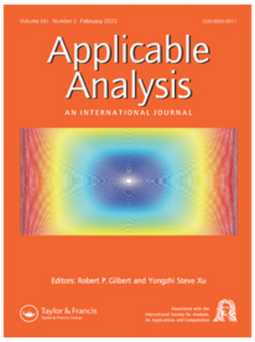

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Applicable Analysis
An International Journal

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/gapa20

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APPLICABLE ANALYSIS
2022, VOL. 101, NO. 2, 437–453
https://doi.org/10.1080/00036811.2020.1745780

Generalized penalty method for semilinear differential variational inequalities

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ABSTRACT

We consider a semilinear differential variational inequality $\mathcal{P}$ in reflexive Banach spaces, governed by a set of constraints K. We associate to $\mathcal{P}$ a sequence of problems $\{\mathcal{P}_n\}$ where, for each $n \in \mathbb{N}$, $\mathcal{P}_n$ is a differential variational inequality governed by a set of constraints K_n and a penalty parameter ρ_n. We use a result in [Liu ZH, Zeng SD. Penalty method for a class of differential variational inequalities. Appl Anal. 2019;1–16. doi:10.1080/00036811.2019.1652736] to prove the unique solvability of problems $\{\mathcal{P}\}$ and $\{\mathcal{P}_n\}$. Then, we prove that, under appropriate assumptions, the sequence of solutions to Problem $\mathcal{P}_n$ converges to the solution of the original problem $\mathcal{P}$. The proof is based on arguments of compactness, pseudomonotonicity and Mosco convergence. We also present two relevant particular case of our convergence result, including a recent result obtained in [Liu ZH, Zeng SD. Penalty method for a class of differential variational inequalities. Appl Anal. 2019;1–16. doi:10.1080/00036811.2019.1652736], in the case $K_n = V$. Finally, we provide an example of initial and boundary value problem for which our abstract results can be applied.

ARTICLE HISTORY
Received 16 December 2019
Accepted 17 March 2020

COMMUNICATED BY
B. Mordukhovich

KEYWORDS
Differential variational inequality; unilateral constraint; generalized penalty method; Mosco convergence; initial and boundary value problem

2010 MATHEMATICS SUBJECT CLASSIFICATIONS
47J20; 47J35; 49J30; 49J40; 35A15

1. Introduction

The differential variational inequality theory provides a powerful mathematical tool in the study of various models which describe the electrical circuits with ideal diodes, the frictional contact of deformable bodies, the dynamic traffic networks and the hybrid engineering systems with variable structures. Since differential variational inequalities were studied by Pang and Stewart [1], they have captured special attention in the mathematical literature. For instance, the global bifurcation for periodic solutions of differential variational inequalities was studied in [2] by using the topological degree theory, and the study of two parameter global bifurcation of periodic solutions for a class of differential variational inequalities was performed in [3]. Significant results in the field can be found in [4–9] and the references therein.

Recently, various classes of differential variational inequalities in infinite dimensional spaces have attracted an increasing interest, including the class of so-called semilinear differential variational inequalities. A semilinear differential variational inequality is a system coupling a semilinear evolution equation with a variational inequality. Semilinear differential variational inequalities have been first studied in [7]. There, basic results concerning the properties of the solution set were obtained,

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under the assumptions of compactness and convexity for the set of constraints K. Additional results have been obtained in [10], by relaxing the compactness of the set K. A general existence theorem for semilinear differential variational inequalities with non-compact sets of constraints and non-local boundary condition can be found in [11]. General existence results for differential variational inequalities with non-local condition and fractional differential variational inequalities in infinite dimensional spaces have been obtained in [5,11,12].

Penalty methods represent a mathematical tool used in the study of a large variety of problems, including the analysis and numerical solution of constrained problems. Reference in the field are [13–15], among others. The idea of penalty methods is to construct a sequence of unconstrained problems which have unique solution which converge to the solution of the original constrained problem, as the penalty parameter tends to zero. Penalty methods for variational inequalities have been studied by many authors, mainly for numerical purposes. Details can be found in [16] and the references therein. However, as far as we known, most of references use penalty methods to study only a single variational inequality and very few works are dealing with penalty methods for differential variational inequalities. Among them we refer the reader to the recent papers [17,18]. There, the authors prove existence, uniqueness and convergence results for a penalty method in the study of differential variational inequalities.

In this paper we introduce a penalty-type method for a class of semilinear differential variational inequalities in abstract Banach spaces. A short description of this method is as follows. First, we consider a differential variational inequality denoted by $\mathcal{P}$, governed by a set of constraints K. Next, we associate to $\mathcal{P}$ a sequence of problems $\{\mathcal{P}_n\}$ such that, for each $n \in \mathbb{N}$, $\mathcal{P}_n$ is a differential variational inequality governed by the set of constraints K_n and penalty parameter ρ_n. Both problems $\mathcal{P}_n$ and $\mathcal{P}$ have a unique solution, denoted by u_n and u, respectively. Finally, we prove the convergence of u_n to u, as the sequence $\{K_n\}$ converge in the sense of Mosco and $\rho_n \to 0$. Note that in constrast with the classical penalty method in which the unilateral constraint $u \in K$ is completely removed, in the method described above this unilateral constraint is replaced with a new unilateral constraint $u_n \in K_n$. For this reason we refer to this method as a generalized penalty method. Its novelty consists in the fact that, compared with the classical penalty method which is governed by the sequence $\rho_n \to 0$, the generalized penalty method is governed by two sequences $\rho_n \to 0$ and $\{K_n\}$, the later being convergent in the sense of Mosco to a set $\widetilde{K}$ which can be different of the whole space. Its generality allows us to apply this method in various situations which we present as examples and particular cases.

The rest of the manuscript is organized in the following way. In Section 2 we introduce some preliminary material, then we state the semilinear differential variational inequality problem, together with its unique solvability. In Section 3 we construct the sequence of generalized penalty problems, then we prove their unique solvability together with our main convergence result. Next, in Section 4 we present two relevant particular cases for which our results work. Finally, in Section 5 we illustrate the generalized penalty method in the study of an initial and boundary value problem with unilateral constraints.

2. Preliminaries

Everywhere below $T > 0$ and $(E, \|\cdot\|_E)$, $(V, \|\cdot\|_V)$ are reflexible Banach spaces. We denote by $\mathcal{L}(E)$ the space of linear continuous operators on E, equipped with the norm $\|\cdot\|_{\mathcal{L}(E)}$. We also denote by V^* the dual of V, by 0_{V^*} the zero element of V^* and by $\langle\cdot,\cdot\rangle$ the duality pairing between V^* and V. Moreover, we use $E \times V$ for the product of the spaces E and V, endowed with the canonical product topology. In addition, we denote by $C([0, T]; E)$ the space of continuous functions defined on $[0, T]$ with values in E and, for any subset $U \subset V$, we use the notation $C([0, T]; U)$ for the set of continuous functions defined on $[0, T]$ with values in U. All the limits, lower limits and upper limits are considered as $n \to \infty$, even if we do not mention it explicitly. We now recall the following definitions that we need in the rest of the paper.

Definition 2.1: An operator $G : V \to V^*$ is said to be:

(i) monotone, if for all $u, v \in V$, we have $\langle Gu - Gv, u - v\rangle \geq 0$;
(ii) strongly monotone, if there exists $m_G > 0$ such that $\langle Gu - Gv, u - v\rangle \geq m_G\|u - v\|_V^2$ for all $u, v \in V$;
(iii) bounded, if it maps bounded sets in V into bounded sets in V^*;
(iv) pseudomonotone, if G is bounded and for every sequence $\{u_n\} \subseteq V$ converging weakly to $u \in V$ such that $\limsup \langle Gu_n, u_n - u\rangle \leq 0$, we have

$$\langle Gu, u - v\rangle \leq \liminf \langle Gu_n, u_n - v\rangle \quad \text{for all } v \in V;$$

(v) hemicontinuous, if for all $u, v, w \in V$, the function $t \mapsto \langle G(u + tv), w\rangle$ is continuous on $[0, 1]$;
(vi) demicontinuous, if $u_n \to u$ in V implies $Gu_n \to Gu$ weakly in V^*.

Definition 2.2: An operator $P : V \to V^*$ is said to be a penalty operator of the set $K \subset V$ if P is bounded, demicontinuous, monotone and $K = \{\, u \in V : Pu = 0_{V^*} \,\}$.

Definition 2.3: A function $\phi : V \to \mathbb{R}$ is said to be lower semicontinuous if $\liminf \phi(u_n) \geq \phi(u)$ for any sequence $\{u_n\} \subset V$ with $u_n \to u$in V.

Definition 2.4: Let $\{K_n\}$ be a sequence of non-empty subsets of V and let $\widetilde{K}$ a non-empty subset of V. We say that the sequence $\{K_n\}$ converges to $\widetilde{K}$ in the sense of Mosco if the following conditions hold:

(i) For each $u \in \widetilde{K}$, there exists a sequence $\{u_n\}$ such that $u_n \in K_n$ for each $n \in \mathbb{N}$ and $u_n \to u$ in V.
(ii) For each sequence $\{u_n\}$ such that $u_n \in K_n$ for each $n \in \mathbb{N}$ and $u_n \to u$ weakly in V, we have $u \in \widetilde{K}$.

We shall denote the convergence in the sense of Mosco by $K_n \xrightarrow{M} K$ and we recall that it has been introduced in [19].

We now turn to the statement of the differential variational inequality in which our interest is. Let $A : D(A) \subset E \to E$, $x_0 \in E$ and $K \subset V$. Moreover, let $f : [0, T] \times E \times V \to E$, $g : [0, T] \times E \times V \to V^*$ and $\varphi : V \to \mathbb{R}$. With these data we consider the following problem.

Problem $\mathcal{P}$. Find two functions $x : [0, T] \to E$ and $u : [0, T] \to V$ such that

$$\begin{aligned} &x'(t) = Ax(t) + f(t, x(t), u(t)) \quad \text{a.e. } t \in [0, T],\\ &u(t) \in SOL(K, g(t, x(t), u(t)), \varphi) \quad \text{for all } t \in [0, T],\\ &x(0) = x_0. \end{aligned} \tag{1}$$

Here and below in this paper x' denotes the derivative of x with respect to the time variable and, for each $t \in [0, T]$, the inclusion $u(t) \in SOL(K, g(t, x(t), u(t)), \varphi)$ is a short hand notation which means that $u(t)$ satisfies the variational inequality

$$u(t) \in K, \quad \langle g(t, x(t), u(t)), v - u(t)\rangle + \varphi(v) - \varphi(u(t)) \geq 0 \quad \forall\, v \in K. \tag{2}$$

In the study of Problem $\mathcal{P}$ we consider the following hypotheses on the data.

$$\begin{cases} A : D(A) \subset E \to E \text{ is the generator of a } C_0\text{-semigroup of} \\ \text{linear and continuous operators } \{S(t)\}_{t\geq 0} \text{ on the space } E. \end{cases} \tag{3}$$

$$\begin{cases} f : [0,T] \times E \times V \to E \text{ is such that :} \\ (a) \text{ for all } (x,u) \in E \times V, \text{the function } t \mapsto f(t,x,u) \text{ is measurable on} \\ \qquad [0,T]; \\ (b) \text{ the function } t \mapsto f(t,0,0) \text{ belongs to } L^1([0,T];E); \\ (c) \text{ there exists a positive function } \psi \in L^1([0,T];\mathbb{R}^+) \text{ such that} \\ \qquad \|f(t,x_1,u_1) - f(t,x_2,u_2)\|_E \leq \psi(t)(\|x_1 - x_2\|_E + \|u_1 - u_2\|_V) \\ \qquad \text{for a.e. } t \in [0,T] \text{ and all } (x_1,u_1),(x_2,u_2) \in E \times V. \end{cases} \tag{4}$$

$$K \text{ is a non} - \text{empty closed convex subset of } V. \tag{5}$$

$$\begin{cases} g : [0,T] \times E \times V \to V^* \text{ is such that :} \\ (a) \text{ for all } (t,x) \in [0,T] \times E, \text{the operator } u \mapsto g(t,x,u) : V \to V^* \text{ is} \\ \qquad \text{bounded, hemicontinuous and strongly monotone} \\ \qquad \text{with constant } m_g > 0; \\ (b) \text{ there exists a constant } L_g > 0 \text{ such that} \\ \qquad \|g(t_1,x_1,u) - g(t_2,x_2,u)\|_{V^*} \leq L_g(|t_1 - t_2| + \|x_1 - x_2\|_E) \\ \qquad \text{for all } t_1,t_2 \in [0,T], u \in V \text{ and } x_1,x_2 \in E. \end{cases} \tag{6}$$

$$\varphi : V \to \mathbb{R} \text{ is a convex and lower semicontinuous function.} \tag{7}$$

Moreover, we recall that

$$x_0 \in E. \tag{8}$$

Following the references [1,7,20] we adopt the following definition.

Definition 2.5: A pair of functions (x,u) is said to be a mild solution of system (1) if $x \in C([0,T];E)$, $u \in C([0,T];K)$,

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s,x(s),u(s))\,\mathrm{d}s \quad \text{for all } t \in [0,T] \tag{9}$$

and $u(t) \in SOL(K, g(t,x(t),u(t)),\varphi)$ for all $t \in [0,T]$. If (x,u) is a mild solution of problem (1), then x is called the mild trajectory and u the variational control trajectory.

Note that, for simplicity, below in this paper we sometimes use the terminology 'solution' for the mild solution of any the system of the form (1). We now are in the position to state the following existence and uniqueness result.

Theorem 2.6: *Assume that (3)–(8) hold. Then there exists a unique mild solution $(x,u) \in C([0,T];E) \times C([0,T];K)$ to Problem $\mathcal{P}$.*

Proof: Theorem 2.6 was proved in [17] under the following additional assumption: either K is a bounded subset in V or

$$\begin{cases} \text{there exists an element } v^* \in K \text{ such that} \\ \liminf_{u\in K, \|u\|_V\to\infty} \dfrac{\langle g(t,x,u), u - v^*\rangle + \varphi(u) - \varphi(v^*)}{\|u\|_V} = +\infty \\ \text{for all } (t,x) \in [0,T] \times E. \end{cases} \tag{10}$$

Nevertheless, a careful analysis reveals the fact that, if K is unbounded, then assumptions (6)(a) and (7) guarantee the validity of the condition (10), with any $v \in K$.

Indeed, let $u,\ v \in K$ and $(t,x) \in [0,T] \times E$ be fixed. We write

$$\langle g(t,x,u), u - v\rangle = \langle g(t,x,u) - g(t,x,v), u - v\rangle + \langle g(t,x,v), u - v\rangle.$$

Then, using the strong monotonicity of $g(t,x,\cdot)$ and some elementary inequalities we obtain that

$$\begin{aligned} \langle g(t,x,u), u - v\rangle &\ge m_g\|u - v\|_V^2 - \|g(t,x,v)\|_{V^*}\|u - v\|_V \\ &\ge m_g(\|u\|_V - \|v\|_V)^2 - \|g(t,x,v)\|_{V^*}(\|u\|_V + \|v\|_V) \\ &= m_g\|u\|_V^2 - (2m_g\|v\|_V + \|g(t,x,v)\|_{V^*})\|u\|_V \\ &\quad + m_g\|v\|_V^2 - \|g(t,x,v)\|_{V^*}\|v\|_V. \end{aligned} \tag{11}$$

In the meantime, since φ is convex and lower semicontinuous, there exist an element $l \in V^*$ and a constant $\beta \in \mathbb{R}$ such that

$$\varphi(w) \ge \langle l, w\rangle + \beta \quad \text{for all } w \in V, \tag{12}$$

see [21, Proposition 5.2.25], for example. Therefore, from (11) and (12), we deduce that

$$\begin{aligned} \langle g(t,x,u), u - v\rangle + \varphi(u) - \varphi(v) \ge m_g\|u\|_V^2 - (2m_g\|v\|_V + \|g(t,x,v)\|_{V^*} + \|l\|_{V^*})\|u\|_V \\ + m_g\|v\|_V^2 - \|g(t,x,v)\|_{V^*}\|v\|_V - \varphi(v) - \beta. \end{aligned} \tag{13}$$

Inequality (13) shows that, in the case when K is unbounded, we have

$$\liminf_{u\in K, \|u\|_V\to\infty} \frac{\langle g(t,x,u), u - v\rangle + \varphi(u) - \varphi(v)}{\|u\|_V} = +\infty$$

and, therefore, condition (10) holds with any $v \in K$, as claimed. We now use [17, Theorem 3.1] to conclude the proof. ■

3. The generalized penalty method

In this section we introduce the generalized penalty method in the study of Problem $\mathcal{P}$. As already mentioned, it consists of defining the approximating problems, to prove their unique solvability and the convergence of the sequence of their solutions to the unique solution of $\mathcal{P}$, obtained in Theorem 2.6. To this end, we consider an operator $P : V \to V^*$, two sequences $\{K_n\} \subset V$, $\{\rho_n\} \subset \mathbb{R}$ and, for each $n \in \mathbb{N}$, we introduce the following differential variational inequality problem.

Problem $\mathcal{P}_n$. Find a pair of functons (x_n, u_n) with $x_n : [0, T] \to E$ and $u_n : [0, T] \to K_n$ such that

$$x_n'(t) = Ax_n(t) + f(t, x_n(t), u_n(t)) \quad \text{a.e. } t \in [0, T],$$
$$u_n(t) \in SOL(K_n, g(t, x_n(t), u_n(t)) + \frac{1}{\rho_n} P(u_n(t)), \varphi) \text{ for all } t \in [0, T],$$
$$x_n(0) = x_0. \tag{14}$$

Here and below the inclusion $u_n(t) \in SOL(K_n, g(t, x_n(t), u_n(t)) + \frac{1}{\rho_n} P(u_n(t)), \varphi)$ is a short hand notation which means that $u_n(t)$ satisfies the variational inequality

$$u_n(t) \in K_n, \quad \langle g(t, x_n(t), u_n(t)), v - u_n(t)\rangle + \frac{1}{\rho_n}\langle Pu_n(t), v - u_n(t)\rangle$$
$$+ \varphi(v) - \varphi(u_n(t)) \geq 0 \quad \forall\, v \in K_n. \tag{15}$$

In the study of Problems $\mathcal{P}_n$, we consider the following hypotheses on the data.

$$\begin{cases} \text{For every } n \in \mathbb{N}, K_n \text{ is a nonempty closed convex subset of } V \\ \text{and, moreover, } K_n \supset K. \end{cases} \tag{16}$$

$$\text{For every } n \in \mathbb{N}, \ \rho_n > 0. \tag{17}$$

$$P : V \to V^* \text{ is a bounded, demicontinuous and monotone operator.} \tag{18}$$

$$\begin{cases} \text{There exists a set } \widetilde{K} \text{ such that} \\ \text{(a) } K_n \subset \widetilde{K} \subset V \text{ for each } n \in \mathbb{N}. \\ \text{(b) } K_n \xrightarrow{M} \widetilde{K} \text{ as } n \to \infty. \\ \text{(c) } \langle Pu, v - u\rangle \leq 0 \text{ for all } u \in \widetilde{K} \text{ and } v \in K. \\ \text{(d) if } u \in \widetilde{K} \text{ and } \langle Pu, v - u\rangle = 0 \text{ for all } v \in K, \text{then } u \in K. \end{cases} \tag{19}$$

$$\rho_n \to 0 \text{ as } n \to \infty. \tag{20}$$

Folowing Definition 2.5 we recall that a pair of functions (x_n, u_n) is said to be a mild solution of system (14) if $x_n \in C([0, T]; E)$, $u_n \in C([0, T]; K_n)$,

$$x_n(t) = S(t)x_0 + \int_0^t S(t - s)f(s, x_n(s), u_n(s))\, \mathrm{d}s \quad \text{ for all } t \in [0, T] \tag{21}$$

and $u_n(t) \in SOL(K_n, g(t, x(t), u_n(t)) + \frac{1}{\rho_n} P(u_n(t)), \varphi)$ for all $t \in [0, T]$.

Our main result in this paper is the following.

Theorem 3.1: *Assume that* (3), (4), (6)–(8), (16)–(18) *hold. Then:*

(1) *For each* $n \in \mathbb{N}$*, there exists a unique mild solution* $(x_n, u_n) \in C([0, T]; E) \times C([0, T]; K_n)$ *to Problem* $\mathcal{P}_n$*.*
(2) *If, moreover,* (5), (19)–(20) *hold, then the mild solution* (x_n, u_n) *of Problem* $\mathcal{P}_n$ *converges to the mild solution* (x, u) *of Problem* $\mathcal{P}$ *obtained in Theorem* 2.6*, i.e.*

$$(x_n(t), u_n(t)) \to (x(t), u(t)) \quad \text{in } E \times V, \text{ as } n \to \infty, \tag{22}$$

for all $t \in [0, T]$.

Proof: (1) Let $n \in \mathbb{N}$ and consider function $g_n : [0,T] \times E \times V \to V^*$ defined by

$$g_n(t,x,u) = g(t,x,u) + \frac{1}{\rho_n} Pu.$$

Using hypotheses (6), (17) and (18) it is easy too see that g_n satisfies condition (6) with the constants m_g and L_g. Therefore, since (16) holds, we are in a position to use Theorem 2.6 with $K = K_n$. We deduce in this way that there exists a unique solution $(x_n, u_n) \in C([0,T];E) \times C([0,T];K_n)$ to Problem $\mathcal{P}_n$, which concludes the first part of the proof.

(2) For the second part we assume that, in addition, (5) and (19)–(20) hold. We fix $n \in \mathbb{N}$ and we consider the auxiliary problem of finding a function $\widetilde{u}_n \in C([0,T];K_n)$ such that

$$\langle g(t,x(t),\widetilde{u}_n(t)), v - \widetilde{u}_n(t)\rangle + \frac{1}{\rho_n}\langle P\widetilde{u}_n(t), v - \widetilde{u}_n(t)\rangle + \varphi(v) - \varphi(\widetilde{u}_n(t)) \geq 0 \tag{23}$$

for all $v \in K_n$ and $t \in [0,T]$. Recall that here and below x is the mild trajectory of Problem $\mathcal{P}$. Using a standard result on time-dependent variational inequalities we see that problem (23) has a unique solution $\widetilde{u}_n \in C([0,T];K_n)$. We now divide the rest of the proof into four steps.

Step (i). We claim that for any $t \in [0,T]$ there exists a subsequence of the sequence $\{\widetilde{u}_n(t)\}$, again denoted by $\{\widetilde{u}_n(t)\}$, which converges weakly to an element $\widetilde{u}(t) \in \widetilde{K}$, i.e.

$$\widetilde{u}_n(t) \to \widetilde{u}(t) \text{ weakly in } V, \text{as } n \to \infty. \tag{24}$$

To prove this claim we fix $t \in [0,T]$ and $u_0 \in K$. We use the strong monotonicity of g, inequality (23) with $v = u_0 \in K \subset K_n$ and assumption (19)(c) to obtain

$$\begin{aligned} m_g\|\widetilde{u}_n(t) - u_0\|_V^2 &\leq \langle g(t,x(t),\widetilde{u}_n(t)) - g(t,x(t),u_0), \widetilde{u}_n(t) - u_0\rangle \\ &\leq \frac{1}{\rho_n}\langle P\widetilde{u}_n(t), u_0 - \widetilde{u}_n(t)\rangle + \varphi(u_0) - \varphi(\widetilde{u}_n(t)) + \langle g(t,x(t),u_0), u_0 - \widetilde{u}_n(t)\rangle \\ &\leq \varphi(u_0) - \varphi(\widetilde{u}_n(t)) + \langle g(t,x(t),u_0), u_0 - \widetilde{u}_n(t)\rangle. \end{aligned}$$

Next, using inequality (12) we find that

$$\begin{aligned} m_g\|\widetilde{u}_n(t) - u_0\|_V^2 &\leq \varphi(u_0) - \langle l, \widetilde{u}_n(t)\rangle - \beta + \|g(t,x(t),u_0)\|_{V^*}\|\widetilde{u}_n(t) - u_0\|_V \\ &\leq \varphi(u_0) + (\|g(t,x(t),u_0)\|_{V^*} + \|l\|_{V^*})\|\widetilde{u}_n(t) - u_0\|_V + \|l\|_{V^*}\|u_0\|_V + |\beta|. \end{aligned}$$

Further, we denote

$$M_1(t) = \|g(t,x(t),u_0)\|_{V^*} + \|l\|_{V^*}$$

and apply the elementary inequality $ab \leq \frac{1}{2}(a^2 + b^2)$, valid for all $a, b \in \mathbb{R}$, to get

$$\frac{m_g}{2}\|\widetilde{u}_n(t) - u_0\|_V^2 \leq \varphi(u_0) + \frac{(M_1(t))^2}{2m_g} + \|l\|_{V^*}\|u_0\|_V + |\beta|.$$

This inequality implies that the sequence $\{\widetilde{u}_n(t) - u_0\}$ bounded in V and, therefore, the sequence $\{\widetilde{u}_n(t)\}$ is bounded in V, too. Next, from the reflexivity of V, we deduce that there exists an element $\widetilde{u}(t) \in V$ such that, passing to a subsequence if necessary, again denoted by $\{\widetilde{u}_n(t)\}$, the weak convergence (24) holds. Recall that $\widetilde{u}_n(t) \in K_n$ for each $n \in \mathbb{N}$. Therefore, assumption (19) (b) and Definition 2.4 (ii) of the Mosco convergence imply that $\widetilde{u}(t) \in \widetilde{K}$, as claimed.

Step (ii). We show that $\widetilde{u}(t) = u(t)$ for all $t \in [0,T]$.

Let $t \in [0,T]$ and $v \in \widetilde{K}$. Then, Definition 2.4 (i) guarantees that there exists a sequence $\{v_n\}$ such that $v_n \in K_n$ for each $n \in \mathbb{N}$ and $v_n \to v$ in V as $n \to \infty$. We now use inequality (23) to see that

$$\frac{1}{\rho_n}\langle P\widetilde{u}_n(t), \widetilde{u}_n(t) - v_n\rangle \le \langle g(t,x(t),v_n), v_n - \widetilde{u}_n(t)\rangle + \varphi(v_n) - \varphi(\widetilde{u}_n(t))$$
$$\le \|g(t,x(t),v_n)\|_{V^*}\|v_n - \widetilde{u}_n(t)\|_V + \varphi(v_n) + \|l\|_{V^*}\|\widetilde{u}_n(t)\|_V + |\beta|.$$

Hence, properties (6) and (7) of the functions g and φ, respectively, combined with the convergence $v_n \to v$ in V and boundedness of the sequence $\{\widetilde{u}_n(t)\}$ show that there exists a positive constant c which does not depend on n, such that

$$\frac{1}{\rho_n}\langle P\widetilde{u}_n(t), \widetilde{u}_n(t) - v_n\rangle \le c.$$

Then, since $\rho_n \to 0$, we deduce that

$$\limsup\, \langle P\widetilde{u}_n(t), \widetilde{u}_n(t) - v_n\rangle \le 0, \tag{25}$$

Now, since the sequence $\{P\widetilde{u}_n(t)\}$ is bounded in V^* and $v_n \to v$ in V have

$$\limsup\, \langle P\widetilde{u}_n(t), \widetilde{u}_n(t) - v\rangle \le \limsup\, \langle P\widetilde{u}_n(t), \widetilde{u}_n(t) - v_n\rangle$$
$$+ \limsup\, \langle P\widetilde{u}_n(t), v_n - v\rangle = \limsup\, \langle P\widetilde{u}_n(t), \widetilde{u}_n(t) - v_n\rangle$$

and, therefore, (25) yields

$$\limsup\, \langle P\widetilde{u}_n(t), \widetilde{u}_n(t) - v\rangle \le 0 \quad \forall\, v \in \widetilde{K}. \tag{26}$$

Moreover, the regularity $\widetilde{u}(t) \in \widetilde{K}$ allows us to take $v = \widetilde{u}(t)$ in the (26) to obtain

$$\limsup\, \langle P\widetilde{u}_n(t), \widetilde{u}_n(t) - \widetilde{u}(t)\rangle \le 0. \tag{27}$$

On the other hand, recall that assumption (18) guarantees that the operator $P : V \to V^*$ is bounded, demicontinuous and monotone. Then, by a standard result ([22, Theorem 3.69], for instance) we deduce that P is pseudomonotone. Thus, inequality (27) combined with the pseudomonotonicity of P implies that

$$\langle P\widetilde{u}(t), \widetilde{u}(t) - v\rangle \le \liminf\, \langle P\widetilde{u}_n(t), \widetilde{u}_n(t) - v\rangle \le \limsup\, \langle P\widetilde{u}_n(t), \widetilde{u}_n(t) - v\rangle$$

for all $v \in V$ and, therefore, (26) yields

$$\langle P\widetilde{u}(t), \widetilde{u}(t) - v\rangle \le 0 \quad \forall\, v \in \widetilde{K}. \tag{28}$$

Next, since (16) and (19)(a) guarantee that $K \subset \widetilde{K}$, we use (28) to deduce that

$$\langle P\widetilde{u}(t), \widetilde{u}(t) - v\rangle \le 0 \quad \forall\, v \in K. \tag{29}$$

We now combine inequality (29) with assumption (19)(c) to find that

$$\langle P\widetilde{u}(t), \widetilde{u}(t) - v\rangle = 0 \quad \forall\, v \in K,$$

then we use assumption (19)(d) to obtain the regularity

$$\widetilde{u}(t) \in K. \tag{30}$$

Assume now that $w \in K$. Then, using inequality (23) we find that

$$\langle g(t, x(t), \widetilde{u}_n(t)), \widetilde{u}_n(t) - w\rangle \leq \frac{1}{\rho_n}\langle P\widetilde{u}_n(t), w - \widetilde{u}_n(t)\rangle + \varphi(w) - \varphi(\widetilde{u}_n(t))$$

and, therefore, assumption 19)(c) yields

$$\langle g(t, x(t), \widetilde{u}_n(t)), \widetilde{u}_n(t) - w\rangle \leq \varphi(w) - \varphi(\widetilde{u}_n(t)). \tag{31}$$

On the other hand, by the monotonicity of g we deduce that

$$\langle g(t, x(t), w), \widetilde{u}_n(t) - w\rangle \leq \langle g(t, x(t), \widetilde{u}_n(t)), \widetilde{u}_n(t) - w\rangle$$

and, using (31) we find that

$$\langle g(t, x(t), w), \widetilde{u}_n(t) - w\rangle \leq \varphi(w) - \varphi(\widetilde{u}_n(t)). \tag{32}$$

We pass to the upper limit in this inequality and use assumption (7) and the convergence (24) to obtain that

$$\langle g(t, x(t), w), w - \widetilde{u}(t)\rangle + \varphi(w) - \varphi(\widetilde{u}(t)) \geq 0.$$

Next, we take $w = (1 - \lambda)\widetilde{u}(t) + \lambda v$ where v is an arbitrary element in K and $\lambda \in (0, 1)$. Then, (30) and (5) imply that $w \in K$ and, therefore, the previous inequality combined with the convexity of the function φ yields

$$\langle g(t, x(t), (1 - \lambda)\widetilde{u}(t) + \lambda v), v - \widetilde{u}(t)\rangle + \varphi(v) - \varphi(\widetilde{u}(t)) \geq 0.$$

We now use pass to the limit when $\lambda \to 0$ and use assumption (6)(a) to see that

$$\langle g(t, x(t), \widetilde{u}(t)), v - \widetilde{u}(t)\rangle + \varphi(v) - \varphi(\widetilde{u}(t)) \geq 0 \qquad \forall\, v \in K.$$

Note that assumption (6)(a) guarantees the uniqueness of the solution of this inequality. Therefore, since $u(t) \in K$ is the unique solution of the above inequality, we deduce that $\widetilde{u}(t) = u(t)$, as claimed.

Step (*iii*). We prove that $\widetilde{u}_n(t) \to u(t)$ in V, for all $t \in [0, T]$.

Let $t \in [0, T]$. First, a careful examination of the proofs in steps (i) and (ii) reveals that any weakly convergent subsequence of the sequence $\{\widetilde{u}_n(t)\}$ converges weakly in V to $u(t)$, as $n \to \infty$. Moreover, the sequence $\{\widetilde{u}_n(t)\}$ is bounded and, therefore, the whole sequence $\{\widetilde{u}_n(t)\}$ converges weakly in V to $u(t)$.

Next, taking $w = u(t)$ in (31) and passing to the upper limit we see that

$$\limsup\, \langle g(t, x(t), \widetilde{u}_n(t)), \widetilde{u}_n(t) - u(t)\rangle \leq 0. \tag{33}$$

This inequality together with (24) and the pseudomonotonicity of the function g, guaranteed by assumption (6)(a), yields

$$\langle g(t, x(t), \widetilde{u}(t)), \widetilde{u}(t) - v\rangle \leq \liminf \langle g(t, x(t), \widetilde{u}_n(t)), \widetilde{u}_n(t) - v\rangle \quad \forall\, v \in X.$$

Next, we take $v = u(t)$ in the previous inequality to deduce that

$$\liminf \langle g(t, x(t), \widetilde{u}_n(t)), \widetilde{u}_n(t) - u(t)\rangle \geq 0. \tag{34}$$

Using now inequalities (33) and (34) it folows that

$$\langle g(t, x(t), \widetilde{u}_n(t)), \widetilde{u}_n(t) - u(t)\rangle \to 0, \quad \text{as } n \to \infty. \tag{35}$$

Therefore, using the strong monotonicity of g combined with the convergence $\widetilde{u}_n(t) \to u(t)$ weakly in V and (35), we have

$$\begin{aligned} m_g\|\widetilde{u}_n(t) - u(t)\|_V^2 &\leq \langle g(t, x(t), \widetilde{u}_n(t)) - g(t, x(t), u(t)), \widetilde{u}_n(t) - u(t)\rangle \\ &= \langle g(t, x(t), \widetilde{u}_n(t)), \widetilde{u}_n(t) - u(t)\rangle - \langle g(t, x(t), u(t)), \widetilde{u}_n(t) - u(t)\rangle \to 0, \end{aligned}$$

as $n \to \infty$. This ends the proof of Step (iii).

Step (iv). Finally, we show that $(x_n(t), u_n(t)) \to (x(t), u(t))$ as $n \to \infty$, for all $t \in [0, T]$ where, recall, $(x_n, u_n) \in C([0, T]; E) \times C([0, T]; K_n)$ is the unique solution of Problem $\mathcal{P}_n$.

Let $n \in \mathbb{N}$ and $t \in [0, T]$. We have

$$\langle g(t, x_n(t), u_n(t)), v - u_n(t)\rangle + \frac{1}{\rho_n}\langle Pu_n(t), v - u_n(t)\rangle + \varphi(v) - \varphi(u_n(t)) \geq 0, \tag{36}$$

for all $v \in K_n$. We take $v = \widetilde{u}_n(t)$ in (36) and $v = u_n(t)$ in (23), then we add the resulting inequalities and use the monotonicity of the operator P to find that

$$\langle g(t, x_n(t), u_n(t)) - g(t, x(t), \widetilde{u}_n(t)), u_n(t) - \widetilde{u}_n(t)\rangle \leq 0.$$

This inequality combined with assumption (6) on g shows that

$$\begin{aligned} m_g\|u_n(t) - \widetilde{u}_n(t)\|_V^2 &\leq \langle g(t, x_n(t), u_n(t)) - g(t, x_n(t), \widetilde{u}_n(t)), u_n(t) - \widetilde{u}_n(t)\rangle \\ &\leq \langle g(t, x(t), \widetilde{u}_n(t)) - g(t, x_n(t), \widetilde{u}_n(t)), u_n(t) - \widetilde{u}_n(t)\rangle \\ &\leq L_g\|x(t) - x_n(t)\|_E\|u_n(t) - \widetilde{u}_n(t)\|_V \end{aligned}$$

and, therefore, writing $\|u_n(t) - u(t)\|_V \leq \|u_n(t) - \widetilde{u}_n(t)\|_V + \|\widetilde{u}_n(t) - u(t)\|_V$ yields

$$\|u_n(t) - u(t)\|_V \leq \frac{L_g}{m_g}\|x(t) - x_n(t)\|_E + \|\widetilde{u}_n(t) - u(t)\|_V. \tag{37}$$

On the other hand, using (9) and (21) we obtain that

$$\|x_n(t) - x(t)\|_E \leq M_A\int_0^t \|f(s, x_n(s), u_n(s)) - f(s, x(s), u(s))\|_E\, ds$$

where M_A is a positive constant such that $\|S(s)\|_{\mathcal{L}(E)} \leq M_A$ for all $s \in [0, T]$. Using now assumption (4) and inequality (37) we find that

$$\|x_n(t) - x(t)\|_E \leq M_A\int_0^t \psi(s)\|\widetilde{u}_n(s) - u(s)\|_V\, ds + M_A\int_0^t \psi(s)\left(1 + \frac{L_g}{m_g}\right)\|x_n(s) - x(s)\|_E\, ds.$$

We now use the Gronwall inequality to deduce that there exists a positive constant C_0 which does not depend on n such that

$$\|x_n(t) - x(t)\|_E \leq C_0\int_0^t \psi(s)\|\widetilde{u}_n(s) - u(s)\|_V\, ds.$$

This inequality combined with the convergence $\widetilde{u}_n(s) \to u(s)$ in V, for all $s \in [0, T]$, the boundedness result obtained in the proof of Step (i) and the Lebesgue convergence theorem implies that

$$\limsup \|x_n(t) - x(t)\|_E \leq C_0\int_0^t \lim \psi(s)\|\widetilde{u}_n(s) - u(s)\|_V\, ds = 0.$$

Therefore, we deduce that

$$x_n(t) \to x(t) \quad \text{in } E, \text{as } n \to \infty. \tag{38}$$

On the other hand, inequality (37), the convergence $\widetilde{u}_n(t) \to u(t)$ in V obtained in Step (iii) and the convergence (38) show that

$$u_n(t) \to u(t) \quad \text{in } V, \quad \text{as } n \to \infty. \tag{39}$$

The convergence (22) is now a direct consequence of (38) and (39) and this concludes the proof. ■

4. Relevant particular cases

In this section we present some particular cases which lead to relevant consequences of Theorem 3.1. To this end, below in this section we assume that (3)–(8) hold and we denote by (x, u) the mild solution of Problem $\mathcal{P}$ provided by Theorem 2.6. Recall that this solution satisfies

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s, x(s), u(s))\,ds \quad \forall\, t \in [0, T], \tag{40}$$

$$u(t) \in K, \quad \langle g(t, x(t), u(t)), v - u(t)\rangle + \varphi(v) - \varphi(u(t)) \geq 0 \quad \forall\, v \in K,\ t \in [0, T]. \tag{41}$$

(a) The case when K_n does not depend on n. In this case we can take $\widetilde{K} = K_n$ to deduce the following consequence of Theorem 3.1.

Corollary 4.1: *Assume that* (3)–(8), (17), (18), (20) *hold and, moreover, assume that there exists a convex closed subset* $\widetilde{K}$ *of* V *such that* $K \subset \widetilde{K}$ *and* (19)(c), (d) *hold. Then, for each* $n \in \mathbb{N}$, *there exists a unique couple of functions* $(x_n, u_n) \in C([0, T]; E) \times C([0, T]; \widetilde{K})$ *such that*

$$x_n(t) = S(t)x_0 + \int_0^t S(t-s)f(s, x_n(s), u_n(s))\,ds \quad \forall\, t \in [0, T], \tag{42}$$

$$u_n(t) \in \widetilde{K}, \quad \langle g(t, x_n(t), u_n(t)), v - u_n(t)\rangle + \frac{1}{\rho_n}\langle Pu_n(t), v - u_n(t)\rangle$$
$$+ \varphi(v) - \varphi(u_n(t)) \geq 0 \quad \forall\, v \in \widetilde{K},\ t \in [0, T]. \tag{43}$$

Moreover, the convergence (22) *holds, for any* $t \in [0, T]$.

Note that in the particular case when $\widetilde{K} = V$, Corollary 4.1 represents the main convergence result obtained in [17]. In this case inequality (42) is an unconstrained variational inequality. In contrast, in Corollary 4.1 the set $\widetilde{K}$ could be different from the whole space V and inequality (42) could be a time-dependent variational inequality with constraints. In this general case Corollary 4.1 shows that the solution of the differential variational inequality (40)–(41), governed by the set of constraints K, can be approached by the solution of the differential variational inequality (42)–(43), governed by a different set of constraints $\widetilde{K}$, as the penalty parameter ρ_n is small enough.

(b) The case when $K_n \xrightarrow{M} V$ and P is a particular penalty operator of K. Assume that the sets K_n satisfy the following condition.

$$\begin{cases} \text{For each } u \in V, \text{ there exists a sequence } \{u_n\} \text{ such that} \\ u_n \in K_n \text{ for each } n \in \mathbb{N} \text{ and } u_n \to u \text{ in } V. \end{cases} \tag{44}$$

Note that this assumptions implies that $K_n \xrightarrow{M} V$ as $n \to \infty$ and, therefore, it guarantees that condition (19)(a)(b) are satisfied with $\widetilde{K} = V$.

Let $I_V : V \to V$ be the identity map on V, $P_K : V \to K$ the projection operator of K, $J : V \to V^*$ the normalized duality map on V and let P be the operator defined by

$$P : V \to V^*, \quad P = J(I_V - P_K). \tag{45}$$

Then, it is well known that P is a penalty operator on K, see [21, Proposition 1.3.27]. We use Definition 2.2 to see that in this case conditions (18) and (19)(c) are satisfied. We claim that condition (19)(d) holds, too. Indeed, assume that $u \in V$ is such that $\langle Pu, v - u\rangle = 0$ for all $v \in K$. Then, taking $v = P_K u$ in the previous equality we obtain that $\langle J(u - P_K u), P_K u - u\rangle = 0$ and, since $\langle J(w), w\rangle = \|w\|_X^2$ for all $w \in V$, we deduce that $u = P_K u$. This implies that $u \in K$ and concludes the proof of the claim.

The ingredients presented above allow us to deduce the following consequence of Theorem 3.1.

Corollary 4.2: *Assume that* (3)–(8), (16), (17), (20), (44) *and* (45) *hold. Then, for each* $n \in \mathbb{N}$, *there exists a unique couple of functions* $(x_n, u_n) \in C([0,T];E) \times C([0,T];K_n)$ *such that*

$$x_n(t) = S(t)x_0 + \int_0^t S(t-s)f(s, x_n(s), u_n(s))\,\mathrm{d}s \quad \forall\, t \in [0,T], \tag{46}$$

$$\begin{aligned} &u_n(t) \in K_n, \quad \langle g(t, x_n(t), u_n(t)), v - u_n(t)\rangle + \frac{1}{\rho_n}\langle Pu_n(t), v - u_n(t)\rangle \\ &\quad + \varphi(v) - \varphi(u_n(t)) \geq 0 \quad \forall\, v \in K_n,\ t \in [0,T]. \end{aligned} \tag{47}$$

Moreover, the convergence (22) *holds, for any* $t \in [0,T]$.

A first example of sets K_n which satisfies condition (44) is provided by the closed balls $K_n = \{\, v \in V \ :\ \|v\|_V \leq n\,\}$. Indeed, for each $u \in V$ consider the sequence $\{u_n\} \subset V$ defined by

$$u_n = \begin{cases} u & \text{if } \|u\|_V \leq n, \\ n\,\dfrac{u}{\|u\|_V} & \text{if } \|u\|_V > n \end{cases}$$

It is easy to see that $u_n \in K_n$ for each $n \in \mathbb{N}$ and $u_n \to u$ in V. A second example is obtained by taking $K_n = V$. Note that in the first example condition $K \subset K_n$ holds if K is a bounded sequence and n is large enough. This inclusion is obviously satisfied if $K_n = V$.

Remark 4.3: We end this section with the remark that Corollary 4.2 still holds if we replace assumption (45) with its more general version

$$P \text{ is a penalty operator for the set } K. \tag{48}$$

Indeed, (48) and Definition 2.2 imply that conditions (18) and (19)(c) hold. Moreover, since $\widetilde{K} = V$, inequality (28) imply the regularity (30) and, therefore, in this case Theorem 2.6 still holds without condition (19) (d). We can use this version of Theorem 2.6 to prove that Corollary 4.2 holds under the assumptions mentioned above. In this way we recover, once more, the main convergence result obtained in [17], in the particular case when $K_n = V$, for each $n \in \mathbb{N}$.

5. An example

The abstract results presented in Sections 3 and 4 can be used in the study of various initial and boundary problems arising in Physics, Mechanics and Engineering Sciences. A large number of examples can be considered. Nevertheless, to keep the paper in a reasonable length, in this section we restrict ourselves to present only a purely academic example.

Let Ω be a bounded domain of $\mathbb{R}^d$ ($d \in \mathbb{N}$) with a smooth boundary Γ, divided in two measurable parts Γ_1 and Γ_2, such that *meas* $\Gamma_1 > 0$. Let $T > 0$ be a finite interval of time. We denote by $z \in \Omega \cup \Gamma$ the spatial variable, by $t \in [0,T]$ the time variable and by ν the outward unit normal at Γ. With these notation we consider the following parabolic-elliptic problem.

Problem $\mathcal{Q}$. Find $x : \Omega \times [0,T] \to \mathbb{R}$ and $u : \Omega \times [0,T] \to \mathbb{R}$ such that

$$x'(z,t) - \Delta x(z,t) = e(z, t, x(z,t), u(z,t)) \quad \text{in } \Omega \times [0,T], \tag{49}$$

$$x(z,t) = 0 \quad \text{on } \Gamma \times [0,T], \tag{50}$$

$$x(z,0) = x_0(z) \quad \text{in } \Omega, \tag{51}$$

$$\left.\begin{array}{l} u(z,t) \geq k, \\ -\Delta u(z,t) + \beta(z)u(z,t) \geq h(z,t,x(z,t)), \\ \big(u(z,t) - k\big)\big(\Delta u(z,t) - \beta(z)u(z,t) + h(z,t,x(z,t))\big) = 0 \end{array}\right\} \text{ in } \Omega \times [0,T], \tag{52}$$

$$u(z,t) = 0 \quad \text{on } \Gamma_1 \times [0,T], \tag{53}$$

$$\left.\begin{array}{l} \left|\dfrac{\partial u(z,t)}{\partial \nu}\right| \leq \phi(z), \\ -\dfrac{\partial u(z,t)}{\partial \nu} = \phi(z)\dfrac{u(z,t)}{|u(z,t)|} \quad \text{if } u(z,t) \neq 0 \end{array}\right\} \quad \text{on } \Gamma_2 \times [0,T]. \tag{54}$$

In the study of Problem Q we use standard notation for Lebesgue and Sobolev spaces. Moreover, we consider the space

$$V = \{\, v \in H^1(\Omega) \ : \ v(z) = 0 \text{ a.e. } z \in \Gamma_1 \,\} \tag{55}$$

endowed with the inner product

$$(u,v)_V = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x \tag{56}$$

and the associated norm $\|\cdot\|_V$. It is well known that $(V, \|\cdot\|_V)$ is a Hilbert space. Moreover, as usual, we use V^* for the dual of V and $\langle\cdot,\cdot\rangle$ for the duality pairing mapping between V^* and V.

We now consider the following assumptions on the data.

$$\begin{cases} e : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ is such that :} \\ (a) \text{ for all } (s,r) \in \mathbb{R}^2, \text{the function } (z,t) \mapsto e(z,t,s,r) \\ \qquad \text{is measurable on } \Omega \times [0,T]; \\ (b) \text{ there exists two positive functions} \\ \qquad \theta \in L^1(\Omega \times (0,T)) \text{ and } \psi \in L^1(0,T) \text{ such that} \\ \qquad |e(z,t,0,0)| \leq \theta(z,t), \\ \qquad |e(z,t,s_1,r_1) - e(z,t,s_2,r_2)| \leq \psi(t)(|s_1 - s_2| + |r_1 - r_2|), \\ \qquad \text{for a.e. } z \in \Omega,\ t \in [0,T] \text{ and all } (s_1,r_1),\ (s_2,r_2) \in \mathbb{R}^2. \end{cases} \tag{57}$$

$$\begin{cases} h : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R} \text{ is such that :} \\ (a) \text{ the function } z \mapsto h(z,0,0) \text{ is belongs to } L^2(\Omega); \\ (b) \text{ there exists a constant } L_h > 0 \text{ such that} \\ \qquad |h(z,t_1,s_1) - h(z,t_2,s_2)| \leq L_h(|t_1 - t_2| + |s_1 - s_2|), \\ \qquad \text{for all } t_1,\ t_2 \in [0,T],\ s_1,\ s_2 \in \mathbb{R},\ \text{a.e. } z \in \Omega. \end{cases} \tag{58}$$

$$\beta \in L^\infty(\Omega), \quad \beta(z) \geq 0 \quad \text{a.e. } z \in \Omega, \tag{59}$$

$$\phi \in L^2(\Gamma_2), \quad \phi(z) \geq 0 \quad \text{a.e. } z \in \Gamma_2, \tag{60}$$

$$x_0 \in L^2(\Omega), \tag{61}$$

$$k > 0. \tag{62}$$

Under these assumptions, let $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$, $f : [0,T] \times L^2(\Omega) \times V \to L^2(\Omega)$, $K \subset V, g : [0,T] \times L^2(\Omega) \times V \to V^*$ and $\varphi : V \to \mathbb{R}$ be defined as follows:

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega), \quad Ax = \Delta x \quad \forall\, x \in D(A), \tag{63}$$

$$f(t,x,u)(z) = e(z,t,x(z),u(z))$$
$$\forall\, t \in [0,T],\ x \in L^2(\Omega),\ u \in V,\ \text{a.e. } z \in \Omega, \tag{64}$$

$$K = \{ u \in V \ : \ u(z) \geq k \ \text{a.e.}\ z \in \Omega \}, \tag{65}$$

$$\langle g(t,x,u), v\rangle = \int_\Omega \nabla u \cdot \nabla v \, dz + \int_\Omega \beta u v \, dz - \int_\Omega h(t,x) v \, dz$$
$$\forall\, t \in [0,T],\ x \in L^2(\Omega),\ u,\ v \in V, \tag{66}$$

$$\varphi(u) = \int_{\Gamma_2} k|u| \, da \qquad \forall\, u \in V. \tag{67}$$

We now perform integration by parts and use the previous notation to deduce the following variational formulation of Problem $\mathcal{Q}$.

Problem $\mathcal{Q}^V$. Find $x : [0,T] \to E$ and $u : [0,T] \to V$ such that

$$x'(t) = Ax(t) + f(t, x(t), u(t)) \qquad \text{a.e. } t \in [0,T], \tag{68}$$

$$u(t) \in K, \quad \langle g(t, x(t), u(t)), v - u(t)\rangle + \varphi(v) - \varphi(u(t)) \geq 0 \quad \forall\, v \in K,\ t \in [0,T], \tag{69}$$

$$x(0) = x_0. \tag{70}$$

It is well known that the operator A defined by (63) satisfies condition (3) on the space $E = L^2(\Omega)$ (see [20], for instance). Therefore, following Definition 2.5 we can consider the concept of mild solution for the differential variational inequality (68)–(69). Moreover, we have the following existence and uniqueness result.

Theorem 5.1: *Assume that* (57)–(62) *hold. Then Problem* $\mathcal{Q}^V$ *has a unique mild solution* $(x, u) \in C([0,T]; L^2(\Omega)) \times C([0,T]; K)$.

Proof: The proof of Theorem 5.1 can be obtained by using Theorem 2.6 with $E = L^2(\Omega)$, V defined in (55), (56) and A, f, h, K, g, φ given by (63)–(67). Indeed, it is easy to check that in this case conditions (3)–(8) are satisfied. The details of the proof are similar to those presented in [17] and, therefore, we omit them. ■

Next, for each $n \in \mathbb{N}$ we consider the following initial and boundary value problem.

Problem $\mathcal{Q}_n$. Find $x_n : \Omega \times [0,T] \to \mathbb{R}$ and $u_n : \Omega \times [0,T] \to \mathbb{R}$ which satisfy (49)–(51), (53)–(54) and, moreover

$$\left.\begin{aligned} & u_n(z,t) \geq k_n, \\ & \Delta u_n(z,t)) + h(z,t,x_n(z,t)) = \beta(z) u_n(z,t) + \frac{1}{\rho_n} p(z, u_n(z,t) - k_n) \end{aligned}\right\} \text{ in } \Omega \times [0,T]. \tag{71}$$

A brief comparison between Problems $\mathcal{Q}$ and $\mathcal{Q}_n$ reveals the fact that in Problem $\mathcal{P}_n$ we replaced the conditions (52) with conditions (71). There, $\rho_n > 0$, $k_n \in \mathbb{R}$ and p is a function assumed to have the following properties.

$$\begin{cases} p\colon \Omega \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ \text{(a) } |p(z,r) - p(z,s)| \leq L_p |r - s| \\ \qquad \text{for all } r,\ s \in \mathbb{R},\ \text{a.e. } z \in \Omega,\ \text{with } L_p > 0; \\ \text{(b) } (p(z,r) - p(z,s))\,(r - s) \geq 0 \\ \qquad \text{for all } r,\ s \in \mathbb{R},\ \text{a.e. } z \in \Omega; \\ \text{(c) } z \mapsto p(z,r) \ \text{ is measurable on } \Omega \text{ for all } r \in \mathbb{R}; \\ \text{(d) } p(z,r) = 0 \text{ if and only if } r \geq 0, \text{a.e. } z \in \Omega. \end{cases} \tag{72}$$

A typical example of such function is given by

$$p_0(z,r) = -cr^- \quad \text{for all } r \in \mathbb{R},\ z \in \Omega$$

where $c > 0$ and r^- represents the negative part of r, i.e. $r^- = max\{-r, 0\}$.

Let $\widetilde{k} \in \mathbb{R}$ and assume that

$$\widetilde{k} \geq k_n \geq k \quad \text{for all } n \in \mathbb{N}, \tag{73}$$

$$k_n \to \widetilde{k} \quad \text{as } n \to \infty. \tag{74}$$

We now define the sets $\widetilde{K}$, K_n and the operator $P : V \to V^*$ by equalities

$$\widetilde{K} = \{\, u \in V \ :\ u(z) \geq \widetilde{k} \text{ a.e. } z \in \Omega \,\}, \tag{75}$$

$$K_n = \{\, u \in V \ :\ u(z) \geq k_n \text{ a.e. } z \in \Omega \,\} \quad \forall\, n \in \mathbb{N}, \tag{76}$$

$$\langle Pu, v\rangle = \int_\Omega p(u-k)v \,\mathrm{d}z \quad \forall\, u,\ v \in V. \tag{77}$$

Then, the variational formulation of Problem $\mathcal{Q}_n$, is the following

Problem $\mathcal{Q}_n^V$. Find $x : [0,T] \to E$ and $u : [0,T] \to V$ such that

$$x_n'(t) = Ax_n(t) + f(t, x_n(t), u_n(t)) \quad \text{a.e. } t \in [0,T], \tag{78}$$

$$u_n(t) \in K_n, \quad \langle g(t, x_n(t), u_n(t), v - u_n(t)\rangle + \frac{1}{\rho_n}\langle Pu_n(t), v - u_n(t)\rangle$$
$$+ \varphi(v) - \varphi(u_n(t)) \geq 0 \quad \forall\, v \in K_n,\ t \in [0,T], \tag{79}$$

$$x_n(0) = x_0. \tag{80}$$

Our main result in this section is the following.

Theorem 5.2: *Assume that* (57)–(62), (72)–(74), (17) *and* (20) *hold. Then, for each $n \in \mathbb{N}$, there exists a unique mild solution $(x_n, u_n) \in C([0,T]; E) \times C([0,T]; K_n)$ to Problem $\mathcal{Q}_n^V$; Moreover the solution converges to the mild solution (x,u) of Problem $\mathcal{Q}^V$ obtained in Theorem* 5.1*, i.e.*

$$(x_n(t), u_n(t)) \to (x(t), u(t)) \quad \text{in } L^2(\Omega) \times V \text{ as } n \to \infty, \tag{81}$$

for all $t \in [0,T]$.

Proof: We use Theorem 3.1 on the spaces $E = L^2(\Omega)$ and V given by (55), with notation (63)–(67), (75)–(77). First, we note that assumption (73) implies that condition (16) and (19)(a) are satisfied. Second, using the properties (72) of the function p it is easy to see that the operator (77) satisfies condition (18). Next, using assumption (73) we deduce that $K_n = \frac{k_n}{\widetilde{k}}\widetilde{K}$ which implies that condition (19)(b) holds, too.

Assume now that $u \in \widetilde{K}$ and $v \in K$. Then, using (72) we see that

$$p(u-k)(v-k) \leq 0 \quad \text{and} \quad p(u-k)(k-u) \leq 0 \quad \text{a.e. in } \Omega \tag{82}$$

which imply that $p(u-k)(v-u) \leq 0$ a.e. in Ω. We conclude from here that

$$\int_\Omega p(u-k)(v-u)\,\mathrm{d}z \leq 0$$

and, therefore, condition (19)(c) holds.

Next, we assume that $u \in \widetilde{K}$ and $\langle Pu, v-u\rangle = 0$ for all $v \in K$, which implies that

$$\int_\Omega p(u-k)(u-k)\,\mathrm{d}z = \int_\Omega p(u-k)(v-k)\,\mathrm{d}z \quad \forall\, v \in K. \tag{83}$$

We now use inequalities (82) to deduce that

$$\int_\Omega p(u-k)(u-k)\,\mathrm{d}z = 0. \tag{84}$$

Therefore, the implication

$$h \geq 0, \quad \int_\Omega h\,\mathrm{d}z = 0 \quad \Longrightarrow \quad h = 0 \quad \text{a.e. on } \Omega \tag{85}$$

combined with (82) and (84) shows that $p(u-k)(u-k) = 0$ a.e. in Ω. This equality together with condition (72)(d) implies that $u \geq k$ a.e. in Ω. Therefore, $u \in K$, which shows that condition (19)(d) holds.

Recall that, by hypothesis, (17) and (20) hold. Moreover, the validity of the rest of conditions in Theorem 3.1 follows from the proof of Theorem 5.1. We are now in a position to use Theorem 3.1 in the study of Problems $\mathcal{P}$ and $\mathcal{P}_n$ to conclude the proof. ■

Acknowledgments

This project has received funding from the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie Grant Agreement No. 823731 CONMECH. It is also supported by the National Natural Science Foundation of China under the Grant Nos. 11671101 and 11961074, the Guangxi Natural Science Foundation under the Grant No. 2017GXNSFBA198031, and the High Level Innovation Team Program from Guangxi Higher Education Institutions of China (Document No. [2018] 35).

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This project has received funding from the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie Grant Agreement No. 823731 CONMECH. It is also supported by the National Natural Science Foundation of China [grant numbers 11671101 and 11961074], the Guangxi Natural Science Foundation [grant number 2017GXNSFBA198031], and the High Level Innovation Team Program from Guangxi Higher Education Institutions of China (Document No. [2018] 35).

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