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A Class of Nonlinear Inclusions and Sweeping Processes in Solid Mechanics

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Abstract We consider a new class of inclusions in Hilbert spaces for which we provide an existence and uniqueness result. The proof is based on arguments of monotonicity, convexity and fixed point. We use this result to establish the unique solvability of an associated class of Moreau's sweeping processes. Next, we give two applications in Solid Mechanics. The first one concerns the study of a time-dependent constitutive law with unilateral constraints and memory term. The second one is related to a frictional contact problem for viscoelastic materials. For both problems we describe the physical setting, list the assumptions on the data and provide existence and uniqueness results.

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1 Introduction

Nonlinear inclusions arise in the study of various boundary value problems and have important applications in Mechanics, Physics, Engineering and Economy. Expressed in terms of multivalued operators, their solvability requires arguments coming from set-valued, convex and nonsmooth analysis. Time-dependent and evolutionary inclusions represent an important ingredient in the study of various classes of variational and hemivariational inequalities, as illustrated in the books [18, 27–29, 31, 32]. There, various existence results have been developed, based on surjectivity properties for pseudomonotone operators.

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A convex sweeping process is a differential inclusion governed by the normal cone of a convex moving set. Such evolution inclusions are strongly involved in the study of unilateral problems in Solid Mechanics, where the convex sets are related to the elastic-visco-plastic constitutive law or, alternatively, to the frictional unilateral contact conditions. Sweeping process problems have been introduced in early seventy's in the pioneering works of Moreau [21–23]. Later, many variants of the so-called Moreau's sweeping process have been developed in the literature: stochastic ([7]), in bounded variation framework ([11, 24]), with perturbations ([11, 20]), nonconvex (complement of a convex set ([36]), closed set in $\mathbb{R}^n$ ([3, 8, 33]) prox-regular ([8, 11, 26]), subsmooth ([14]), α-far ([15]), state-dependent ([13, 16, 25]), in Banach spaces and manifolds ([4, 5]), truncated ([25, 34]), with velocity constraint and/or history-dependent operators ([1, 2, 19]), subject to a control ([6]).

Contact process between deformable bodies abound in industry and everyday life. A few simple example are brake pads with wheels, tires on road, and piston with skirts. Common industrial processes such as metal forming and metal extrusion involve contact evolution, too. Stated as strongly nonlinear boundary value problems which usually do not have classical solutions, the mathematical models of contact lead to a large variety of weak formulations, expressed in terms of variational or hemivariational inequalities. These inequalities could be elliptic, time-dependent or evolutionary, in function of the type of the mechanical process, the constitutive laws and the interface laws used in the construction of each model. Usually, the corresponding unknowns are the displacement or the velocity field and, on occasion, the stress field. Employing such kind of formulations allows the use of standard arguments from the theory of variational and hemivariational inequalities which can be found in various books, including [18, 27–29]. Currently, there is an interest in variational formulation of contact models in the form of a time-dependent inclusion or a sweeping process, see [1, 2], for instance. Nevertheless, using such kind of formulations in the study of contact models requires to adapt the arguments of abstract stationary or differential inclusions and, very often, to develop new arguments in their analysis and control.

The current paper signs up in this direction and its aim is two folds. The first one is to provide an existence and uniqueness result for a new class of time-dependent inclusions and sweeping processes in a real Hilbert space. On this concern, the novelty lies in the special structure of the considered problems which, inspired by potential applications in Mechanics of Solids, is governed by two nonlinear operators, and are defined on a possibly unbounded time interval. The second aim of the present paper is to illustrate the theoretical results in the study of mathematical models arising in Solid and Contact Mechanics. Thus, we consider a viscoelastic constitutive law for which we show that the "irregular" part of the stress field can be determined in a unique way when the evolution of the stress field in time in known. This result, based on the solution of a time-dependent inclusion, provides a better knowledge of viscoelastic constitutive laws with long memory term and looking property which describe the behavior of some real materials like metals, rocks and polymers. Then, we consider a frictional contact model with viscoelastic materials for which we provide a new variational formulation, in terms of a sweeping process in which the unknown is the strain field. At the best of our knowledge, this variational formulation, together with the corresponding existence and uniqueness result, is new and nonstandard. It has the merit to show that the sweeping process theory can be used in the study of contact problems and, on this matter, it could represent an alternative to the classical tools provided by the theory of variational inequalities.

The paper is organized as follows. Section 2 is devoted to the notation and the preliminaries of Convex and Nonlinear Analysis needed throughout the paper. In Sect. 3, we introduce the time-dependent inclusion we are interested in, then we prove its unique solvability (see Theorem 3.1). Such a well-posedness is then apply (Theorem 4.1) to a variant of Moreau's sweeping process in Sect. 4. The inclusion and the sweeping process considered are described through a family of time-dependent convex sets $\{K(t)\}_{t\in I}$ satisfying a specific assumption denoted by $(\mathcal{K})$. Various examples of convex moving set enjoying such an assumption $(\mathcal{K})$ are provided in Sect. 5. Note that part of these examples are useful in various applications arising in Solid and Contact Mechanics. Next, in Sect. 6 we consider a viscoelastic constitutive law with unilateral constraints for the strain field and illustrate the use of Theorem 3.1 in the study of this law. Finally, in Sect. 7, we introduce a mathematical model which describes the frictional contact of a nonlinear viscoelastic body. We provide a variational formulation of the model then we illustrate the use of Theorem 4.1 to obtain its unique weak solvability.

2 Preliminaries

The material presented in this section is standard and, for this reason, we present it without proofs. More details can be found in the books [12, 17, 31, 32], for instance.

In the whole paper, all vector spaces will be real vector spaces. We use $\mathbb{R}_+$ for the set of nonnegative reals, that is, $\mathbb{R}_+ := [0, +\infty[$. The letter T stands for an extended nonnegative real, i.e., $T \in \mathbb{R}_+ \cup \{+\infty\}$ and $I := [0, T] \cap \mathbb{R}$. Throughout the paper, X represents a Hilbert space endowed with an inner product $(\cdot,\cdot)_X$ and its associated norm $\|\cdot\|_X := \sqrt{(\cdot,\cdot)_X}$. The set of parts of X is denoted by 2^X.

Convex sets and nonlinear operators. Let K be a nonempty closed convex subset of X and $f \in X$ be a vector. It is well known that there is a unique element $u \in X$ such that

$$u \in K \quad \text{and} \quad d_K(f) = \|f - u\|_X,$$

or, equivalently,

$$u \in K \quad \text{and} \quad \|f - u\|_X \le \|f - v\|_X \quad \text{for all } v \in K.$$

Here and everywhere below, we use d_S for the *distance function* to a subset $S \subset X$, that is

$$d_S(x) := \inf_{y\in S} \|x - y\|_X \quad \text{for all } x \in S.$$

The element $u \in K$ is called the *projection* of f on K and is denoted by $P_K f$. Such an element is characterized through the following equivalence

$$u = P_K f \quad \Longleftrightarrow \quad u \in K, \quad (u, v - u)_X \ge (f, v - u)_X \quad \text{for all } v \in K. \tag{2.1}$$

The operator $P_K : X \to K$ is called the *projection operator* on K and, in general, is nonlinear. Moreover, it is nonexpansive, i.e.,

$$\|P_K f_1 - P_K f_2\|_X \le \|f_1 - f_2\|_X \quad \text{for all } f_1, f_2 \in X \tag{2.2}$$

and monotone, that is,

$$(P_K f_1 - P_K f_2, f_1 - f_2)_X \ge 0 \quad \text{for all } f_1, f_2 \in X. \tag{2.3}$$

Inequalities similar to (2.2) and (2.3) are satisfied for various nonlinear operators. Here, we recall that an operator $A : X \to X$ is said to be *Lipschitz continuous* provided that there exists $L_A > 0$ such that

$$\|Au - Av\|_X \le L\|u - v\|_X \quad \text{for all } \ u, v \in X$$

and *strongly monotone* whenever there exists $m_A > 0$ such that

$$(Au - Av, u - v)_X \ge m_A\|u - v\|_X^2 \quad \text{for all } \ u, v \in X.$$

It is known (see, e.g., [31, Theorem 1.24]) that a strongly monotone Lipschitz continuous operator $A : X \to X$ is invertible. In such a case, the inverse A^{-1} is also strongly monotone and Lipschitz continuous.

The (obviously convex) function $\psi_K : X \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\psi_K(u) := \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K \end{cases}$$

is called the *indicator function* of K. Its *subdifferential* in the sense of convex analysis is the multivalued operator $\partial\psi_K : X \to 2^X$ defined for every $u \in X$ through

$$\partial\psi_K(u) := \begin{cases} \{\xi \in X \ : \ (\xi, v - u)_X \le 0 \quad \forall\, v \in K\} & \text{if } u \in K, \\ \emptyset \quad \text{otherwise.} \end{cases} \tag{2.4}$$

As usual, the subdifferential $\partial\psi_K$ of the function ψ_K is called *the outward normal cone* of K in the sense of convex analysis and is denoted by N_K, that is,

$$N_K(u) := \partial\psi_K(u) \quad \text{for all } \ u \in X.$$

We derive from (2.4) that the following equivalence holds for all $u, \xi \in X$:

$$\xi \in N_K(u) \iff u \in K, \quad (\xi, v - u)_X \le 0 \quad \text{for all } \ v \in K. \tag{2.5}$$

Moreover, combining the equivalences (2.1) and (2.5) yield

$$w - P_K w \in N_K\big(P_K w\big) \quad \text{for all } \ w \in X. \tag{2.6}$$

History-dependent and almost history-dependent operators. For a normed space $(Y, \|\cdot\|_Y)$, we denote by $C(I; Y)$ the space of continuous functions defined on I with values in Y, i.e.,

$$C(I; Y) = \{\, v : \ I \to Y : v \text{ is continuous}\}.$$

The case $T \in \mathbb{R}$ (i.e., $I = [0, T]$) leads to the space $C([0, T]; Y)$ which is a normed space equipped with the norm $\|\cdot\|_{C([0,T];Y)}$ defined by

$$\|v\|_{C([0,T];Y)} := \max_{t\in[0,T]} \|v(t)\|_Y \quad \text{for all } v \in C([0, T]; Y).$$

It is well known that $C([0, T]; Y)$ is a Banach space whenever Y is also a Banach space. The case $I = \mathbb{R}_+$ leads to the space $C(\mathbb{R}_+; Y)$. If Y is a Banach space then $C(\mathbb{R}_+; Y)$ can be organized in a canonical way as a Fréchet space, i.e., a complete metric space in which the corresponding topology is induced by a countable family of seminorms.

The vector space of continuously differentiable functions on I with values in Y is denoted by $C^1(I;Y)$. Obviously, for any function $v : I \to Y$, the inclusion $v \in C^1(I;Y)$ holds if and only if $v \in C(I;Y)$ and $\dot{v} \in C(I;Y)$. Here and below, $\dot{v}(\cdot)$ stands for the derivative of the function $v(\cdot)$. For a function $v \in C^1(I;Y)$, the equality below will be used in various places of this manuscript:

$$v(t) = \int_0^t \dot{v}(s)\,ds + v(0) \ \text{ for all } \ t \in I.$$

Everywhere in this paper, given two normed spaces Y and Z and an operator $\mathcal{S}\colon C(I;Y) \to C(I;Z)$, for any function $u \in C(I;Y)$ we use the shorthand notation $\mathcal{S}u(t)$ to represent the value of the function $\mathcal{S}u$ at the point $t \in I$, that is, $\mathcal{S}u(t) := (\mathcal{S}u)(t)$.

We end this section with two important classes of operators defined on the space of continuous functions.

Definition 2.1 Let $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be two normed spaces. An operator $\mathcal{S}\colon C(I;Y) \to C(I;Z)$ is called:

a) history-dependent (h.d. for short), if for any nonempty compact set $\mathcal{J} \subset I$, there exists a real $L^{\mathcal{S}}_{\mathcal{J}} > 0$ such that for all $u_1, u_2 \in C(I;Y)$ and all $t \in \mathcal{J}$,

$$\|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_Z \le L^{\mathcal{S}}_{\mathcal{J}} \int_0^t \|u_1(s) - u_2(s)\|_Y\,ds.$$

b) almost history-dependent (a.h.d. for short), if for any nonempty compact set $\mathcal{J} \subset I$, there exist $l^{\mathcal{S}}_{\mathcal{J}} \in [0,1)$ and a real $L^{\mathcal{S}}_{\mathcal{J}} > 0$ such that for all $u_1, u_2 \in C(I;Y)$ and all $t \in \mathcal{J}$,

$$\|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_Z \le l^{\mathcal{S}}_{\mathcal{J}}\, \|u_1(t) - u_2(t)\|_Y + L^{\mathcal{S}}_{\mathcal{J}} \int_0^t \|u_1(s) - u_2(s)\|_Y\,ds. \tag{2.7}$$

It is readily seen that any h.d. operator is an a.h.d. operator. Let us mention here that such operators are deeply involved in Contact Mechanics and Nonlinear Analysis. Indeed, due to their fixed point properties, a.h.d. operators are very useful to establish the well-posedness of various classes of nonlinear equations and variational inequalities. In this paper, we shall use the following fixed point result.

Theorem 2.2 *Let Y be a Banach space and let $\Lambda\colon C(I;Y) \to C(I;Y)$ be an almost history-dependent operator. Then, Λ has a unique fixed point, i.e., there exists a unique element $\eta^* \in C(I;Y)$ such that $\Lambda\eta^* = \eta^*$.*

A proof of Theorem 2.2 can be found in [32, p. 41–45]. There, the main properties of history-dependent and almost history-dependent operators are stated and proved, together with various examples and applications.

We now end this section with the following result on history-dependent operators.

Theorem 2.3 *Let $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be Banach spaces. Assume that $\mathcal{S}\colon C(I;Y) \to C(I;Z)$ is a history-dependent operator and A is an operator which satisfies the following condition*

$$\begin{cases} A\colon Y \to Z \text{ is a linear continuous invertible operator and} \\ \text{there exists } m_A > 0 \text{ such that } \|Ay\|_Z \ge m_A \|y\|_Y \text{ for all } \ y \in Y. \end{cases}$$

Then the operator $A+\mathcal{S}\colon C(I;Y)\to C(I;Z)$ *is invertible and its inverse is of the form* $A^{-1}+\mathcal{R}\colon C(I;Z)\to C(I;Y)$, *where* $\mathcal{R}\colon C(I;Z)\to C(I;Y)$ *is a history-dependent operator.*

A proof of Theorem 2.3 can be found in [32, p. 55–56]. Note that here and below, $A+\mathcal{S}$ represents a shorthand notation for the operator which maps any function $u\in C(I;X)$ to the function $t\mapsto Au(t)+\mathcal{S}u(t)\in C(I;Y)$. Notation $A^{-1}+\mathcal{R}$ has a similar meaning.

3 A Time-Dependent Inclusion

In this section we state and prove an existence and uniqueness result for a time-dependent inclusion involving nonlinear operators. Throughout this section and the following one, we consider a set-valued mapping $K:I\to 2^X$ and two operators $A:X\to X$, $\mathcal{S}:C(I;X)\to C(I;X)$. With the above notation at hands, we introduce the following inclusion problem.

Problem 1 Find a function $u:I\to X$ such that

$$-u(t)\in N_{K(t)}\big(Au(t)+\mathcal{S}u(t)\big)\quad \text{for all }\ t\in I. \tag{3.1}$$

In the study of (3.1) we consider the following assumptions.

($\mathcal{K}$) The set-valued mapping $K:I\to 2^X$ has nonempty closed and convex values and for each $t\in I$ and each sequence $\{t_n\}\subset I$ converging to t, one has

$$P_{K(t_n)}u\to P_{K(t)}u\quad \text{in }\ X,\qquad \text{for any }\ u\in X.$$

($\mathcal{A}$) The operator A is strongly monotone and Lipschitz continuous for some reals $m_A, L_A>0$, respectively.

($\mathcal{H}$) For any nonempty compact set $\mathcal{J}\subset I$, there exist two reals $l_{\mathcal{J}}^{\mathcal{S}}>0$ and $L_{\mathcal{J}}^{\mathcal{S}}>0$ such that for all $u_1,u_2\in C(I;X)$ and $t\in\mathcal{J}$ the inequality (2.7) holds with $Y=Z=X$.

Note that examples of families $\{K(t)\}_{t\in I}$ which satisfies assumption ($\mathcal{K}$) will be presented in Sect. 5 below. We now state and prove our existence and uniqueness result for Problem 1.

Theorem 3.1 *Assume that* ($\mathcal{K}$), ($\mathcal{A}$), ($\mathcal{H}$) *hold. Moreover, assume that for any nonempty compact set* $\mathcal{J}\subset I$ *the following smallness condition holds*:

$$l_{\mathcal{J}}^{\mathcal{S}}<m_A. \tag{3.2}$$

Then, *Problem* 1 *has a unique solution with regularity* $u\in C(I;X)$.

The proof of Theorem 3.1 is carried out in several steps, based on a number of preliminary results that we present in what follows.

Lemma 3.2 *Let* K *be a nonempty closed convex subset of* X, $B:X\to X$ *an operator and* $z,\eta\in X$. *Then*, *the following statements are equivalent*:

(a) $z=P_K\big(z-B(z-\eta)\big)$;

(b) *there exists* $\rho>0$ *such that* $z=P_K\big(z-\rho B(z-\eta)\big)$;

(c) $z=P_K\big(z-\rho B(z-\eta)\big)$ *for all* $\rho>0$.

Proof According to (2.1), for every real $\rho > 0$ we have the equivalence

$$z = P_K\big(z - \rho B(z-\eta)\big) \iff z \in K, \quad (z, v-z)_X \geq \big(z - \rho B(z-\eta), v-z\big)_X \quad \forall v \in K$$

which can be rewritten as

$$z = P_K\big(z - \rho B(z-\eta)\big) \iff z \in K, \quad \rho\big(B(z-\eta), v-z\big)_X \geq 0 \quad \forall v \in K. \tag{3.3}$$

Then, taking $\rho = 1$ leads to the equivalence

$$z = P_K\big(z - B(z-\eta)\big) \iff z \in K, \quad \big(B(z-\eta), v-z\big)_X \geq 0 \quad \forall v \in K. \tag{3.4}$$

On the other hand, it is readily seen that

$$\begin{aligned} &\big(B(z-\eta), v-z\big)_X \geq 0 \quad \forall v \in K \\ \iff &\exists \rho > 0, \quad \rho\big(B(z-\eta), v-z\big)_X \geq 0 \quad \forall v \in K \\ \iff &\forall \rho > 0, \quad \rho\big(B(z-\eta), v-z\big)_X \geq 0 \quad \forall v \in K. \end{aligned} \tag{3.5}$$

It remains to combine the equivalences (3.3)–(3.5) to conclude the proof. □

Lemma 3.3 *Let K a nonempty closed convex subset of X and let $B : X \to X$ be a strongly monotone Lipschitz continuous operator. Then, for each $\eta \in X$ there exists a unique element $z_\eta \in X$ such that*

$$z_\eta = P_K\big(z_\eta - B(z_\eta - \eta)\big).$$

Proof Fix $\eta \in X$. Since B is strongly monotone and Lipschitz continuous, there exists $m_B > 0$ and $L_B > 0$ such that

$$(Bu - Bv, u-v)_X \geq m_B \|u-v\|_X^2 \quad \text{and} \quad \|Bu - Bv\|_X \leq L_B \|u-v\|_X, \tag{3.6}$$

for all $u,\ v \in X$. Pick any real $\rho > 0$ such that

$$0 < \rho < \frac{2m_B}{L_B^2} \tag{3.7}$$

and consider the operator $\Lambda_\rho : X \to X$ defined by

$$\Lambda_\rho z := P_K\big(z - \rho B(z-\eta)\big) \quad \text{for all } z \in X. \tag{3.8}$$

Thanks to the smallness assumption (3.7), we obviously have

$$k_\rho := \sqrt{1 + \rho^2 L_B^2 - 2\rho m_B} \in (0, 1). \tag{3.9}$$

Fix any $z_1,\ z_2 \in X$ and set $u_i := z_i - \eta$, for $i = 1,\ 2$. We use the definition of Λ_ρ in (3.8), the nonexpansivity (2.2) of the projection operator P_K and (3.6) to see that

$$\begin{aligned} \|\Lambda_\rho z_1 - \Lambda_\rho z_2\|_X^2 &\leq \|\big(z_1 - \rho B(z_1-\eta)\big) - \big(z_2 - \rho B(z_2 - \eta)\big)\|_X^2 \\ &= \|(u_1 - u_2) - \rho\big(Bu_1 - Bu_2\big)\|_X^2 \end{aligned}$$

$$= \|u_1 - u_2\|_X^2 - 2\rho\big(u_1 - u_2, Bu_1 - Bu_2\big) + \rho^2\|Bu_1 - Bu_2\|_X^2$$
$$\leq \big(1 + \rho^2 L_B^2 - 2\rho m_B\big)\|u_1 - u_2\|_X^2.$$

Taking into account the equality $u_1 - u_2 = z_1 - z_2$, we obtain

$$\|\Lambda_\rho z_1 - \Lambda_\rho z_2\|_X^2 \leq \big(1 + \rho^2 L_B^2 - 2\rho m_B\big)\|z_1 - z_2\|_X^2. \tag{3.10}$$

Combining (3.10) and (3.9), we then see that Λ_ρ is a contraction on X. Thus, we are in a position to apply the Banach fixed point theorem to get a unique $z_\eta \in X$ such that $\Lambda_\rho z_\eta = z_\eta$. We now combine (3.8) and implication (b) $\Longrightarrow$ (a) in Lemma 3.2 to conclude the proof. □

Lemma 3.4 *Assume that* ($\mathcal{K}$) *holds and let* $B : X \to X$ *be a strongly monotone Lipschitz continuous operator. Then, for each* $\eta \in C(I; X)$, *there exists a unique function* $z_\eta \in C(I; X)$ *such that*

$$z_\eta(t) = P_{K(t)}\Big(z_\eta(t) - B\big(z_\eta(t) - \eta(t)\big)\Big) \quad \textit{for all } t \in I. \tag{3.11}$$

Proof Let $\eta \in C(I; X)$. Note that the existence of an element $z_\eta(\cdot)$ which satisfies (3.11) is a direct consequence of Lemma 3.3. We now prove the continuity of the function $z_\eta(\cdot)$. Fix any $t \in I$ and consider a sequence $(t_n)_{n\in\mathbb{N}}$ of elements of I which converges to t. For each $n \in \mathbb{N}$ denote $K_n := K(t_n)$, $\sigma_n := \eta(t_n)$ $z_n := z_\eta(t_n)$, and $\omega_n := z_n - \rho B(z_n - \sigma_n)$ with $\rho > 0$ given. Set also $K := K(t)$, $\sigma := \eta(t)$, $z := z_\eta(t)$, and $\omega := z - \rho B(z - \sigma)$. With the above notation at hands, using Lemma 3.2 we see that

$$z = P_K\omega \quad \text{and} \quad z_n = P_{K_n}\omega_n \quad \text{for all } n \in \mathbb{N}. \tag{3.12}$$

Let $\rho > 0$ be such that (3.7) hold and let k_ρ be defined by (3.9) where, recall, $m_B, L_B > 0$ are the constants which appear in (3.6). Then, using (3.12), we get

$$\|z_n - z\|_X \leq \big\|P_K\omega - P_{K_n}\omega\big\|_X + \big\|P_{K_n}\omega - P_{K_n}\omega_n\big\|_X. \tag{3.13}$$

We now estimate each of the two terms in the right hand side of (3.13). To this end we set

$$u := z - \sigma \quad \text{and} \quad u_n := z_n - \sigma_n \quad \text{for all } n \in \mathbb{N}. \tag{3.14}$$

Let $n \in \mathbb{N}$. Thanks to (3.14), we see that

$$\big\|P_{K_n}\omega - P_{K_n}\omega_n\big\|_X \leq \|\omega - \omega_n\|_X = \|z - \rho B(z - \sigma) - z_n + \rho B(z_n - \sigma_n)\|_X$$
$$= \|u - u_n - \rho(Bu - Bu_n) + \sigma - \sigma_n\|_X$$
$$\leq \|u - u_n - \rho(Bu - Bu_n)\|_X + \|\sigma - \sigma_n\|_X \tag{3.15}$$

Next, arguments similar to those used in the proof of (3.10) yield

$$\|u - u_n - \rho(Bu - Bu_n)\|_X \leq k_\rho\|u - u_n\|_X. \tag{3.16}$$

We now combine inequalities (3.13), (3.15) and (3.16) to see that

$$\|z_n - z\|_X \leq \big\|P_K\omega - P_{K_n}\omega\big\|_X + k_\rho\|u - u_n\|_X + \|\sigma - \sigma_n\|_X.$$

On the other hand (3.14) implies that

$$\|u_n - u\|_X \le \|z_n - z\|_X + \|\sigma_n - \sigma\|_X$$

and, therefore, the last two inequalities yield

$$(1 - k_\rho)\|z_n - z\|_X \le \left\| P_K\omega - P_{K_n}\omega \right\|_X + (1 + k_\rho)\|\sigma_n - \sigma\|_X. \tag{3.17}$$

We now use (3.17), the inclusion $k_\rho \in (0, 1)$, the assumption $(\mathcal{K})$ and the continuity of the function $\eta : I \to X$ to obtain that $z_n = z_\eta(t_n) \to z_\eta(t) = z$ in X, as $n \to \infty$. This shows that the function $z_\eta : I \to X$ is continuous. The existence part of the lemma is then established. The uniqueness part is a direct consequence of the uniqueness property provided by Lemma 3.3. □

Lemma 3.5 *Assume that* $(\mathcal{K})$ *and* $(\mathcal{A})$ *hold. Then, for each* $\eta \in C(I; X)$, *there exists a unique function* $u_\eta \in C(I; X)$ *such that*

$$-u_\eta(t) \in N_{K(t)}\big(Au_\eta(t) + \eta(t)\big) \quad \textit{for all } t \in I.$$

Proof Let $\eta \in C(I; X)$. Denote by $z_\eta \in C(I; X)$ the function obtained in Lemma 3.4 with $B := A^{-1}$, where A^{-1} represents the inverse operator of A. Consider the function $u_\eta : I \to X$ defined by

$$u_\eta(t) := A^{-1}\big(z_\eta(t) - \eta(t)\big) \quad \text{for all } t \in I$$

and note that $u_\eta \in C(I; X)$. It is readily seen that

$$z_\eta(t) = P_{K(t)}\big(z_\eta(t) - u_\eta(t)\big) \quad \text{for all } t \in I. \tag{3.18}$$

Next, (3.18) and (2.6) entail that

$$-u_\eta(t) \in N_{K(t)}\big(z_\eta(t)\big) \quad \text{for all } t \in I.$$

The existence part of the lemma is then established.

We now focus on the uniqueness part. Let $u_1, u_2 \in C(I; X)$ be two functions such that

$$-u_1(t) \in N_{K(t)}\big(Au_1(t) + \eta(t)\big) \quad \text{and} \quad -u_2(t) \in N_{K(t)}\big(Au_2(t) + \eta(t)\big)$$

for every $t \in I$. Fix any $t \in I$. Then, for $i = 1, 2$, we have

$$Au_i(t) + \eta(t) \in K(t), \quad \big(u_i(t), Au_i(t) + \eta(t) - v\big)_X \le 0 \quad \text{for all } v \in K(t).$$

We derive from this

$$\big(u_1(t), Au_1(t) + \eta(t) - (Au_2(t) + \eta(t))\big)_X \le 0$$

and

$$\big(u_2(t), Au_2(t) + \eta(t) - (Au_1(t) + \eta(t))\big)_X \le 0.$$

Adding the latter inequalities yields

$$\big(u_1(t) - u_2(t), Au_1(t) - Au_2(t)\big)_X \le 0.$$

Finally, we use the strong monotonicity of the operator A to get that $u_1(t) = u_2(t)$ which completes the proof. □

Lemma 3.5 allows us to consider the operator $\Lambda : C(I;X) \to C(I;X)$ defined by

$$\Lambda\eta := \mathcal{S}u_\eta \quad \text{for all } \eta \in C(I;X). \tag{3.19}$$

We have the following result.

Lemma 3.6 *Assume that* $(\mathcal{K})$, $(\mathcal{A})$, $(\mathcal{H})$ *and* (3.2) *hold. Then, the operator* Λ *has a unique fixed point* $\eta^* \in C(I;X)$.

Proof According to Theorem 2.2, it is enough to prove that the operator $\Lambda : C(I;X) \to C(I;X)$ is an almost history-dependent operator. Let $\eta_1, \eta_2 \in C(I;X)$. Using Lemma 3.5, we find two continuous functions $u_1 := u_{\eta_1} : I \to X$ and $u_2 := u_{\eta_2} : I \to X$ such that

$$-u_1(t) \in N_{K(t)}\big(Au_1(t) + \eta_1(t)\big) \quad \text{and} \quad -u_2(t) \in N_{K(t)}\big(Au_2(t) + \eta_2(t)\big) \tag{3.20}$$

for all $t \in I$. Let $\mathcal{J}$ be a nonempty compact subset of I and $t \in \mathcal{J}$. Using (3.19) and assumption $(\mathcal{H})$ yield

$$\begin{aligned} \|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_X &= \|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_X \\ &\le l_{\mathcal{J}}^{\mathcal{S}}\|u_1(t) - u_2(t)\|_X + L_{\mathcal{J}}^{\mathcal{S}}\int_0^t \|u_1(s) - u_2(s)\|_X ds \end{aligned} \tag{3.21}$$

On the other hand, from (3.20) we see that

$$Au_1(t) + \eta_1(t) \in K(t), \quad \big(u_1(t), Au_1(t) + \eta_1(t) - v\big)_X \le 0 \quad \forall v \in K(t) \tag{3.22}$$

and

$$Au_2(t) + \eta_2(t) \in K(t), \quad \big(u_2(t), Au_2(t) + \eta_2(t) - w\big)_X \le 0 \quad \forall w \in K(t). \tag{3.23}$$

Taking $v := Au_2(t) + \eta_2(t)$ in (3.22) and $w := Au_1(t) + \eta_1(t)$ in (3.23) and adding the corresponding inequalities, we arrive to

$$\big(u_1(t) - u_2(t), Au_1(t) - Au_2(t)\big)_X \le \big(u_1(t) - u_2(t), \eta_2(t) - \eta_1(t)\big)_X.$$

Therefore, the strong monotonicity of the operator A guarantees that

$$\|u_1(t) - u_2(t)\|_X \le \frac{1}{m_A}\|\eta_1(t) - \eta_2(t)\|_X.$$

Substituting this inequality in (3.21) yields

$$\begin{aligned} \|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_X &= \|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_X \\ &\le \frac{l_{\mathcal{J}}^{\mathcal{S}}}{m_A}\|\eta_1(t) - \eta_2(t)\|_X + \frac{L_{\mathcal{J}}^{\mathcal{S}}}{m_A}\int_0^t \|\eta_1(s) - \eta_2(s)\|_X\, ds. \end{aligned}$$

We now invoke the smallness assumption (3.2) to obtain that the operator Λ is an almost history-dependent operator. It remains to apply Theorem 2.2 to complete the proof. □

We are now in a position to provide the proof of Theorem 3.1.

Proof Let $\eta^* \in C(I;X)$ be the fixed point of the operator Λ and let $u^\star := u_{\eta^\star} \in C(I;X)$ be the function given by Lemma 3.5 with $\eta := \eta^\star$. So, we have

$$-u^\star(t) \in N_{K(t)}\big(Au(t) + \eta(t)\big) \quad \text{for all } t \in I.$$

This inclusion combined with equalities $\eta^\star = \Lambda\eta^\star = \mathcal{S}u^\star$ give

$$-u^\star(t) \in N_{K(t)}\big(Au^\star(t) + \mathcal{S}u^\star(t)\big) \quad \text{for all } t \in I,$$

that is, $u^\star$ is a solution to Problem 1. This proves the existence part of Theorem 3.1. The uniqueness part is a consequence of the uniqueness of the fixed point of the operator Λ, guaranteed by Lemma 3.3.

Note that the uniqueness part in Theorem 3.1 can also be obtained through a Gronwall-type argument. Indeed, let $\eta_1,\ \eta_2 \in C(I;X)$ be two solutions to Problem 1. Let $\mathcal{J}$ be a nonempty compact subset of I and $t \in \mathcal{J}$. Then, we have

$$-u_1(t) \in N_{K(t)}\big(Au_1(t) + \mathcal{S}u_1(t)\big), \quad -u_2(t) \in N_{K(t)}\big(Au_2(t) + \mathcal{S}u_2(t)\big)$$

or, equivalently,

$$Au_1(t) + \mathcal{S}u_1(t) \in K(t), \qquad \big(u_1(t), Au_1(t) + \mathcal{S}u_1(t) - v\big)_X \le 0 \quad \forall v \in K(t), \tag{3.24}$$

$$Au_2(t) + \mathcal{S}u_2(t) \in K(t), \qquad \big(u_2(t), Au_2(t) + \mathcal{S}u_2(t) - v\big)_X \le 0 \quad \forall v \in K(t). \tag{3.25}$$

Taking $v = Au_2(t) + \mathcal{S}u_2(t)$ in (3.24), $v = Au_1(t) + \mathcal{S}u_1(t)$ in (3.25) and adding the corresponding inequalities, we deduce that

$$\big(u_1(t) - u_2(t), Au_1(t) - Au_2(t)\big)_X \le \big(u_1(t) - u_2(t), \mathcal{S}u_2(t) - \mathcal{S}u_1(t)\big)_X.$$

Therefore, using the strong monotonicity of the mapping A, we get

$$\|u_1(t) - u_2(t)\|_X \le \frac{1}{m_A}\|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_X.$$

Thanks to assumption $(\mathcal{H})$ we obtain

$$\|u_1(t) - u_2(t)\|_X \le \frac{l_{\mathcal{J}}^{\mathcal{S}}}{m_A}\|u_1(t) - u_2(t)\|_X + \frac{L_{\mathcal{J}}^{\mathcal{S}}}{m_A}\int_0^t \|u_1(s) - u_2(s)\|_X\, ds.$$

We now use the smallness assumption (3.2) to see that there exists a constant $C > 0$ which depends on A, $\mathcal{S}$ and $\mathcal{J}$ such that

$$\|u_1(t) - u_2(t)\|_X \le C\int_0^t \|u_1(s) - u_2(s)\|_X\, ds.$$

It follows now from the Gronwall lemma that $u_1(t) = u_2(t)$ and, since $t \in \mathcal{J}$ has been arbitrarily chosen, we find that $u_1 = u_2$. □

Since any history-dependent operator satisfies the smallness condition (3.2) (keep in mind that $l_{\mathcal{J}}^{\mathcal{S}} = 0$ for any nonempty compact $\mathcal{J} \subset I$), we derive from Theorem 3.1 the following result.

Corollary 3.7 *Assume that* $(\mathcal{K})$ *and* $(\mathcal{A})$ *hold. Assume also that* $\mathcal{S}: C(I;X) \to C(I;X)$ *is a history-dependent operator. Then, Problem* 1 *has a unique solution* $u \in C(I;X)$.

We end this section with some additional comments on the smallness condition (3.2). First, we note that it represents a sufficient condition which guarantees the unique solvability of the time-dependent inclusion (3.1). It does not represent a necessary condition as it results from the following counterexample.

Example 1 Let $X = \mathbb{R}$, $K(t) = (-\infty, 0]$ for all $t \in I$, $Au = u$ for all $u \in \mathbb{R}$, $\mathcal{S}u = 2u$ for all $u \in C(I;X)$. Then, we note that assumptions $(\mathcal{K})$, $(\mathcal{A})$, $(\mathcal{H})$ hold with $m_A = 1$ and $l_{\mathcal{J}}^{\mathcal{S}} = 2$. It follows from here that the smallness assumption (3.2) does not hold. Nevertheless, it is easy to see that, with the notation above, the inclusion (3.1) has the unique solution $u \equiv 0$.

We conclude from above that assumption (3.2) is a technical one and it represents only a limitation of the mathematical tools we use in the proof of Theorem 3.1. More precisely, this assumption is used in Lemma 3.6 at is imposed by Theorem 2.2 and Definition 2.1 b). Removing this assumption in the statement of Theorems 3.1 and 4.1 below is left open and represents a question which deserves to be studied in the future.

4 A Sweeping Process

In this section, we use Theorem 3.1 to derive an existence and uniqueness results for a first order sweeping processes. Besides the data K, A and $\mathcal{S}$ and their associated assumptions $(\mathcal{K})$, $(\mathcal{A})$ and $(\mathcal{H})$ introduced in the Sect. 3, we consider an operator $B : X \to X$ and an element u_0 such that:

$(\mathcal{B})$ B is a Lipschitz continuous operator with Lipschitz constant $L_B > 0$.
$(\mathcal{U})$ $u_0 \in X$.

We are now in a position to introduce the following sweeping process.

Problem 2 Find a function $u : I \to X$ such that

$$\begin{cases} -\dot{u}(t) \in N_{K(t)}(A\dot{u}(t) + Bu(t) + \mathcal{S}\dot{u}(t)) \quad \text{for all } t \in I, \\ u(0) = u_0. \end{cases}$$

Our first result in this section is the following.

Theorem 4.1 *Assume that* $(\mathcal{K})$, $(\mathcal{A})$, $(\mathcal{H})$, $(\mathcal{B})$, $(\mathcal{U})$ *and* (3.2) *hold. Then, Problem* 2 *has a unique solution with regularity* $u \in C^1(I;X)$.

Proof We first introduce the operator $\widetilde{\mathcal{S}}: C(I;X) \to C(I;X)$ defined by

$$\widetilde{\mathcal{S}}v(t) := B\Big(\int_0^t v(s)\,ds + u_0\Big) + \mathcal{S}v(t) \quad \text{for all } t \in I, \text{ all } v \in C(I;X). \tag{4.1}$$

Next, we consider the auxiliary problem of finding a function $v : I \to X$ such that

$$-v(t) \in N_{K(t)}\big(Av(t) + \widetilde{\mathcal{S}}v(t)\big) \quad \text{for all } t \in I. \tag{4.2}$$

We use assumptions ($\mathcal{H}$) and ($\mathcal{B}$) to see that for any nonempty compact set $\mathcal{J} \subset I$, any functions $v_1, v_2 \in C(I; X)$ and any $t \in I$, the inequality below holds:

$$\|\widetilde{\mathcal{S}}v_1(t) - \widetilde{\mathcal{S}}v_2(t)\|_X \le l_{\mathcal{J}}^{\mathcal{S}} \|v_1(t) - v_2(t)\|_X + (L_B + L_{\mathcal{J}}^{\mathcal{S}}) \int_0^t \|v_1(s) - v_2(s)\|_X \, ds.$$

Therefore, we are in a position to apply Theorem 3.1 in order to obtain the existence of a unique function $v \in C(I; X)$ which satisfies the time-dependent inclusion (4.2). Denote by $u : I \to X$ the function defined by

$$u(t) := u_0 + \int_0^t v(s)ds \quad \text{for all } \ t \in I. \tag{4.3}$$

Then, (4.1)–(4.3) imply that u is a solution of Problem 2 with regularity $u \in C^1(I; X)$. This proves the existence part of the theorem. The uniqueness part follows from the unique solvability of the auxiliary problem (4.2), guaranteed by Theorem 3.1. □

A direct consequence of Theorem 4.1 is the following.

Corollary 4.2 *Assume that the assumptions* ($\mathcal{K}$), ($\mathcal{A}$), ($\mathcal{B}$) *and* ($\mathcal{U}$) *hold. Assume also that* $\mathcal{S} : C(I; X) \to C(I; X)$ *is a history-dependent operator. Then, Problem 2 has a unique solution with regularity* $u \in C^1(I; X)$.

The proof of Corollary 4.2 follows from arguments similar to those used in the proof of Corollary 3.7 and, therefore, we skip it. Finally, we note that, as in the case of Problem 1, the smallness condition (3.2) represents only a sufficient condition which guarantees the unique solvability of Problem 2.

5 Relevant Examples of Convex Moving Sets

In this section we state and prove additional results which provide examples of families of convex sets which satisfy assumption ($\mathcal{K}$). A first example is the following.

Proposition 5.1 *Let M be a closed linear subspace of X, $A : X \to M$ the projector onto M and $k : I \to \mathbb{R}_+$ a continuous function. Let $K : I \to 2^X$ be the set-valued mapping defined by*

$$K(t) := \{u \in X : \|Au\| \le k(t)\} \quad \textit{for all } \ t \in I.$$

Then, the set-valued mapping $K(\cdot)$ satisfies assumption ($\mathcal{K}$).

Proof It is routine to check that $K(t)$ is nonempty closed and convex for every $t \in I$. The rest of the proof is carried out in three steps that we present in what follows. Everywhere below we denote by $M^\perp$ the orthogonal of M, use the decomposition $X = M \oplus M^\perp$ (keep in mind that M is closed) and the fact that A is a linear, continuous and idempotent operator, i.e., $A^2 = A$.

Claim 1. *For every $t \in I$ and every $u \in X$, one has*

$$P_{K(t)}(u) = \begin{cases} u - Au + \frac{k(t)}{\|Au\|_X} Au & \text{if } u \notin K(t), \\ u & \text{otherwise.} \end{cases} \tag{5.1}$$

To prove the claim we fix $t \in I$ and $u \in X$ and, for simplicity, we set $K := K(t)$. We may assume that $u \notin K$, i.e., $\|Au\|_X > k(t)$. Set $v := u - Au + \frac{k(t)}{\|Au\|_X} Au$. It is readily seen that

$$Av = \frac{k(t)}{\|Au\|_X} Au,$$

and, therefore, $\|Av\| = k(t)$. Hence, we deduce that $v \in K$. Fix any $w \in K$. Through elementary computations we see that

$$(v - u, w - v)_X = \Big(\frac{k(t)}{\|Au\|} - 1\Big)(Au, w - v)_X \tag{5.2}$$

and

$$(Au, w - v)_X = (Au, w - Aw)_X + (Au, Aw)_X - (Au, u - Au)_X - (Au, \frac{k(t)}{\|Au\|_X} Au)_X.$$

On the other hand, the equalities $u = u - Au + Au$ and $w = w - Aw + Aw$ yield

$$u - Au \in M^{\perp} \quad \text{and} \quad w - Aw \in M^{\perp}.$$

Therefore,

$$\begin{aligned}(Au, w - v)_X &= (Au, Aw)_X - (Au, \frac{k(t)}{\|Au\|_X} Au)_X \\ &\le \|Au\|_X \|Aw\|_X - k(t)\|Au\|_X\end{aligned}$$

which shows that

$$(Au, w - v)_X \le \|Au\|_X (\|Aw\|_X - k(t)). \tag{5.3}$$

Combining now (5.2), (5.3) and using inequality $\|Aw\|_X - k(t) \le 0$, guaranteed by inclusion $w \in K$, we find that

$$\Big(\frac{k(t)}{\|Au\|} - 1\Big)^{-1} (v - u, w - v)_X = (Au, w - v)_X \le 0.$$

Now, since $\|Au\|_X > k(t)$, we deduce that $(v - u, w - v)_X \ge 0$. Finally, we use (2.1) to complete the proof of the claim.

Claim 2. *For any $s, t \in I$ and every $u \in X$, one has*

$$\|P_{K(s)}u - P_{K(t)}u\| \le |k(t) - k(s)|\,. \tag{5.4}$$

Let $s, t \in I$ and $u \in X$. If $k(s) = k(t)$ we have $K(s) = K(t)$ and, therefore, (5.4) holds. Assume now that $k(s) \ne k(t)$ and, without loosing the generality, assume that $k(s) < k(t)$. We distinguish the following three cases.

Case a): $\|Au\| \le k(s) < k(t)$. We use (5.1) to see that in this case $P_{K(s)}u = P_{K(t)}u = u$ which implies (5.4).

Case b): $k(s) < \|Au\| \le k(t)$. In this case (5.1) yields

$$P_{K(s)}u = u - Au + \frac{k(s)}{\|Au\|} Au \quad \text{and} \quad P_{K(t)}u = u$$

which implies that

$$\left\| P_{K(s)}u - P_{K(t)}u \right\|_X = \Big(1 - \frac{k(s)}{\|Au\|_X}\Big)\|Au\|_X = \|Au\|_X - k(s) \le |k(t) - k(s)|\,.$$

We conclude from here that inequality (5.4) is satisfied.

Case c): $k(s) < k(t) < \|Au\|$. Then, using (5.1) we have

$$P_{K(s)}u = u - Au + \frac{k(s)}{\|Au\|}\,Au \quad \text{and} \quad P_{K(t)}u = u - Au + \frac{k(t)}{\|Au\|}\,Au$$

and, therefore,

$$\left\| P_{K(s)}u - P_{K(t)}u \right\|_X = |k(t) - k(s)|\,,$$

which completes the proof of this claim.

We now use inequality (5.4) and the continuity of the function k to conclude the proof of Proposition 5.1. □

We proceed with a specific example provided by Proposition 5.1 and, to this end, we need the following notation. Let $d \in \{1, 2, 3\}$ and denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. Recall that the inner product and the Euclidean norm on $\mathbb{S}^d$ are defined by

$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}\ , \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,$$

where the indices i, j run between 1 and d and the summation convention over repeated indices is used. Let M be the subspace of $\mathbb{S}^d$ defined by

$$M = \{\, \boldsymbol{\sigma} = (\sigma_{ij}) \in \mathbb{S}^d \ :\ \sigma_{ii} = 0 \,\}$$

and, for each $\boldsymbol{\varepsilon} \in \mathbb{S}^d$, denote by $\boldsymbol{\varepsilon}^D$ its orthogonal projection of $\boldsymbol{\varepsilon}$, called also the deviator of $\boldsymbol{\varepsilon}$. With these notation we have the following direct consequence of Proposition 5.1.

Corollary 5.2 *Let $k : I \to \mathbb{R}_+$ be a continuous function and let $K : I \to 2^{\mathbb{S}^d}$ be the set-valued mapping defined by*

$$K(t) = \{\, \boldsymbol{\varepsilon} \in \mathbb{S}^d \ :\ \|\boldsymbol{\varepsilon}^D\| \le k(t) \,\} \quad \textit{for all}\ \ t \in I. \tag{5.5}$$

Then, the set-valued mapping $K(\cdot)$ satisfies assumption $(\mathcal{K})$ on the space $X = \mathbb{S}^d$.

The convex set defined by (5.5) is called the *von Mises convex*. It is intensively used in Solid Mechanics (see, e.g., [32] and the references therein).

We now move to another class of examples provided by the following result.

Proposition 5.3 *Let K_0 be a closed convex nonempty subset of X and let $a : I \to (0, +\infty)$ be a continuous function. Let also $f \in C(I; X)$ and $K : I \to 2^X$ be the set-valued mapping defined by*

$$K(t) := a(t)K_0 + f(t) \quad \textit{for all}\ \ t \in I. \tag{5.6}$$

Then, the set-valued mapping $K(\cdot)$ satisfies assumption $(\mathcal{K})$.

Proof Using (5.6), it is not difficult to check that $K(t)$ is nonempty closed and convex, for any $t \in I$. The rest of the proof is based on the following equality:

$$P_{K(t)}u = a(t)P_{K_0}\Big(\frac{u - f(t)}{a(t)}\Big) + f(t) \quad \text{for all } \ t \in I \text{ and } u \in X. \tag{5.7}$$

To prove (5.7) we fix $t \in I$ and $u \in X$ and denote by $z(t)$ the projection of u on the set $K(t)$. Then, using (5.6) there exists a unique element $z_0 \in K_0$ such that

$$z(t) = P_{K(t)}u = a(t)z_0 + f(t). \tag{5.8}$$

Moreover, we have

$$\|z(t) - u\|_X \le \|v(t) - u\|_X \quad \text{for all } v \in K(t),$$

or equivalently,

$$\|a(t)z_0 + f(t) - u\|_X \le \|a(t)v_0 + f(t) - u\|_X \quad \text{for all } v_0 \in K_0.$$

This implies that

$$\Big\|z_0 - \Big(\frac{u - f(t)}{a(t)}\Big)\Big\|_X \le \Big\|v_0 - \Big(\frac{u - f(t)}{a(t)}\Big)\Big\|_X \quad \text{for all } v_0 \in K_0,$$

and, therefore,

$$z_0 = P_{K_0}\Big(\frac{u - f(t)}{a(t)}\Big). \tag{5.9}$$

Equality (5.7) is now a direct consequence of (5.8) and (5.9).

Assume now that $t \in I$ and $t_n \to t$. Then, using (5.7) for t_n and t and the continuity of the function a and f it is easy to see that $P_{K(t_n)}u \to P_K(u)$ in X which concludes the proof. □

A direct consequence of Proposition 5.3 follows.

Corollary 5.4 *Let K_0 be a nonempty closed convex subset of X, $f \in C(I; X)$ and let $K : I \to 2^X$ be the set-valued mapping defined by*

$$K(t) = K_0 + f(t) \quad \textit{for all } \ t \in I.$$

Then, the set-valued mapping $K(\cdot)$ satisfies assumption $(\mathcal{K})$.

Besides the various examples above, we point out that the assumption $(\mathcal{K})$ is strongly related to several concepts of convergence of sets. In order to recall them, we need the *Pompeiu-Hausdorff* distance defined by equality

$$\text{haus}\,(C_1, C_2) := \max\Big\{\sup_{u \in C_2} d_{C_1}(u), \sup_{u \in C_1} d_{C_2}(u)\Big\},$$

for any two nonempty subsets C_1, C_2 of the Hilbert space X.

Definition 5.5 Let $\{C_n\}$ be a sequence of nonempty subsets of X and let C a nonempty subset of X. One says that the sequence $\{C_n\}$ converges to C:

(i) in the sense of Mosco, if the following conditions hold.
 (a) For each $u \in C$, there exists a sequence $\{u_n\}$ such that $u_n \in C_n$ for each $n \in \mathbb{N}$ and $u_n \to u$ in X.
 (b) For each sequence $\{u_n\}$ such that $u_n \in C_n$ for each $n \in \mathbb{N}$ and $u_n \to u$ weakly in X, we have $u \in C$.

(ii) in the sense of Wijsman, if

$$\lim_{n\to\infty} d_{C_n}(u) = d_C(u) \quad \text{for any } \ u \in X.$$

(iii) in the sense of Pompeiu-Hausdorff, if

$$\lim_{n\to\infty} \text{haus}\,(C_n, C) = 0.$$

The following results are related to assumption ($\mathcal{K}$) and could be useful in applications.

Proposition 5.6 *Let $\{C_n\}$ be a sequence of nonempty closed and convex subsets of X and let K be a nonempty closed convex subset of X. Then, the following statement are equivalent*:
 (a) *$\{C_n\}$ converges to C in the sense of Mosco.*
 (b) *$\{C_n\}$ converges to C in the sense of Wijsman.*
 (c) *$P_{C_n}u \to P_C u$ for any $u \in X$.*

Proposition 5.7 *Assume that $K : I \to 2^X$ has nonempty closed and convex values and, for each $t \in I$ and each sequence $\{t_n\} \subset I$ converging to t, the sequence $\{K(t_n)\}$ converges to $K(t)$ in the sense of Pompeiu-Hausdorff. Then the family $\{K(t)\}$ satisfies assumption ($\mathcal{K}$).*

A proof of the equivalences (a), (b), (c) in Proposition 5.6 can be found in [35] and, therefore, we skip it. The proof of Proposition 5.7 follows from the classical equality

$$\text{haus}(K(t_n), K) = \sup_{u\in X} \left| d_{K(t_n)}(u) - d_{K(t)}(u) \right|,$$

valid for every integer $n \in \mathbb{N}$, Definition 5.5 (ii) and the implication (b) $\Longrightarrow$ (c) above.

In the context of Definition 5.5, it follows from Proposition 5.7 and Proposition 5.6 that the convergence is the sense of Pompeiu-Hausdorff implies the convergence in the sense of Mosco. The converse of this statement does not hold. Indeed, we refer the reader to [9, Example 4.7.11] for an example of sequence of nonempty bounded closed convex sets in the Hilbert space $l^2(\mathbb{N})$ which converge in the sense of Mosco but fails to converge in the sense of Pompeiu-Hausdorff.

6 A Viscoelastic Constitutive Law

In this section we use the space $(\mathbb{S}^d, \|\cdot\|)$ introduced in Sect. 5 to apply Corollary 3.7 in the study of a viscoelastic constitutive law of the form

$$\boldsymbol{\sigma}(t) \in \mathcal{A}\boldsymbol{\varepsilon}(t) + \int_0^t b(t-s)\boldsymbol{\varepsilon}(s)\,ds + \partial\psi_{K(t)}\boldsymbol{\varepsilon}(t) \quad \text{for all } t \in I. \tag{6.1}$$

In (6.1) and everywhere below in this section, $\boldsymbol{\sigma} : I \to \mathbb{S}^d$ represents the stress tensor, $\boldsymbol{\varepsilon} : I \to \mathbb{S}^d$ the stain tensor, $\mathcal{A}$ is the fourth order tensor, b is a relaxation function, $\psi_{K(t)}$

is the indicator function of a time-dependent set $K(t) \subset \mathbb{S}^d$ and $\partial\psi_{K(t)}$ represents its subdifferential (in the sense of convex analysis). Such kind of constitutive laws model the behavior of real materials like metals, rocks and polymers and can be derived by rheological arguments, as explained in [10, 30]. Here, we restrict ourselves to mention that they are obtained by connecting in parallel an elastic element with looking with a viscoelastic element with long memory.

In the study of (6.1), we assume that the tensor $\mathcal{A}$ is symmetric and positively defined, i.e.,

$$\begin{cases} \text{(a) } \mathcal{A} : \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) } \mathcal{A}\boldsymbol{\varepsilon} = (a_{ijkl}\varepsilon_{kl}) \text{ for all } \boldsymbol{\varepsilon} = (\varepsilon_{ij}) \in \mathbb{S}^d. \\ \text{(c) } a_{ijkl} = a_{jikl} = a_{klij} \in \mathbb{R}. \\ \text{(d) There exists } m_A > 0 \text{ such that} \\ \qquad \mathcal{A}\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} \geq m_A\|\boldsymbol{\varepsilon}\|^2 \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \end{cases} \tag{6.2}$$

Here the indices $i,\ j,\ k$, run between 1 and d and the summation convention is used. Moreover, for the relaxation function and the set of constraints, we assume that

$$b : I \to \mathbb{R} \quad \text{is a continuous function,} \tag{6.3}$$

$$\begin{cases} K(t) = \{\, \boldsymbol{\varepsilon} \in \mathbb{S}^d \ : \ \|\boldsymbol{\varepsilon}^D\| \leq k(t) \,\} \text{ for all } t \in I \\ \text{where } k : I \to \mathbb{R}_+ \text{ is a continuous function.} \end{cases} \tag{6.4}$$

We proceed with the following comments on these assumptions. First, assumption (6.2) guarantees that the tensor $\mathcal{A}$ is invertible. Moreover, its inverse (denoted in what follows by $\mathcal{A}^{-1}$) is also symmetric and positively defined. Next, we stress that we chose assumption (6.3) for simplicity and point out that more general cases in which b is a fourth-order tensor valued can be considered. In addition, notation $\boldsymbol{\varepsilon}^D$ in (6.4) represents the deviator of the tensor $\boldsymbol{\varepsilon}$, introduced in Sect. 5. Therefore, (6.4) shows that the family $\{K(t)\}_{t\in I}$ represents a family of time-dependent von Mises convexes. Here, the dependence of the function k on t arises if we assume that $k(t) = \widetilde{k}(\theta(t))$ where $\widetilde{k} : \mathbb{R} \to \mathbb{R}$ is a given function and $\theta : I \to \mathbb{R}$ is a parameter, say the temperature.

Now, a direct analysis of (6.1) shows that at each moment t the stress field satisfies the equality

$$\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}_1(t) + \boldsymbol{\sigma}_2(t) \tag{6.5}$$

where $\boldsymbol{\sigma}_1(t)$ and $\boldsymbol{\sigma}_2(t)$ represent the "regular" and "irregular" part of the stress, given by

$$\boldsymbol{\sigma}_1(t) = \mathcal{A}\boldsymbol{\varepsilon}(t) + \int_0^t b(t-s)\boldsymbol{\varepsilon}(s)\,ds, \tag{6.6}$$

$$\boldsymbol{\sigma}_2(t) \in \partial\psi_{K(t)}\boldsymbol{\varepsilon}(t). \tag{6.7}$$

Denote $\boldsymbol{\omega} := -\boldsymbol{\sigma}_2$. Our aim in what follows is to determinate a relation between the functions $\boldsymbol{\omega}$ and $\boldsymbol{\sigma}$. First, we use assumption (6.3) to see that the operator $\widetilde{S} : C(I;\mathbb{S}^d) \to C(I;\mathbb{S}^d)$ defined by equality

$$\widetilde{\mathcal{S}}\boldsymbol{\varepsilon}(t) = \int_0^t b(t-s)\boldsymbol{\varepsilon}(s)\,ds \quad \text{for all } \boldsymbol{\varepsilon} \in C(I;\mathbb{S}^d),\ t \in I$$

is a history-dependent operator. Then, using Theorem 2.3 and equality (6.6) we deduce that there exists a history-dependent operator $\mathcal{R} : C(I;\mathbb{S}^d) \to C(I;\mathbb{S}^d)$ such that

$$\boldsymbol{\varepsilon}(t) = \mathcal{A}^{-1}\boldsymbol{\sigma}_1(t) + \mathcal{R}\boldsymbol{\sigma}_1(t). \tag{6.8}$$

We now combine equalities (6.5) and (6.8) and use notation $\boldsymbol{\omega} = -\boldsymbol{\sigma}_2$ to find that

$$\boldsymbol{\varepsilon}(t) = \mathcal{A}^{-1}\boldsymbol{\omega}(t) + \mathcal{A}^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}\big(\boldsymbol{\sigma}(t) + \boldsymbol{\omega}(t)\big)$$

and, therefore (6.7) shows that

$$-\boldsymbol{\omega}(t) \in \partial\psi_{K(t)}\Big(\mathcal{A}^{-1}\boldsymbol{\omega}(t) + \mathcal{A}^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}(\boldsymbol{\sigma}(t) + \boldsymbol{\omega}(t))\Big).$$

This inclusion combined with notation $N_{K(t)} = \partial\psi_K(t)$ leads to the following problem.

Problem 3 Find a function $\boldsymbol{\omega} : I \to \mathbb{S}^d$ such that

$$-\boldsymbol{\omega}(t) \in N_{K(t)}\Big(\mathcal{A}^{-1}\boldsymbol{\omega}(t) + \mathcal{A}^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}(\boldsymbol{\sigma}(t) + \boldsymbol{\omega}(t))\Big) \quad \text{for all } t \in I.$$

Our main result in this section is the following.

Theorem 6.1 *Assume that* (6.2)–(6.4) *hold. Assume also that* $\boldsymbol{\sigma} \in C(I;\mathbb{S}^d)$. *Then, Problem* 3 *has a unique solution* $\boldsymbol{\omega} \in C(I;\mathbb{S}^d)$.

Proof We use Corollary 3.7 with $X = \mathbb{S}^d$, $K(t)$ given by (6.4), $A = \mathcal{A}^{-1}$ and $\mathcal{S} : C(I;\mathbb{S}^d) \to C(I;\mathbb{S}^d)$ given by

$$\mathcal{S}\boldsymbol{\tau}(t) := \mathcal{A}^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}(\boldsymbol{\sigma}(t) + \boldsymbol{\tau}(t)) \quad \text{for all } \boldsymbol{\tau} \in C(I;\mathbb{S}^d),\ t \in I.$$

First, we note that assumptions (6.4) and Corollary 5.2 show that condition $(\mathcal{K})$ is satisfied. Next, we note that assumption (6.2) implies condition $(\mathcal{A})$. Finally, condition $(\mathcal{H})$ is satisfied since $\mathcal{R}$ is a history-dependent operator and $\boldsymbol{\sigma} \in C(I;\mathbb{S}^d)$. Theorem 6.1 is now a direct consequence of Corollary 3.7. □

In addition to the mathematical interest in the existence and uniqueness result in Theorem 6.1, it is important from mechanical point of view. Indeed, consider a given stress function $\boldsymbol{\sigma} \in C(I;\mathbb{S}^d)$; then, using Theorem 6.1 we can determinate in a unique way its "irregular" part $\boldsymbol{\sigma}_2$; next, using equalities (6.5) and (6.8) we can determinate the "regular" part of $\boldsymbol{\sigma}$, denoted $\boldsymbol{\sigma}_1$, as well as the strain function $\boldsymbol{\varepsilon}$ associated to $\boldsymbol{\sigma}$ within the constitutive law (6.1).

7 A Frictional Contact Problem

In this section, we apply Corollary 4.2 in the study of a mathematical model which describes the equilibrium of a viscoelastic body in frictional contact of with a foundation. Throughout this section $d \in \{2, 3\}$, $\mathbb{S}^d$ stands for the space of second order symmetric tensors on $\mathbb{R}^d$ and "$\cdot$", $\|\cdot\|$ will represent the inner product and the Euclidean norm on the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively.

The physical setting is as follows. A viscoelastic body occupies, in the reference configuration, a bounded domain $\Omega \subset \mathbb{R}^d$ with a Lipschitz continuous boundary Γ. The boundary Γ is divided intro three mutually disjoint measurable sets Γ_1, Γ_2 and Γ_3, such that the

Lebesgue measure of Γ_1 is positive. The body is held fixed on Γ_1, is acted upon by surface tractions of density $\boldsymbol{f}_2$ on Γ_2 and is in frictional contact on Γ_3 with a foundation. The contact is bilateral, i.e., there is no between separation the body and the foundation. We denote by $\boldsymbol{f}_0$ the density of body forces, by $\boldsymbol{\nu}$ the outward unit to Γ and, as usual, we use I for the time interval of interest. Moreover, the indices ν and τ will represent the normal and tangential components of vectors and tensors. Then, the contact model described above can be formulated as follows.

Problem 4 Find a displacement field $\boldsymbol{u}\colon \Omega \times I \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma}\colon \Omega \times I \to \mathbb{S}^d$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{C}(t-s)\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s))\,ds \qquad \text{in } \Omega, \tag{7.1}$$

$$\operatorname{Div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \qquad \text{in } \Omega, \tag{7.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{7.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \qquad \text{on } \Gamma_2, \tag{7.4}$$

$$u_\nu(t) = 0 \qquad \text{on } \Gamma_3, \tag{7.5}$$

$$\|\boldsymbol{\sigma}_\tau(t)\| \le g, \quad -\boldsymbol{\sigma}_\tau(t) = g\,\frac{\dot{\boldsymbol{u}}_\tau(t)}{\|\dot{\boldsymbol{u}}_\tau(t)\|} \quad \text{if} \quad \dot{\boldsymbol{u}}_\tau(t) \neq 0 \qquad \text{on } \Gamma_3 \tag{7.6}$$

for all $t \in I$ and, moreover,

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{in } \Omega. \tag{7.7}$$

Note that Problem 4 describes contact processes which arise in various industrial settings like metal forming and metal extrusion, in which there is no separation between the contact surfaces. Details can be found in [28, 29, 31, 32], for instance. A brief description of the equations and boundary conditions in this problem is the following. First, (7.1) is the constitutive law in which $\mathcal{A}$ represents the viscosity operator, $\mathcal{B}$ is the elasticity operator and $\mathcal{C}$ denotes the relaxation tensor. Equation (7.2) is the equation of equilibrium in which Div denotes the divergence operator and conditions (7.3), (7.4) are the displacement and traction condition, respectively. Condition (7.5) represents the bilateral contact condition and (7.6) represents the quasistatic version of Tresca's friction law in which g denotes a positive function, the friction bound. Finally, condition (7.7) is the initial condition in which $\boldsymbol{u}_0$ denotes a given initial displacement field.

In the study of Problem 4 we use the standard notation for Sobolev and Lebesgue spaces, endowed with their canonical inner products and associated norms. Moreover, for an element $\boldsymbol{v} \in H^1(\Omega)^d$ we write $\boldsymbol{v}$ for the trace $\gamma\boldsymbol{v} \in L^2(\Gamma)^d$ of $\boldsymbol{v}$ to Γ and use by v_ν, $\boldsymbol{v}_\tau$ for its normal and tangential components given by $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$, respectively. In addition, we consider the following spaces:

$$V = \{\,\boldsymbol{v} \in H^1(\Omega)^d : \ \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma_1,\ v_\nu = 0 \ \text{ on } \ \Gamma_3\,\},$$

$$Q = \{\,\boldsymbol{\sigma} = (\sigma_{ij}) : \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\,\}.$$

The spaces V and Q are real Hilbert spaces endowed with their canonical inner products given by

$$(\boldsymbol{u},\boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx, \qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx. \tag{7.8}$$

Here and below $\boldsymbol{\varepsilon}$ represents the deformation operator, that is,

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

the index that follows a comma denoting the partial derivative with respect to the corresponding component of the spatial variable $\boldsymbol{x}$, i.e., $u_{i,j} = \partial u_i / \partial x_j$. The associated norms on these spaces are denoted by $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively.

Next, we recall that for a regular stress function $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ the normal and tangential components at Γ are given by $\sigma_\nu = \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$ and, moreover, the following Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \int_\Omega \mathrm{Div}\, \boldsymbol{\sigma} \cdot \boldsymbol{v}\, dx = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}\, da \quad \text{for all } \boldsymbol{v} \in H^1(\Omega)^d. \tag{7.9}$$

Finally, we introduce the space of fourth order tensors defined by

$$\mathbf{Q}_\infty = \{\, \mathcal{C} = (c_{ijkl}) \mid c_{ijkl} = c_{jikl} = c_{klij} \in L^\infty(\Omega) \,\}.$$

It is known that $\mathbf{Q}_\infty$ endowed with

$$\|\mathcal{C}\|_{\mathbf{Q}_\infty} := \max_{0 \le i,j,k,l \le d} \|c_{ijkl}\|_{L^\infty(\Omega)}$$

is a Banach space. Moreover, it is not difficult to see that

$$\|\mathcal{C}\boldsymbol{\tau}\|_Q \le d\, \|\mathcal{C}\|_{\mathbf{Q}_\infty} \|\boldsymbol{\tau}\|_Q \quad \text{for all } \mathcal{C} \in \mathbf{Q}_\infty,\ \boldsymbol{\tau} \in Q. \tag{7.10}$$

In the study of the mechanical problem (7.1)–(7.7) we assume that the viscosity operator $\mathcal{A}$ and the elasticity operator $\mathcal{B}$ satisfy the following conditions.

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\
\qquad \|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\
\qquad\quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\
\qquad (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\
\qquad\quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\
\qquad \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\
\text{(e) } \mathcal{A}(\boldsymbol{x}, \mathbf{0}) = \mathbf{0} \text{ a.e. } \boldsymbol{x} \in \Omega.
\end{array}\right. \tag{7.11}$$

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{B}} > 0 \text{ such that} \\
\qquad \|\mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{B}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\
\qquad\quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\
\qquad \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\
\text{(d) } \mathcal{B}(\boldsymbol{x}, \mathbf{0}) = \mathbf{0} \text{ a.e. } \boldsymbol{x} \in \Omega.
\end{array}\right. \tag{7.12}$$

We also assume that the relaxation tensor $\mathcal{C}$ and the densities of body forces and tractions are such that

$$\mathcal{C} \in C(I; \mathbf{Q}_\infty). \tag{7.13}$$

$$\boldsymbol{f}_0 \in C(I; L^2(\Omega)^d). \tag{7.14}$$

$$\boldsymbol{f}_2 \in C(I; L^2(\Gamma_2)^d). \tag{7.15}$$

Finally, for the friction bound and the initial displacement we assume that

$$g \in L^\infty(\Gamma_3) \quad \text{and} \quad g(\boldsymbol{x}) \geq 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3.$$

$$\boldsymbol{u}_0 \in V. \tag{7.16}$$

These assumptions allow us to consider the operators $A : Q \to Q$, $B : Q \to Q$, $\mathcal{S} : C(I, Q) \to C(I, Q)$, the functions $j\colon V \to \mathbb{R}$, $\boldsymbol{f}\colon I \to V$, the family of sets $\{\Sigma(t)\}_{t\in I}$ and the element $\boldsymbol{\omega}_0$ defined by

$$(A\boldsymbol{\omega}, \boldsymbol{\tau})_Q = \int_\Omega \mathcal{A}\boldsymbol{\omega} \cdot \boldsymbol{\tau}\, dx \quad \text{for all } \boldsymbol{\omega},\ \boldsymbol{\tau} \in Q, \tag{7.17}$$

$$(B\boldsymbol{\omega}, \boldsymbol{\tau}) = (\mathcal{B}\boldsymbol{\omega}, \boldsymbol{\tau})_Q \quad \text{for all } \boldsymbol{\omega}, \boldsymbol{\tau} \in Q, \tag{7.18}$$

$$(\mathcal{S}\boldsymbol{\omega}(t), \boldsymbol{\tau})_Q = (\int_0^t \mathcal{C}(t-s)\boldsymbol{\omega}(s))\, ds, \boldsymbol{\tau})_Q \tag{7.19}$$

$$\text{for all } \ \boldsymbol{\omega} \in C(I; Q),\ \boldsymbol{\tau} \in Q.$$

$$j(\boldsymbol{v}) = \int_{\Gamma_3} g\, \|\boldsymbol{v}_\tau\|\, da \quad \text{for all } \ \boldsymbol{v} \in V, \tag{7.20}$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, da \tag{7.21}$$

$$\text{for all } \boldsymbol{v} \in V,\ t \in I,$$

$$\Sigma(t) = \{\, \boldsymbol{\tau} \in Q \ : \ (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + j(\boldsymbol{v}) \geq (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V\,\} \tag{7.22}$$

$$\text{for all } \ t \in I.$$

$$\boldsymbol{\omega}_0 = \boldsymbol{\varepsilon}(\boldsymbol{u}_0). \tag{7.23}$$

Assume in what follows that $(\boldsymbol{u}, \boldsymbol{\sigma})$ represents a regular solution of Problem 4 and let $\boldsymbol{v} \in V$, $t \in I$ be arbitrary fixed. Then, using standard arguments based on the Green formula (7.9) we find that

$$\int_\Omega \boldsymbol{\sigma}(t) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))\, dx + \int_{\Gamma_3} g\|\boldsymbol{v}_\tau(s)\|\, da - \int_{\Gamma_3} g\|\dot{\boldsymbol{u}}_\tau(s)\|\, da$$

$$\geq \int_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \dot{\boldsymbol{u}}(t))\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot (\boldsymbol{v} - \dot{\boldsymbol{u}}(t))\, da.$$

We now use the notation (7.20) and (7.21) to see that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q + j(\boldsymbol{v}) - j(\dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \tag{7.24}$$

and, taking successively $\boldsymbol{v} = 2\dot{\boldsymbol{u}}(t)$ and $\boldsymbol{v} = \boldsymbol{0}_V$ in this inequality we obtain that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q + j(\dot{\boldsymbol{u}}(t)) = (\boldsymbol{f}(t), \dot{\boldsymbol{u}}(t))_V. \tag{7.25}$$

Then, using (7.24), (7.25) and (7.22) yields

$$\boldsymbol{\sigma}(t) \in \Sigma(t), \qquad (\boldsymbol{\tau} - \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q \geq 0. \tag{7.26}$$

Therefore, substituting the constitutive law (7.5) in (7.26) and using notation (7.17)–(7.19) we find that

$$-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) \in N_{\Sigma(t)}(A\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + B\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{S}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))). \tag{7.27}$$

We now introduce the notation $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\omega}$ and use inclusion (7.27) together with equalities (7.7), (7.23) to obtain the following variational formulation of Problem 4.

Problem 5 Find a strain field $\boldsymbol{\omega}\colon I \to V$ such that

$$-\dot{\boldsymbol{\omega}}(t) \in N_{\Sigma(t)}\big(A\dot{\boldsymbol{\omega}}(t) + B\boldsymbol{\omega}(t) + \mathcal{S}\dot{\boldsymbol{\omega}}(t)\big) \quad \text{for all } t \in I, \tag{7.28}$$

$$\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0. \tag{7.29}$$

Note that Problem 5 represents a sweeping process in which the unknown is the strain field. At the best of our knowledge, this problem is new and nonstandard since, usually, the variational formulation of Problem 4 is provided by a history-dependent variational inequality for the displacement field, as shown in [31] and the references therein. Nevertheless, we stress that the arguments in [31] work for inequalities in which the set of constraints does not depend on the time. Therefore, the arguments there can not be applied in the study of problem (7.28)–(7.29) since, here, the set of constraints is $\Sigma(t)$ which, obviously, is time-dependent.

We now state and prove the following existence and uniqueness result.

Theorem 7.1 *Assume that* (7.11)–(7.16) *hold. Then Problem* 5 *has a unique solution* $\boldsymbol{\omega} \in C^1(I; Q)$.

Proof We use Corollary 4.2 with $X = Q$ and $K(t) = \Sigma(t)$ for all $t \in I$ and, to this end, we check in what follows the validity of the assumptions $(\mathcal{K})$, $(\mathcal{A})$, $(\mathcal{H})$, $(\mathcal{B})$ and $(\mathcal{U})$.

First, we note that assumptions (7.14) and (7.15) imply that the element $\boldsymbol{f}$ given by (7.21) has the regularity $\boldsymbol{f} \in C(I; V)$ and, therefore, $\boldsymbol{\varepsilon}(\boldsymbol{f}) \in C(I; Q)$. On the other hand, it is obviously to see that the set

$$\Sigma_0 = \{\, \boldsymbol{\tau} \in Q \;:\; (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + j(\boldsymbol{v}) \geq 0 \quad \forall\, \boldsymbol{v} \in V \,\}$$

is a nonempty closed convex subset of Q and, since $(\boldsymbol{f}(t), \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{f}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q$, we deduce that

$$\Sigma(t) = \Sigma_0 + \boldsymbol{\varepsilon}(\boldsymbol{f}(t)) \qquad \text{for all } t \in I.$$

These ingredients and Corollary 5.4 show that condition $(\mathcal{K})$ is satisfied.

Next, note that assumptions (7.11) and (7.12) imply that the operators (7.17) and (7.18) satisfy conditions $(\mathcal{A})$ and $(\mathcal{B})$, respectively. Moreover, we use assumption (7.13) and inequality (7.10) to see that for any nonempty compact $\mathcal{J}$, any functions $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and any $t \in \mathcal{J}$

we have

$$\|\mathcal{S}\boldsymbol{u}_1(t)-\mathcal{S}\boldsymbol{u}_2(t)\|_V \le d \max_{r\in[0,b(\mathcal{J})]} \|\mathcal{C}(s)\|_{\mathbf{Q}_\infty} \int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V\, ds \tag{7.30}$$

where $b(\mathcal{J}) = \max\{s : s \in \mathcal{J}\}$. Inequality (7.30) proves that the operator $\mathcal{S}$ is a history-dependent operator. Finally, assumptions and (7.16) guarantee that $\boldsymbol{\omega}_0 = \boldsymbol{\varepsilon}(\boldsymbol{u}_0) \in Q$ and, therefore, condition ($\mathcal{U}$) is satisfied.

It follows from above that we are in a position to apply Corollary 4.2 to conclude the proof. □

We complete the statement of Theorem 7.1 with the following result.

Proposition 7.2 *Assume that* (7.11)–(7.16) *hold and denote by* $\boldsymbol{\omega} \in C^1(I;Q)$ *the solution of Problem* 5 *obtained in Theorem* 7.1. *Then, there exists a unique displacement fields* $\boldsymbol{u} \in C^1(I;V)$ *such that* $\boldsymbol{\omega} = \boldsymbol{\varepsilon}(\boldsymbol{u})$.

Proof Denote by $\boldsymbol{\theta} \in C(I;Q)$ the function defined by $\boldsymbol{\theta} = A\dot{\boldsymbol{\omega}} + B\boldsymbol{\omega} + \mathcal{S}\dot{\boldsymbol{\omega}}$ and let $t \in I$. Then, using (7.28) we find that

$$\boldsymbol{\theta}(t) \in \Sigma(t), \quad (\dot{\boldsymbol{\omega}}(t), \boldsymbol{\tau} - \boldsymbol{\theta}(t))_Q \ge 0 \quad \text{for all } \boldsymbol{\tau} \in \Sigma(t). \tag{7.31}$$

Let $\boldsymbol{z} \in \boldsymbol{\varepsilon}(V)^\perp$ where, here and below, $\boldsymbol{\varepsilon}(V)$ represents the range of the deformation operator $\boldsymbol{\varepsilon}: V \to Q$ and $M^\perp$ denotes the orthogonal of the subset $M \subset Q$. We have $(\boldsymbol{z}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q = 0$ for all $\boldsymbol{v} \in V$ which entails that $\boldsymbol{\theta}(t) \pm \boldsymbol{z} \in \Sigma(t)$. Therefore, testing with $\boldsymbol{\tau} = \boldsymbol{\theta}(t) \pm \boldsymbol{z}$ in (7.31) we deduce that $(\dot{\boldsymbol{\omega}}(t), \boldsymbol{z})_Q = 0$ which shows that $\dot{\boldsymbol{\omega}}(t) \in (\boldsymbol{\varepsilon}(V)^\perp)^\perp$. We now recall that $\boldsymbol{\varepsilon}(V)$ is a closed subspace of Q and we refer the reader to [32, p.212] for a proof of this result. Thus, $(\boldsymbol{\varepsilon}(V)^\perp)^\perp = \boldsymbol{\varepsilon}(V)$ and, therefore, the inclusion $\dot{\boldsymbol{\omega}}(t) \in (\boldsymbol{\varepsilon}(V)^\perp)^\perp$ implies that $\dot{\boldsymbol{\omega}}(t) \in \boldsymbol{\varepsilon}(V)$. This ensures that there exists a element $\boldsymbol{v}(t) \in V$ such that

$$\dot{\boldsymbol{\omega}}(t) = \boldsymbol{\varepsilon}(\boldsymbol{v}(t)). \tag{7.32}$$

Moreover, since (7.8) implies that $\|\boldsymbol{v}(t)\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v}(t))\|_Q$, we deduce that the element $\boldsymbol{v}(t) \in V$ which satisfies (7.32) is unique and, in addition, the function $t \mapsto \boldsymbol{v}(t)$ has the regularity $\boldsymbol{v} \in C(I;V)$. Consider now the function $\boldsymbol{u}: I \to V$ defined by

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{v}(s)\, ds + \boldsymbol{u}_0 \qquad \forall\, t \in I \tag{7.33}$$

and note that $\boldsymbol{u} \in C^1(I;V)$. Then, we use equalities (7.29), (7.32), (7.23) and (7.33) to see that

$$\boldsymbol{\omega}(t) = \int_0^t \dot{\boldsymbol{\omega}}(s)ds + \boldsymbol{\omega}(0) = \int_0^t \boldsymbol{\varepsilon}(\boldsymbol{v}(s))ds + \boldsymbol{\varepsilon}(\boldsymbol{u}_0) = \boldsymbol{\varepsilon}\Big(\int_0^t \boldsymbol{v}(s)ds + \boldsymbol{u}_0\Big) = \boldsymbol{\varepsilon}(\boldsymbol{u}(t))$$

for each $t \in I$, which concludes the existence part of the proposition. The uniqueness part follows from equality $\|\boldsymbol{u}(t)\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\|_Q$, valid for all $t \in I$. □

Let $\boldsymbol{\omega}$ be a the solution of the sweeping process (7.28), (7.29). Then, a couple of functions $(\boldsymbol{u}, \boldsymbol{\sigma})$ which satisfies the equalities (7.1) and $\boldsymbol{\omega} = \boldsymbol{\varepsilon}(\boldsymbol{u})$ is called a weak solution to the contact problem (7.1)–(7.7). We conclude from above that Theorem 7.1 and Proposition 5.7 provide the unique weak solvability of this contact problem.

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References

1. Adly, S., Haddad, T.: An implicit sweeping process approach to quasistatic evolution variational inequalities. SIAM J. Math. Anal. **50**, 761–778 (2018)
2. Adly, S., Sofonea, M.: Time-dependent inclusions and sweeping processes in contact mechanics. J. Appl. Math. Phys. (ZAMP) **70**, 39 (2019). https://doi.org/10.1007/s00033-019-1084-4
3. Benabdellah, H.: Existence of solutions to the nonconvex sweeping process. J. Differ. Equ. **164**, 286–295 (2000)
4. Bernicot, F., Venel, J.: Sweeping process by prox-regular sets in Riemannian Hilbert manifolds. J. Differ. Equ. **259**, 4086–4121 (2015)
5. Bounkhel, M., Rabab, A.-Y.: First and second order convex sweeping processes in reflexive smooth Banach spaces. Set-Valued Var. Anal. **18**, 151–182 (2010)
6. Cao, T.H., Mordukhovich, B.S.: Optimal control of a nonconvex perturbed sweeping process. J. Differ. Equ. **266**, 1003–1050 (2019)
7. Castaing, C.: Version aléatoire de problème de rafle par un convexe variable. C. R. Acad. Sci. Paris, Sér. A **277**, 1057–1059 (1973)
8. Colombo, G., Goncharov, V.V.: The sweeping processes without convexity. Set-Valued Anal. **7**, 357–374 (1999)
9. Denkowski, Z., Migórski, S., Papageorgiou, N.S.: An Introduction to Nonlinear Analysis: Theory. Kluwer Academic/Plenum, Dordrecht/New York (2003)
10. Drozdov, A.D.: Finite Elasticity and Viscoelasticity–a Course in the Nonlinear Mechanics of Solids. World Scientific, Singapore (1996)
11. Edmond, J.F., Thibault, L.: BV solutions of nonconvex sweeping process differential inclusions with perturbation. J. Differ. Equ. **226**, 135–179 (2006)
12. Ekeland, I., Temam, R.: Convex Analysis and Variational Problems. North-Holland, Amsterdam (1976)
13. Haddad, T.: Nonconvex differential variational inequality and state-dependent sweeping process. J. Optim. Theory Appl. **159**, 386–398 (2013)
14. Haddad, T., Noel, J., Thibault, L.: Perturbed sweeping process with a subsmooth set depending on the state. Linear Nonlinear Anal. **2**, 155–174 (2016)
15. Jourani, A., Vilches, E.: Positively α-far sets and existence results for generalized perturbed sweeping processes. J. Convex Anal. **23**, 775–821 (2016)
16. Kunze, M., Monteiro Marques, M.D.P.: On parabolic quasi-variational inequalities and state-dependent sweeping processes. Topol. Methods Nonlinear Anal. **12**, 179–191 (1998)
17. Kurdila, A.J., Zabarankin, M.: Convex Functional Analysis. Birkhäuser, Basel (2005)
18. Migórski, S., Ochal, A., Sofonea, M.: Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems. Advances in Mechanics and Mathematics, vol. 26. Springer, New York (2013)
19. Migórski, S., Sofonea, M., Zeng, S.: Well-posedness of history-dependent sweeping processes. SIAM J. Math. Anal. **51**, 1082–1107 (2019)
20. Monteiro Marques, M.D.P.: Differential Inclusions in Nonsmooth Mechanical Problems. Shocks and Dry Friction. Progress in Nonlinear Differential Equations and Their Applications, vol. 9. Birkhäuser, Basel (1993)
21. Moreau, J.J.: Rafle par un convexe variable I. Travaux Sém. Anal. Convexe Montpellier (1971). Exposé 15
22. Moreau, J.J.: Sur l'évolution d'un système élastoplastique. C. R. Acad. Sci. Paris, Sér A-Bn **273**, A118–A121 (1971)
23. Moreau, J.J.: On unilateral constraints, friction and plasticity. In: Capriz, G., Stampacchia, G. (eds.) New Variational Techniques in Mathematical Physics. C.I.M.E. II, Ciclo 1973, pp. 173–322. Edizione Cremonese, Roma (1974)
24. Moreau, J.J.: Evolution problem associated with a moving convex in a Hilbert space. J. Differ. Equ. **26**, 347–374 (1977)
25. Nacry, F.: Truncated nonconvex state-dependent sweeping process: implicit and semi-implicit adapted Moreau's catching-up algorithms. J. Fixed Point Theory Appl. **20**, 121 (2018)

26. Nacry, F., Thibault, L.: Regularization of sweeping process: old and new. Pure Appl. Funct. Anal. **4**, 59–117 (2019)
27. Naniewicz, Z., Panagiotopoulos, P.D.: Mathematical Theory of Hemivariational Inequalities and Applications. Dekker, New York (1995)
28. Panagiotopoulos, P.D.: Inequality Problems in Mechanics and Applications. Birkhäuser, Boston (1985)
29. Panagiotopoulos, P.D.: Hemivariational Inequalities, Applications in Mechanics and Engineering. Springer, Berlin (1993)
30. Sofonea, M.: History-dependent inequalities for contact problems with locking materials. J. Elast. **134**, 127–148 (2019)
31. Sofonea, M., Matei, A.: Mathematical Models in Contact Mechanics. London Mathematical Society Lecture Note Series, vol. 398. Cambridge University Press, Cambridge (2012)
32. Sofonea, M., Migórski, S.: Variational-Hemivariational Inequalities with Applications. Pure and Applied Mathematics. Chapman & Hall/CRC Press, London/Boca Raton (2018)
33. Thibault, L.: Sweeping process with regular and nonregular sets. J. Differ. Equ. **193**, 1–26 (2003)
34. Tolstonogov, A.A.: Sweeping process with unbounded nonconvex perturbation. Nonlinear Anal. **108**, 291–301 (2014)
35. Tsukada, M.: Convergence of best approximations in a smooth Banach space. J. Approx. Theory **40**, 301–309 (1984)
36. Valadier, M.: Quelques problèmes d'entraînement unilatéral en dimension finie. Travaux Sém. Anal. Convexe Montpellier (1988). Exposé 8