbol{v}_2 - P\boldsymbol{u}_2 \rangle_H \geq 0.$$

Therefore,

$$\begin{aligned}\langle \boldsymbol{w}_1 - \boldsymbol{w}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2 \rangle_H &\geq -\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_H \|P\boldsymbol{u}_1 - P\boldsymbol{u}_2\|_H \\ &\geq -\|\gamma\|^2 L_\nu \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_H \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_H.\end{aligned}$$

So, we conclude that condition $(\mathcal{H}'_{Cx})$(b) holds with $\tilde{L}_r = \|\gamma\|^2 L_\nu \ \forall r \in \mathbb{R}_+$.

We are now in position to apply Theorem 3.1 (see Remark 3.2) to conclude that Problem 5.2 has a unique solution and the map $\boldsymbol{u}_0 \mapsto \boldsymbol{u}(\boldsymbol{u}_0)$ is locally Lipschitz continuous from H into AC $([0, T]; H)$, which ends the proof. □

6 Conclusions

In this paper, we prove the well-posedness and parametric stability for history/state dependent implicit sweeping processes. Our method relies on the equivalence between the implicit sweeping process and a nonlinear differential equation. An important continuation of this work is to study problems of optimal control for implicit sweeping processes, which, up to our knowledge, have not been considered yet. In the mechanical context, these problems consist of leading the stress tensor as close as possible to a given target (see [32–34] and the references therein). It seems that our approach could be fruitful, especially, to obtain optimality conditions. This research will be pursued in the future.

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