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Existence of solutions for non-coercive variational-hemivariational inequalities involving the nonlocal fractional p-Laplacian

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ABSTRACT
In this paper, we study the existence of solutions for non-coercive variational-hemivariational inequalities involving nonlocal fractional p-Laplacian. Our approach is based on the theory of pseudomonotone operators in the sense of Brézis, recession analysis and the properties of nonlocal fractional p-Laplace operators recently established.The innovation of this paper is that we do not assume the usual coercivity or smallness assumptions as stated in most existing literature. The absence of such assumptions will make the existence of solutions very complicated. We have to proceed from the specific problem itself and seek specific methods based on the inherent characteristics of different problems.

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1. Introduction

In this paper,we study the following fractional p-Laplacian elliptic variational-hemivariational inequality. Find $u \in \mathcal{C}$ such that $\forall v \in \mathcal{C}$

$$\langle \mathcal{L}_K u - f, v - u\rangle_{W_0} + J^0(u, v - u) \geq \phi(u) - \phi(v), \tag{1}$$

where $J^0(u, v)$ stands for the generalized directional derivative of J at the point u in the direction v for a locally Lipschitz functional $J(\cdot)$ (cf. [1,2]) and $\phi(\cdot)$ denotes a convex functional, $\mathcal{C}$ a nonempty,closed and convex subset of a fractional Sobolev space W_0 (see below for the definition) and $f \in W_0^*$ (the dual space of W_0).

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Throughout the paper, without further mention, we always assume that $0 < s < 1$ and $1 < p < N/s$. $\mathcal{L}_K$ stands for a nonlocal operator defined as follows:

$$\mathcal{L}_K u(x) := \lim_{\epsilon \to 0^+} 2 \int_{\mathbb{R}^N \setminus B_\epsilon(x)} |u(x) - u(y)|^{p-2}(u(x) - u(y))K(x - y)\, \mathrm{d}y, \quad \forall x \in \mathbb{R}^N,$$

where $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is a function satisfying the following assumption (**K**):

(i) $\gamma K \in L^1(\mathbb{R}^N)$, where $\gamma(x) = \min\{|x|^p, 1\}$;
(ii) there exists $\lambda > 0$ such that $K(x) \geq \lambda |x|^{-(N+ps)}, \forall x \in \mathbb{R}^N \setminus \{0\}$;
(iii) $K(x) = K(-x), \forall x \in \mathbb{R}^N \setminus \{0\}$.

A typical example for K is given by $K(x) = |x|^{-(N+ps)}$ for $s \in (0, 1)(N > ps)$. In this case $\mathcal{L}_K$ is the fractional p-Laplace operator $(-\Delta_p)^s$,which (up to normalization factor) is defined as

$$(-\Delta_p)^s u(x) := \lim_{\epsilon \to 0^+} 2 \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}}\, \mathrm{d}y.$$

See [3] for more details.

When $p = 2$, this definition coincides (up to normalization constant depending on N and s, see [4]) with the linear fractional Laplace operator $(-\Delta)^s$, defined by,

$$(-\Delta)^s = \mathcal{F}^- \circ \mathcal{M}_s \circ \mathcal{F},$$

where $\mathcal{F}$ is the Fourier transform operator and $\mathcal{M}_s$ is the multiplication by $|\xi|^{2s}$.

Partial differential equations involving nonlocal operators have attracted a lot of attention, because nonlocal operators can accurately describe the complex systems in our real life, for example, anomalous diffusion phenomenon, dynamical networks behaviours, and geophysical flows etc., see [4–15].

Many efforts have been devoted to the study of problems involving the fractional p-Laplacian operator, among which we mention existence of solutions within the framework of Morse theory and the mountain pass theorem [16–18]. For the motivations that lead to the study of such operators, we refer the reader to the contribution [6] of Caffarelli.

The theory of hemivariational inequalities as generalization of variational inequalities is based on properties of the Clarke subgradient for locally Lipschitz functions. Recently, the study of hemivatiaional inequalities involving nonlocal operators has drawn a wide range of interest, for instance, Teng [19] and Xi-Huang-Zhou [20] applied the nonsmooth critical theory and nonsmooth version the three-critical-points theorem to prove the multiplicity of weak solution to nonlocal elliptic hemivariational inequalities evolved the fractional Laplace operator $(-\Delta)^s$. But, it is worth mentioning that in some certain situations, the formulated problems have not variational structure. This results in that the nonsmooth critical point theory and the nonsmooth variational approaches cannot be carried out. Motivated by this reason, more recently, Liu-Tan [10] considered the nonlocal elliptic hemivariational inequality (1) with the linear fractional Laplace operator and $\phi \equiv 0$ by using the surjectivity result for pseudomonotone and coercive operators.

The research work in this paper is the continuation of our paper [10]. Specifically, our aim in this study is to establish the existence of solutions for problem (1) without any coercivity condition. To the best of our knowledge, the mathematical literature dedicated to the existence of solutions for nonlocal elliptic hemivariational inequalities (1) without any coercivity assumptions is still untreated topics and this fact is the motivation of the present work. Even for local problems without any coercivity assumptions, the theory of existence of solution that we have obtained is new. The absence of such assumptions will make the existence of solutions very complicated. We have to proceed from the specific problem itself and seek specific methods based on the inherent characteristics of different problems. Our approach is based on the theory of pseudomonotone operators in the sense of Brézis, recession analysis and the properties of nonlocal fractional p-Laplace operators recently established.

2. Preliminaries and mathematical framework

In this section, we first recall some basic results, which will be used in the next section. Let $s \in (0,1), 1 < p < \infty$ and the fractional critical exponent p_s^* be defined as

$$p_s^* = \begin{cases} \frac{Np}{N-sp} & \text{if } sp < N, \\ \infty & \text{if } sp \geq N. \end{cases}$$

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Denote $\mathcal{Q} = \mathbb{R}^{2N} \setminus \mathcal{O}$, where $\mathcal{O} = (\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega) \subset \mathbb{R}^{2N}$. In the sequel, we always assume that $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ satisfies hypothesis (**K**). W is a linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function u in W belongs to $L^p(\Omega)$ and

$$\int_{\mathcal{Q}} |u(x) - u(y)|^p K(x-y)\, \mathrm{d}x\, \mathrm{d}y < \infty.$$

The space W is equipped with the norm

$$\|u\|_W = \|u\|_{L^p(\Omega)} + \left(\int_{\mathcal{Q}} |u(x) - u(y)|^p K(x-y)\, \mathrm{d}x\, \mathrm{d}y \right)^{1/p}. \tag{2}$$

We shall work in the closed linear subspace

$$W_0 = \{u \in W : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}.$$

In W_0, we may also use the norm

$$\|u\|_{W_0} = \left(\int_Q |u(x) - u(y)|^p K(x-y)\, dx\, dy \right)^{1/p}. \tag{3}$$

We collect the useful facts on the space W_0 (see [18,21], for more details) as follows.

Proposition 2.1: *$C_0^\infty(\Omega)$ is dense in W_0 and W_0 is a uniformly convex Banach space. Furthermore, for $q \in [1, p_s^*]$, there exists a positive constant $c(q)$ such that*

$$\|u\|_{L^q(\mathbb{R}^N)} \le c(q) \|u\|_{W_0}, \quad \forall u \in W_0. \tag{4}$$

Furthermore, the embedding is compact if $q \in [1, p_s^*)$.

Now we are in the position to define the nonlocal operator $\mathcal{L}_K : W_0 \to W_0^*$ (the dual space of W_0) as follows

$$\langle \mathcal{L}_K u, v \rangle_{W_0} =: \int_Q |u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))K(x-y)\, dx\, dy$$
$$\forall u, v \in W_0.$$

We recall some preliminary material of operators of type (S_+) and pseudomonotone operators. Let V be a reflexive Banach space and V^* be its dual space with the dual paring $\langle \cdot, \cdot \rangle_V$.

Definition 2.2: We say that the multivalued operator $A : V \to 2^{V^*}$ is pseudomonotone in the sense of Brézis iff the following conditions hold:

(a) for each $u \in V$, the set Au is nonempty, bounded, closed and convex in V^*.
(b) A is upper semicontinuous from each finite-dimensional subspace of V to V^* endowed with the weak topology.
(c) if $\{u_k\} \subset V$ with $u_k \to u$ weakly in V, and $u_k^* \in Au_k$ is such that

$$\limsup_{n \to \infty} \langle u_k^*, u_k - u \rangle_V \le 0,$$

then for every $v \in V$, there exists $u^*(v) \in Au$ such that

$$\liminf_{k \to \infty} \langle u_k^*, u_k - v \rangle_V \ge \langle u^*(v), u - v \rangle_V.$$

We say that the multivalued operator $A : V \to 2^{V^*}$ is of type (S_+) if the above assumptions (a) and (b) are satisfied as well as

(d) if $\{u_k\} \subset V$ with $u_k \to u$ weakly in V, and $u_k^* \in Au_k$ is such that

$$\limsup_{n \to \infty} \langle u_k^*, u_k - u \rangle_V \le 0,$$

then $u_k \to u$ in V and there exists a subsequence $\{u_{k_j}^*\}$ of $\{u_k^*\}$ such that $\{u_{k_j}^*\}$ converges weakly to $u^* \in A(u)$.

Obviously, an operator $A : V \to 2^{V^*}$ of type (S_+) is also a pseudomonotone operator in the sense of Brézis.

Let us recall $h^0(u, v)$ the Clarke generalized directional derivative of a locally Lipschitz functional $h : V \to \mathbb{R}$ at $u \in V$ in the direction $v \in V$

$$h^0(u, v) = \limsup_{\lambda \to 0^+, w \to u} \frac{h(w + \lambda v) - h(w)}{\lambda}$$

and the generalized Clarke subdifferential of h at $u \in V$

$$\partial_c h(u) := \{u^* \in V^* \mid h^0(u, v) \geq \langle u^*, v\rangle_V \quad \text{for all } v \in V\}.$$

The next proposition provides basic properties of the generalized directional derivative and the generalized gradient.

Proposition 2.3 ([1,2]): *Let V be a Banach space. If $h : U \to \mathbb{R}$ is a locally Lipschitz functional on a subset U of V, then*

(i) *For every $u \in U$ the gradient $\partial_c h(u)$ is a nonempty, convex, and weakly* compact subset of V^* which is bounded by the Lipschitz constant $K_u > 0$ of h near u.*
(ii) *The graph of the generalized gradient $\partial_c h$ is closed in $V \times (w^* - V^*)$ topology, i.e. if $\{u_k\} \subset U$ and $\{\zeta_k\} \subset V^*$ are sequences such that $\zeta_k \in \partial_c h(u_k)$ and $u_k \to u$ in V, $\zeta_k \to \zeta$ weakly* in V^*, then $\zeta \in \partial_c h(u)$ where, recall, $w^* - V^*$ denotes the space V^* equipped with weak* topology.*
(iii) *The multifunction $U \ni u \to \partial_c h(u) \subseteq V^*$ is upper semicontinuous from U into $w^* - V^*$.*
(iv) *for each $v \in V$, there exists $z_v \in \partial_c h(u)$ such that*

$$h^0(u, v) = \max\{\langle z, v\rangle_V, \mid z \in \partial_c h(u)\} = \langle z_v, v\rangle_V.$$

(v) *The function $v \to h^0(u, v)$ is finite, positively homogeneous, and subadditive on V, and satisfies $|h^0(u, v)| \leq K_u \|v\|_V$.*
(vi) *$h^0(u, v)$ is upper semicontinuous as a function of (u, v) and a function of v alone, is Lipschitz of rank K_u on U.*
(vii) *$h^0(u, -v) = (-h)^0(u, v)$.*

We deal with the functional $J : L^p(\Omega) \to \mathbb{R}$ of type

$$J(u) = \int_\Omega j(x, u(x))\, \mathrm{d}x, \quad u \in L^p(\Omega). \tag{5}$$

For the integrand $j : \Omega \times \mathbb{R} \to \mathbb{R}$, we make the following hypothesis (H):

(1) $j(\cdot, t)$ is measurable on Ω for all $t \in \mathbb{R}$ and there exists $e \in L^p(\Omega)$ such that $j(\cdot, e(\cdot)) \in L^1(\Omega)$.

(2) $j(x,\cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $x \in \Omega$.
(3) $|\xi| \le \bar{a} + \bar{b}|t|^{p-1}, \forall \xi \in \partial_c j(x,t), \forall t \in \mathbb{R}$, a.e. $x \in \Omega$ with $\bar{a}, \bar{b} \ge 0$.

Proposition 2.4 ([2]): *Assume that assumption (H) holds. Then the functional J defined by (5) has the following properties:*

(a) *J is well defined and finite on $L^p(\Omega)$.*
(b) *J is Lipschitz continuous on bounded subsets of $L^p(\Omega)$ and, therefore, it is also locally Lipschitz on $L^p(\Omega)$.*
(c) *For all $u, v \in L^p(\Omega)$, we have*

$$\int_\Omega j^0(x, u(x); v(x))\,\mathrm{d}x \ge J^0(u, v), \quad \forall u, v \in L^p(\Omega). \tag{6}$$

(d) *For all $v \in W_0$, we have $\|\partial J(v)\|_{W_0^*} \le a + b\|v\|_{W_0}^{p-1}$ with $a = \bar{a}c(p)\sqrt{2|\Omega|}$ and $b = \bar{b}(c(p))^p\sqrt{2}$, where $c(p)$ denotes the embedding constant in (4).*

Furthermore, if there exists $C_j \ge 0$ such that for all $t_1, t_2 \in \mathbb{R}$, a.e.$x \in \Omega$

$$j^0(x, t_1; t_2 - t_1) + j^0(x, t_2; t_1 - t_2) \le C_j|t_2 - t_1|^p.$$

Then, $\forall u, v \in W_0$, one has

$$J^0(v_1; v_2 - v_1) + J^0(v_2; v_1 - v_2) \le C_J\|v_1 - v_2\|_{W_0}^p, \tag{7}$$

with $C_J = C_j(c(p))^p$.

Remark 2.5: It is obvious that (7) is equivalent to the following inequaliy

$$\langle w_1 - w_2, v_1 - v_2\rangle_{W_0} \ge -C_J\|v_1 - v_2\|_{W_0}^p, \tag{8}$$

for all $v_i \in W_0$, $w_i \in \partial_c J(v_i)$, $i = 1, 2$. The latter is the so-called relaxed monotonicity condition (see [2,22]).

The following proposition will be useful (see for instance, [23]) in the sequel.

Proposition 2.6: *Let $T : V \to 2^{V^*}$ be a pseudomonotone operator in the sense of Brézis, $\mathcal{C} \subseteq V$ be nonempty, bounded,closed and convex and $\varphi : V \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and l.s.c.. Then, for a given $f \in V^*$, there exist $u \in \mathcal{C}$ and $u^* \in T(u)$ such that*

$$\langle u^* - f, v - u\rangle \ge \varphi(u) - \varphi(v); \quad \forall v \in \mathcal{C}.$$

Now we also introduce some concepts we will need in what follows, along with their main properties. Let $\mathcal{C}$ be a nonempty convex subset of the topological vector space X. The recession cone of $\mathcal{C}$ is defined by

$$\mathcal{C}_\infty = \{u \in X | \forall v \in \mathcal{C}, \forall t > 0 : v + tu \in \mathcal{C}\}. \tag{9}$$

When $\mathcal{C}$ is a closed convex set we have that

$$\mathcal{C}_\infty = \bigcap_{t>0} \left[\frac{\mathcal{C} - u_0}{t}\right], \tag{10}$$

where u_0 is arbitrary chosen in $\mathcal{C}$. Equivalently,this amounts to say that u belongs to $\mathcal{C}_\infty$ if and only if there exist sequences $\{t_n\}_{n\in N}$ and $\{u_n\}_{n\in N} \subseteq \mathcal{C}$ such that $\lim_{n\to\infty} t_n = \infty$ and $u = \lim_{n\to\infty} t_n^{-1} u_n$. In this case it is obvious that $\mathcal{C}_\infty$ is a closed convex cone.

Let $\phi : X \to R \bigcup \{+\infty\}$ be a proper, convex and l.s.c. the behaviour at infinity of ϕ can be described by the recession function ϕ_∞ of ϕ which is defined by the formula

$$\phi_\infty(x) := \lim_{t\to\infty} \frac{\phi(x_0 + tx)}{t},$$

where x_0 is any element of $Dom\phi = \{x \in X : \phi(x) < +\infty\}$. Equivalently, in terms of epigraph, it amounts to say that

$$epi\phi_\infty = (epi\phi)_\infty.$$

The functional ϕ_∞ turns out to be proper, convex, l.s.c and positively homogeneous of degree 1, say

$$\phi_\infty(\lambda x) = \lambda \phi_\infty(x), \quad \forall x \in X, \lambda \geq 0.$$

It is known that

$$\phi(x + y) \leq \phi(x) + \phi_\infty(y), \quad \forall x \in Dom\phi, y \in X, \tag{11}$$

$$[\phi \leq \lambda]_\infty = \{u \in X | \phi_\infty(u) \leq 0\}, \quad \forall \lambda \in R \text{ with } [\phi \leq \lambda] \neq \emptyset, \tag{12}$$

and

$$\phi_\infty(x) \leq \lim \inf_{n\to\infty} \frac{\phi(t_n x_n)}{t_n}, \tag{13}$$

where $\{x_n\}$ is any sequence in X converging weakly to x and $t_n \to +\infty$ for more details, one can see [24].

3. Main results

Since $C_0^\infty(\Omega)$ is dense in W_0, so W_0 is dense in $L^q(\Omega), q \in (1, p^*]$. Applying the corollary in [1, p .47], we have

$$\partial_c(J|_{W_0}(u)) = \partial_c(J|_{L^q(\Omega)}(u)), \quad \forall u \in W_0, \tag{14}$$

in the sense that every element $w \in \partial_c(J|_{W_0}(u))$ admits a unique extension to an element $\bar{w} \in \partial_c(J|_{L^q(\Omega)}(u))$ such that

$$\langle w, v\rangle_{W_0} = \langle \bar{w}, v\rangle_{L^q(\Omega)}, \quad \forall v \in W_0. \tag{15}$$

First, we show the following Lemma:

Lemma 3.1: *Under the assumption (H), the multivalued operator $\mathcal{L}_K + \partial_c J|_{W_0} : W_0 \to 2^{W_0^*}$ is bounded and of type (S_+).*

Proof: Note that, by the Hölder inequality, $\forall u, v \in W_0$, we get that

$$\begin{aligned}
&|\langle \mathcal{L}_K u, v\rangle_{W_0}| \\
&\quad = \left| \int_Q |u(x) - u(y)|^{p-2}(u(x) - u(y)(v(x) - v(y))K(x-y)\,\mathrm{d}x\,\mathrm{d}y \right| \\
&\quad \le \|u\|_{W_0}^{p-1} \|v\|_{W_0}.
\end{aligned}$$

Therefore, we easily obtain that $\mathcal{L}_K$ is continuous and bounded.

$\forall u \in W_0, \forall w \in \partial_c J|_{W_0}(u)$, we denote the extension of w in $L^p(\Omega)$ by $\bar{w}$. One has from Proposition 2.3(iv) and 2.4(d)

$$\begin{aligned}
\langle w, v\rangle_{W_0} &\le J^0(u, v) \\
&= \sup\{\langle w, v\rangle_{W_0}, w \in \partial_c J(u))\} \\
&\le (a + b\|u\|_{W_0}^{p-1})\|v\|_{W_0},
\end{aligned}$$

which implies the boundedness of the multivalued operator $\partial_c J|_{W_0}$.

Since W_0 is a reflexive Banach space, by Proposition 2.3(i) we obtain that for any $u \in W_0$, $\partial_c J|_{W_0}(u)$ is a nonempty, convex, weak-compact subset of W_0^*. Therefore, the sum operator $\mathcal{L}_K u + \partial_c J|_{W_0}(u)$ is a nonempty, convex, bounded, closed subset of W_0^*. i. e. condition (a) in Definition 2.2 is true.

In virtue of the continuity of $\mathcal{L}_K$ and Proposition 2.3(iii), the sum operator $\mathcal{L}_K + \partial_c J|_{W_0}$ is upper semicontinuous from W_0 into W_0^* endowed with the weak topology. So condition (b) holds.

To complete the assertion that the sum operator $\mathcal{L}_K + \partial_c J|_{W_0}$ is of type (S_+), let $\{u_n\}$ be a sequence in W_0 converging weakly to u, and $w_n \in \partial_c J|_{W_0}(u_n)$ such that

$$\limsup_{n\to\infty}\langle \mathcal{L}_K u_n + w_n, u_n - u\rangle_{W_0} \le 0,$$

which implies

$$\limsup_{n\to\infty}\langle \mathcal{L}_K u_n, u_n - u\rangle_{W_0} + \liminf_{n\to\infty}\langle w_n, u_n - u\rangle_{W_0} \le 0. \tag{16}$$

Let $\bar{w}_n \in \partial_c(J|_{L^q(\Omega)}(u_n))$ be the extension of $w_n \in \partial_c(J|_{W_0}(u_n))$.

By (15), one has

$$\langle w_n, u_n - u\rangle_{W_0} = \langle \bar{w}_n, u_n - u\rangle_{L^p(\Omega)}.$$

The weak convergence of $\{u_n\}$ in W_0 and the compactness of the embedding $W_0 \subseteq L^p(\Omega)$ imply the convergence of $\{u_n\}$ in $L^p(\Omega)$ i.e.

$$u_n \to u \quad \text{strongly in } L^p(\Omega).$$

Hence, we may assume that $\{u_n\}$ are in a neighbourhood of u. While $J|_{L^p}$ is locally Lipschitz, by Proposition 2.3(i), one deduces that $\{\bar{w}_n\}$ is a bounded sequence in $L^{p'}(\Omega)$. Therefore, we have

$$\lim_{n\to\infty}\langle w_n, u_n - u\rangle_{W_0} = \lim_{n\to\infty}\langle \bar{w}_n, u_n - u\rangle_{L^p(\Omega)} = 0. \tag{17}$$

By (16) and (17), one has

$$\limsup_{n\to\infty}\langle \mathcal{L}_K u_n, u_n - u\rangle_{W_0} \le 0.$$

Since $u_n \to u$ weakly in W_0, we get from the above inequality

$$\limsup_{n\to\infty}\langle \mathcal{L}_K u_n - \mathcal{L}_K u, u_n - u\rangle_{W_0} \le 0. \tag{18}$$

By use of the well-known Simon inequality (cf. [18]): for all $\xi, \eta \in \mathbb{R}$, there exists $A_p > 0$ such that

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) \ge \begin{cases} A_p|\xi - \eta|^p, & \text{if } p \ge 2,\\ \dfrac{A_p|\xi - \eta|^2}{(|\xi|^p + |\eta|^p)^{(2-p)/p}}, & \text{if } 1 < p < 2. \end{cases} \tag{19}$$

It follows from the above inequalities that the operator $\mathcal{L}_K$ is strictly monotone. By (3.5) one has

$$\lim_{n\to\infty}\langle \mathcal{L}_K u_n - \mathcal{L}_K u, u_n - u\rangle_{W_0} = 0. \tag{20}$$

For $p \ge 2$, we have from (19) and (20)

$$\lim_{n\to\infty}\|u_n - u\|_{W_0}^p \le A_p^{-1}\lim_{n\to\infty}\langle \mathcal{L}_K u_n - \mathcal{L}_K u, u_n - u\rangle_{W_0} = 0.$$

For $1 < p < 2$, we get from (19) and (20)

$$\begin{aligned} &\lim_{n\to\infty} \|u_n - u\|^p_{W_0} \\ &\quad \le A_p^{-p/2} \lim_{n\to\infty} [\langle \mathcal{L}_K u_n - \mathcal{L}_K u, u_n - u\rangle_{W_0}]^{p/2} (\|u_n\|^p_{W_0} + \|u\|^p_{W_0})^{(2-p)/2} \\ &\quad = 0. \end{aligned}$$

Therefore, $u_n \to u$ in W_0. By use of Proposition 2.3(ii), there exists a subsequence of $\{w_n\}$, still denoted by $\{w_n\}$ such that

$$w_n \to w \in \partial_c J|_{W_0}(u), \quad \text{weakly in } W_0^*. \tag{21}$$

Hence, (21) and the continuity of $\mathcal{L}_K$ imply that $\mathcal{L}_K u_n + w_n$ converges weakly to $\mathcal{L}_K u + w \in W_0$, which proves that the condition (d) is satisfied. So the sum operator $\mathcal{L}_K + \partial_c J|_{W_0}$ is of type (S_+). ■

A direct application of Lemma 3.1 and Proposition 2.6 leads to the following basic existence result:

Theorem 3.2: *Let $\mathcal{C}$ be a nonempty, bounded, closed, convex subset of $W_0, f \in W_0^*, \phi : W_0 \to \mathbb{R}\bigcup\{+\infty\}$ a proper, convex, l.s.c. functional. Then under the assumption (H), problem (1) has at least one solution.*

Proof: By Lemma 3.1, the sum operator $\mathcal{L}_K + \partial_c J|_{W_0}$ is of type (S_+). So the sum operator $\mathcal{L}_K + \partial_c J|_{W_0}$ is a pseudomonotone operator in the sense of Brézis. Since $\mathcal{C} \subseteq W_0$ is nonempty, bounded, closed and convex. Therefore, from Proposition 2.6, for any $f \in W_0^*$, there exist $u \in \mathcal{C}$ and $w \in \partial_c J(u)$ such that

$$\langle \mathcal{L}_K u - f, v - u\rangle_{W_0} + \langle w, v - u\rangle_{W_0} \ge \phi(u) - \phi(v), \quad \forall v \in \mathcal{C},$$

which implies that from Proposition 2.3(iv)

$$\langle \mathcal{L}_K u - f, v - u\rangle_{W_0} + J^0(u, v - u) \ge \phi(u) - \phi(v), \quad \forall v \in \mathcal{C}.$$

Hence, problem (1) has at least one solution. This ends the proof of the theorem. ■

Similar to [25,26], we introduce the recession function, denoted by $r_{\mathcal{L}_K,J}$, associated with the nonlocal operator $\mathcal{L}_K : W_0 \to W_0^*$ and the locally Lipschitz functional $J : W_0 \to \mathbb{R}$, that is,

$$r_{\mathcal{L}_K,J}(u) := \liminf\{\langle \mathcal{L}_K(tv), v\rangle_{W_0} - J^0(tv, -v) | t \to \infty, v \to u\}$$
$$= \inf\{\liminf_{n\to\infty}[\langle \mathcal{L}_K(t_n v_n), v_n\rangle_{W_0} - J^0(t_n v_n, -v_n)] | t_n \to \infty, v_n \to u\}.$$

Set $\mathcal{C}_n := \{v \in \mathcal{C} | \|v\|_{W_0} \leq n\}$ in the sequel. We introduce the set $R(\mathcal{L}_K, f, J, \phi, \mathcal{C})$ of asymptotic directions:

$$R(\mathcal{L}_K, f, J, \phi, \mathcal{C}) :$$
$$= \{w \in \mathcal{C}_\infty | \exists u_n \in \mathcal{C}, \|u_n\|_{W_0} \to \infty, w_n := u_n/\|u_n\|_{W_0} \to w \text{ weakly and}$$
$$\forall v \in \mathcal{C}_n, \langle \mathcal{L}_K u_n - f, v - u_n\rangle_{W_0} + J^0(u_n, v - u_n) \geq \phi(u_n) - \phi(v)\}.$$

Theorem 3.3: *Let $\mathcal{C}$ be a nonempty, closed, convex subset of $W_0, f \in W_0^*, \phi : W_0 \to \mathbb{R} \cup \{+\infty\}$ a proper convex, l.s.c. functional. Suppose in addition that assumption (H) holds and the set $R(\mathcal{L}_K, f, J, \phi, \mathcal{C})$ is empty. Then problem (1) has at least one solution.*

Proof: The idea of the proof comes from the one in [26, Theorem 2]. In virtue of Theorem 3.2, there exists $u_n \in \mathcal{C}_n$ such that

$$\langle \mathcal{L}_K u_n - f, v - u_n\rangle_{W_0} + J^0(u_n, v - u_n) \geq \phi(u_n) - \phi(v), \quad \forall v \in \mathcal{C}_n. \tag{22}$$

Claim 1. There exists $n_0 \in N$ such that $\|u_{n_0}\| < n_0$.

Indeed, suppose the contrary: $\|u_n\| = n$ for each solution u_n of (22). On relabelling if necessary, we may assume that $w_n := u_n/\|u_n\|_{W_0} \to w \in \mathcal{C}_\infty$ weakly in W_0. Therefore $w \in R(\mathcal{L}_K, f, J, \phi, \mathcal{C})$, which contradicts the assumptions of Theorem 3.3.

Claim 2. u_{n_0} solves problem (1).

Since $\|u_{n_0}\|_{W_0} < n_0$, we have, for each $v \in \mathcal{C}$, the existence of an $\epsilon > 0$ such that $v_\epsilon = u_{n_0} + \epsilon(v - u_{n_0}) \in \mathcal{C}_{n_0}$. It suffices to take

$$\epsilon < (n_0 - \|u_{n_0}\|)/\|v - u_{n_0}\| \text{ if } v \neq u_{n_0},$$
$$\epsilon = 1 \text{ if } v = u_{n_0}.$$

We obtain

$$\langle \mathcal{L}_K u_{n_0} - f, v_\epsilon - u_{n_0}\rangle_{W_0} + J^0(u_{n_0}, v_\epsilon - u_{n_0}) \geq \phi(u_{n_0}) - \phi(v_\epsilon),$$

which implies that

$$\langle \mathcal{L}_K u_{n_0} - f, \epsilon(v - u_{n_0})\rangle_{W_0} + J^0(u_{n_0}, \epsilon(v - u_{n_0}))$$
$$\geq \phi(u_{n_0}) - \phi(u_{n_0} + \epsilon(v - u_{n_0})).$$

Since $J^0(u, v)$ is positively homogeneous in v by Proposition 2.3(v) and ϕ is convex, we derive

$$\epsilon\langle\mathcal{L}_K u_{n_0} - f, v - u_{n_0}\rangle_{W_0} + \epsilon J^0(u_{n_0}, v - u_{n_0}) \geq \epsilon(\phi(u_{n_0})) - \phi(v)). \tag{23}$$

Dividing (23) by $\epsilon > 0$, we finally obtain

$$\langle\mathcal{L}_K u_{n_0} - f, v - u_{n_0}\rangle_{W_0} + J^0(u_{n_0}, v - u_{n_0}) \geq \phi(u_{n_0}) - \phi(v), \quad \forall v \in \mathcal{C}.$$

This completes the proof. ■

In the sequel, in order to simplify some computations, we shall assume that $0 \in \mathcal{C}$ and $\phi(0) = 0$. Let us introduce the set $R_0(\mathcal{L}_K, f, J, \phi, \mathcal{C})$ of asymptotic directions:

$$\begin{aligned} R_0(\mathcal{L}_K, f, J, \phi, \mathcal{C}) &: \\ &= \{w \in \mathcal{C}_\infty | \exists u_n \in \mathcal{C}, \|u_n\|_{W_0} \to \infty, w_n := u_n/\|u_n\|_{W_0} \to w \text{ weakly and} \\ &\quad \langle\mathcal{L}_K u_n - f, u_n\rangle_{W_0} - J^0(u_n, -u_n) + \phi(u_n) \leq 0\}. \end{aligned}$$

Obviously, $R(\mathcal{L}_K, f, J, \phi, \mathcal{C}) \subseteq R_0(\mathcal{L}_K, f, J, \phi, \mathcal{C})$. Therefore, we have

Corollary 3.4: *Let $\mathcal{C}$ be a nonempty, closed, convex subset of W_0, $f \in W_0^*$, $\phi : W_0 \to \mathbb{R} \cup \{+\infty\}$ a proper convex, l.s.c. functional. Suppose in addition that assumption (H) holds and the set $R_0(\mathcal{L}_K, f, J, \phi, \mathcal{C})$ is empty. Then problem (1) has at least one solution.*

We say that $R_0(\mathcal{L}_K, f, J, \phi, \mathcal{C})$ is asymptotically compact if the sequence $\{w_n\}_{n \in N}$ which appears in the definition of this set converges strongly to w, that is, if $u_n \in \mathcal{C}$, $\|u_n\|_{W_0} \to \infty$, $w_n := u_n/\|u_n\|_{W_0} \to w$ weakly, and

$$\langle\mathcal{L}_K u_n - f, u_n\rangle_{W_0} - J^0(u_n, -u_n) + \phi(u_n) \leq 0$$

imply that $w_n \to w$.

Corollary 3.5: *Let $\mathcal{C}$ be a nonempty, closed, convex subset of W_0, the assumption (H) holds, $f \in W_0^*$, $\phi : W_0 \to \mathbb{R} \cup \{+\infty\}$ a proper convex, l.s.c. functional. Assume that:*

(i) *$R(\mathcal{L}_K, f, J, \phi, \mathcal{C})$ is asymptotically compact,*
(ii) *there is subset X of $W_0 - \{0\}$ such that $R(\mathcal{L}_K, f, J, \phi, \mathcal{C}) \subseteq X$ and*

$$r_{\mathcal{K},J}(w) + \phi_\infty(w) > \langle f, w\rangle_{W_0}, \quad w \in X.$$

Then problem (1) has at least one solution.

Proof: Following Corollary 3.4, it is enough to show that $R_0(\mathcal{L}_K,f,J,\phi,\mathcal{C})$ is empty. Suppose by contradiction that $R_0(\mathcal{L}_K,f,J,\phi,\mathcal{C}) \neq \emptyset$. Then we may find a sequence $\{u_n\}_{n\in N} \subset \mathcal{C}$ such that $\|u_n\| \to \infty$, $w_n := u_n/\|u_n\| \to w$ weakly and

$$\langle \mathcal{L}_K u_n, u_n\rangle_{W_0} - J^0(u_n,-u_n) + \phi(u_n) \le \langle f, u_n\rangle_{W_0}. \tag{24}$$

By assumption (i), one has that $w_n \to w$. Dividing (24) by $\|u_n\|$, we obtain

$$\langle \mathcal{L}_K(\|u_n\|w_n, w_n\rangle_{W_0} - J^0(\|u_n\|w_n,-w_n) + \frac{\phi(\|u_n\|w_n)}{\|u_n\|} \le \langle f, w_n\rangle_X. \tag{25}$$

Passing to the *liminf* in (25), we derive from (13)

$$r_{\mathcal{L}_K,J}(w) + \phi_\infty(w) \le \langle f, w\rangle_{W_0},$$

which contradicts assumption (ii) and the proof follows. ■

Moreover, if we denote $B := \{w \in W_0 \mid \|w\|_{W_0} \le 1\}$ and

$$R_{\mathcal{L}_K,J} := \liminf_{\|u\|_{W_0}\to\infty} \frac{\langle \mathcal{L}_K(u), u\rangle_{W_0} - J^0(u,-u)}{\|u\|_{W_0}},$$

then we easily have

Corollary 3.6: *Let $\mathcal{C}$ be a nonempty, closed, convex subset of W_0, the assumption (H) holds, $f \in W_0^*, \phi : W_0 \to \mathbb{R}\cup\{+\infty\}$ a proper convex, l.s.c. functional. Assume that the following inequality holds*

$$R_{\mathcal{L}_K,J} + \phi_\infty(w) > \langle f, w\rangle_{W_0}, \quad \forall w \in B. \tag{26}$$

Then problem (1) has at least one solution.

Proof: Similar to the proof of Corollary 3.5, we suppose by contradiction that $R_0(\mathcal{L}_K,f,J,\phi,K,\mathcal{C}) \neq \emptyset$. Then we may find a sequence $\{u_n\}_{n\in N} \subset K$ such that $\|u_n\| \to \infty$, $w_n := u_n/\|u_n\| \to w \in B$ weakly (note that since W_0 is an infinite-dimensional normed space,then B is the weak closure of $S = \{w \in X|\ \|w\|_{W_0} = 1\}$) and

$$\langle \mathcal{L}_K u_n, u_n\rangle_{W_0} - J^0(ju_n,-ju_n) + \phi(u_n) \le \langle f, u_n\rangle_{W_0},$$

which implies

$$\frac{\langle \mathcal{L}_K u_n, u_n\rangle_{W_0} - J^0(ju_n,-ju_n)}{\|u_n\|} + \frac{\phi(u_n)}{\|u_n\|} \le \langle f, w_n\rangle_{W_0}. \tag{27}$$

Passing to the *liminf* in (27), we derive

$$R_{\mathcal{L}_K,J} + \phi_\infty(w) \le \langle f, w\rangle_{W_0},$$

which contradicts assumption (26) and the proof follows. ■

We call an operator $A : W_0 \to 2^{W_0^*}$, u_0-coercive, if there exists $c : R_+ \to R$ with $\lim_{r\to\infty} c(r) = \infty$ such that

$$\langle w, u - u_0\rangle_{W_0} \geq c(\|u\|)\|u\| \tag{28}$$

for all $u \in D(A)$ and all $w \in A(u)$ with $\|u\|$ large enough.

Corollary 3.7: *Let $\mathcal{C}$ be a nonempty, closed, convex subset of W_0, $f \in W_0^*$, $\phi : W_0 \to \mathbb{R} \cup \{+\infty\}$ a convex, l.s.c. functional. Suppose in addition that assumptions (H) hold and the operator $\mathcal{L}_K + \partial_c J : W_0 \to 2^{W_0^*}$, u_0-coercive with $u_0 \in \mathcal{C} \cap Dom(\phi)$. Then problem (1) has at least one solution.*

Proof: Suppose by contradiction that $R(\mathcal{L}_K, f, J, \phi, \mathcal{C})$ is nonempty. Then we may find a sequence $\{u_n\}_{n\in N}$ such that $\|u_n\|_{W_0} \to \infty$, $w_n := u_n/\|u_n\| \to w$ weakly and

$$\langle \mathcal{L}_K u_n - f, u_n - u_0\rangle_{W_0} - J^0(u_n, u_0 - u_n) \leq \phi(u_0) - \phi(u_n).$$

By Propositon 2.3(iv), there exists $v_n \in \partial_c J(u_n)$ such that

$$\langle \mathcal{L}_K u_n + v_n - f, u_n - u_0\rangle_{W_0} \leq \phi(u_0) - \phi(u_n).$$

By dividing the above inequality by $\|u_n\|$ and u_0-coercivity assumption (28), we obtain

$$\begin{aligned} c(\|u_n\|) &\leq \frac{\langle \mathcal{L}_K u_n + v_n, u_n - u_0\rangle_{W_0}}{\|u_n\|} \\ &\leq \frac{\phi(u_0)}{\|u_n\|} - \frac{\phi(\|u_n\|w_n)}{\|u_n\|} + \langle f, w_n\rangle_{W_0} - \frac{\langle f, u_0\rangle_{W_0}}{\|u_n\|} \\ &\to \langle f, w\rangle_{W_0} - \phi_\infty(w), \quad \text{as } \|u_n\| \to \infty. \end{aligned}$$

Here we have used inequality (13) in the limit. This is a contradiction by the unboundedness of the left side of the above as $\|u_n\| \to \infty$. The proof follows. ■

Remark 3.8: It is known that if the nonlinear nonlocal operator $\mathcal{L}_K$ is the linear fractional Laplace operator $(-\Delta)^s$ and $\phi \equiv 0$, then Corollary 3.7 reduces to the main result in [10, Theorem 3.3], which means that Corollary 3.7 generalizes and extends [10, Theorem 3.3].

For further studies, we introduce a concept of (J, ϕ)-pseudomonotonicity, which has nothing to do with the concept of pseudomonotonicity in the sense of Brézis (cf. [27]).

Definition 3.9: The set-valued mapping $F : W_0 \to 2^{W_0^*}$ is said to be (J, ϕ)-pseudomonotone with respect to $f \in W_0^*$ iff, for all $(u, u^*), (v, v^*) \in Gr(F)$,

$$\langle u^* - f, v - u\rangle + J^0(u, v - u) + \phi(v) - \phi(u) \geq 0$$
$$\Rightarrow$$
$$\langle v^* - f, u - v\rangle + J^0(v, u - v) + \phi(u) - \phi(v) \leq 0.$$

Remark 3.10: If $p \geq 2$ and (7) (or equivalently (8)) holds with $C_J \leq A_p$ where C_J and A_p are the constants in (7) and (3.6),respectively, then $\forall f \in W_0^*$ the operator $\mathcal{L}_K$ is (J, ϕ)-pseudomonotone with respect to f.

In fact, suppose that

$$\langle \mathcal{L}_K u - f, v - u\rangle + J^0(u, v - u) + \phi(v) - \phi(u) \geq 0.$$

From (7), (19), one has

$$\begin{aligned}
&\langle \mathcal{L}_K v - f, u - v\rangle + J^0(v, u - v) + \phi(u) - \phi(v) \\
&\quad \leq -[\langle \mathcal{L}_K u - f, v - u\rangle + J^0(u, v - u) + \phi(v) - \phi(u)] \\
&\qquad - \langle \mathcal{L}_K v - \mathcal{L}_K u, v - u\rangle + J^0(v, u - v) + J^0(u, v - u) \\
&\quad \leq -\langle \mathcal{L}_K v - \mathcal{L}_K u, v - u\rangle + J^0(v, u - v) + J^0(u, v - u) \\
&\quad \leq (C_J - A_p)\|v - u\|_{W_0}^p \\
&\quad \leq 0, \ \forall v \in \mathcal{C},
\end{aligned}$$

which shows that $\mathcal{L}_K$ is (J, ϕ)-pseudomonotone with respect to f.

Lemma 3.11: *The following statements are true:*

(1) *u is a solution of the following variational-hemivariational inequality: find $u \in \mathcal{C}$ such that*

$$\langle \mathcal{L}_K v - f, u - v\rangle + J^0(v, u - v) + \phi(u) - \phi(v) \leq 0, \ \forall v \in \mathcal{C}; \tag{29}$$

(2) *$u \in \mathcal{C}$ is a solution to problem (1).*

Then (1)⇒(2). Furthermore, if the operator $\mathcal{L}_K$ is (J, ϕ)-pseudomonotone with respect to $f \in W_0^*$, then the two statements are equivalent.

Proof: (1)⇒(2) Let $u \in \mathcal{C}$ be a solution of problem (29). For $\forall v \in \mathcal{C}, \forall t \in (0,1]$, we have $v_t = u + t(v-u) \in \mathcal{C}$. From (29) and Proposition 2.3(v), $J(v,\cdot)$ is positively homogeneous,one has

$$t\langle \mathcal{L}_K v_t - f, u - v\rangle + tJ^0(v_t, u-v) + \phi(u) - \phi(v_t) \leq 0.$$

By virtue of the convexity of ϕ we derive

$$\langle \mathcal{L}_K v_t - f, v - u\rangle - J^0(v_t, u-v) + \phi(v) - \phi(u) \geq 0.$$

From Proposition 2.3(iv), there exists $w_t \in \partial J(v_t)$ such that

$$\langle \mathcal{L}_K v_t - f, v - u\rangle + \langle w_t, v-u\rangle + \phi(v) - \phi(u) \geq 0,$$

which implies that

$$\langle \mathcal{L}_K v_t - f, v - u\rangle + J^0(v_t, v-u) + \phi(v) - \phi(u) \geq 0.$$

The continuity of the operator $\mathcal{L}_K$ and Proposition 2.3(vi) enables us to pass to the limit as $t \to 0$. Therefore we get

$$\langle \mathcal{L}_K u - f, v - u\rangle + J^0(u, v-u) + \phi(v) - \phi(u) \geq 0, \quad \forall v \in \mathcal{C}.$$

(2)⇒(1) is obvious if the operator $\mathcal{L}_K$ is (J,ϕ)-pseudomonotone with respect to f. This completes the proof. ■

Corollary 3.12: *The solution set $S_1(f)$ of the variational -hemivariational inequality (29) is convex and closed. Furthermore, if the operator $\mathcal{L}_K$ is (J,ϕ)-pseudomonotone with respect to f, the solution set $S(f)$ of the variational-hemivariational inequality (1) is convex and closed.*

Proof: For any $u_n \in S_1(f)$ with $u_n \to u_0$, one has

$$\langle \mathcal{L}_K v, u_n - v\rangle + J^0(v, u_n - v) + \phi(u_n) - \phi(v) \leq 0, \ \forall v \in \mathcal{C}\}.$$

From the lower semicontinuity of ϕ and Proposition 2.3(vi), it follows that

$$\langle \mathcal{L}_K v, u_0 - v\rangle + J^0(v, u_0 - v) + \phi(u_0) - \phi(v) \leq 0, \ \forall v \in \mathcal{C}\},$$

which implies that $u_0 \in S_1(f)$ and so $S_1(f)$ is closed.

By virtue of the convexity and lower semicontinuity of ϕ and Proposition 2.3(v), we easily conclude $S_1(f)$ is closed and convex. If the operator $\mathcal{L}_K$ is (J,ϕ)-pseudomonotone with respect to f, Lemma 3.9 ensures the equivalence between the problems (1) and (29), which completes our proof. ■

For convenience, we use $barr(\mathcal{C})$ to denote the barrier cone of $\mathcal{C}$ which is defined by $barr(\mathcal{C}) := \{u^* \in W_0^* : \sup_{u \in \mathcal{C}} \langle u^*, u\rangle_{W_0} < \infty\}$.

Theorem 3.13: *Let $\mathcal{C}$ be a nonempty, closed, convex subset of W_0 with barr($\mathcal{C}$) having nonempty interior, $f \in W_0^*, \phi : W_0 \to \mathbb{R} \cup \{+\infty\}$ a convex, l.s.c. functional and assumptions (H) hold. In addition suppose that the operator $\mathcal{L}_K$ is (J, ϕ)-pseudomonotone with respect to f and*

$$\mathcal{C}_\infty \cap \{w \in W_0 : \langle \mathcal{L}_K v, w\rangle_{W_0} + J^0(v, w) + \phi_\infty(w) \leq 0, \forall v \in \mathcal{C}\} = \{0\}.$$

Then problem (1) has at least one solution.

Proof: In terms of Theorem 3.3, we only need to show that $R(\mathcal{L}_K, f, J, \phi, \mathcal{C})$ is empty. In fact, if there exists $w \in R(\mathcal{L}_K, f, J, \phi, \mathcal{C})$, then there is a sequence $\{u_n\}$ such that

$$u_n \in \mathcal{C}, \|u_n\|_{W_0} \to \infty, w_n := u_n / \|u_n\|_{W_0} \to w \in \mathcal{C}_\infty \quad \text{weakly} \tag{30}$$

and

$$\langle \mathcal{L}_K u_n - f, v - u_n\rangle_{W_0} + J^0(u_n, v - u_n) \geq \phi(u_n) - \phi(v), \forall v \in \mathcal{C}_n.$$

By use of barr($\mathcal{C}$) having nonempty interior and [28, Lemma 2.2], we have

$$0 \neq w \in \mathcal{C}_\infty. \tag{31}$$

Since the operator $\mathcal{L}_K$ is (J, ϕ)-pseudomonotone with respect to f, we have

$$\langle \mathcal{L}_K v - f, u_n - v\rangle + J^0(v, u_n - v) + \phi(u_n) - \phi(v) \leq 0, \quad \forall v \in \mathcal{C}_n.$$

Since $J^0(\cdot, \cdot)$ is positively homogeneous and subadditive for the second variable, one has

$$\begin{aligned} 0 &\geq \frac{\langle \mathcal{L}_K v - f, u_n - v\rangle + J^0(v, u_n - v)}{\|u_n\|} + \frac{\phi(u_n)}{\|u_n\|} - \frac{\phi(v)}{\|u_n\|} \\ &\geq \frac{\langle \mathcal{L}_K v - f, u_n - v\rangle + J^0(v, u_n) - J^0(v, v)}{\|u_n\|} + \frac{\phi(u_n)}{\|u_n\|} - \frac{\phi(v)}{\|u_n\|} \\ &= \frac{\langle \mathcal{L}_K v - f, u_n - v\rangle - J^0(v, v)}{\|u_n\|} + \frac{J^0(v, u_n)}{\|u_n\|} + \frac{\phi(u_n)}{\|u_n\|} - \frac{\phi(v)}{\|u_n\|}. \end{aligned}$$

This together with (2.13) implies that

$$\begin{aligned} 0 &\geq \langle \mathcal{L}_K v - f, w\rangle + \liminf_{n\to\infty} J^0\left(v, \frac{u_n}{\|u_n\|}\right) + \liminf_{n\to\infty} \frac{\phi(u_n)}{\|u_n\|} \\ &\geq \langle \mathcal{L}_K v - f, w\rangle + J^0(v, w) + \phi_\infty(w) \quad v \in \mathcal{C}. \end{aligned} \tag{32}$$

Hence, we have from (29) and (30)

$$0 \neq w \in \mathcal{C}_\infty \cap \{w \in W_0 : \langle \mathcal{L}_K v - f, w\rangle_{W_0} + J^0(v, w) + \phi_\infty(w) \leq 0, \forall v \in \mathcal{C}\},$$

which is a contradiction to the assumption of the theorem. ■

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