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# Boundary optimal control for antiplane contact problems with power-law friction

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ABSTRACT

We consider a contact model with power-law friction in the antiplane context. Our study focuses on the boundary optimal control, paying special attention to optimality conditions and computational methods. Depending on the exponent of the power-law friction, we are able to deduce an optimality condition for the original problem or for a regularized version of it. Furthermore, we introduce and analyze a computational technique based on linearization, saddle point theory and a fixed point method. For a slightly modified optimal control problem, some numerical experiments are presented.



## 1. Introduction

In this paper we consider a model describing the antiplane shear deformation of an elastic, isotropic, nonhomogeneous cylindrical body, in frictional contact on a part of the boundary with a rigid foundation. If we refer the cylinder to a cartesian coordinate system $Ox_1x_2x_3$ such that its generators are parallel with the axis $Ox_3$, the cross section of the body is a bounded connected open set $\Omega \subset Ox_1x_2$. The boundary $\Gamma$ of $\Omega$ is Lipschitz continuous and partitioned in three measurable parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ of positive measure. From the mathematical point of view the model consists of the following nonlinear boundary value problem

$$\operatorname{div}(\mu(x)\nabla u(x)) + f_0(x) = 0 \text{ in } \Omega, \tag{1}$$

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$$u(x) = 0 \text{ on } \Gamma_1, \tag{2}$$

$$\mu(x)\partial_\nu u(x) = f_2(x) \text{ on } \Gamma_2, \tag{3}$$

$$\mu(x)\partial_\nu u(x) = -g(x)|u(x)|^{r-1}u(x) \text{ on } \Gamma_3. \tag{4}$$

The unknown of the problem is a function $u = u(x_1, x_2) : \overline{\Omega} \to \mathbb{R}$ that represents the third component of the displacement vector which in the antiplane model has the particular form $(0, 0, u(x_1, x_2))$. The function $\mu = \mu(x_1, x_2) : \overline{\Omega} \to \mathbb{R}$ is a coefficient of the material, the functions $f_0 = f_0(x_1, x_2) : \Omega \to \mathbb{R}$, $f_2 = f_2(x_1, x_2) : \Gamma_2 \to \mathbb{R}$ are related to the density of the body forces and the density of the surface traction, respectively and $g : \Gamma_3 \to \mathbb{R}_+$ is the coefficient of friction. The vector $\nu = \nu(x_1, x_2) = (\nu_1(x_1, x_2), \nu_2(x_1, x_2))$ represents the outward unit normal vector to the boundary and $\partial_\nu u = \nabla u \cdot \nu$. The boundary condition (4) on $\Gamma_3$ depends on the positive parameter $r$ and it is called in the literature *power-law friction*. Formally, Coulomb law with the friction coefficient $g = g(x)$ (also known as friction Tresca law) is obtained from (4) in the limit $r \to 0$. Hence, condition (4) can be seen as a regularization of the Tresca's law, see [10] (page 178). For this reason, our study will be mainly focused on the case $0 < r < 1$ but the case $r \geq 1$ will be also addressed. For details on antiplane contact models and their mathematical analysis we refer, e.g., to [10].

The nonlinear feature of our problem is given by relation (4). By setting $h(u) = |u|^{r-1}u$, (4) can be seen as a nonlinear Robin type boundary condition,

$$\mu\partial_\nu u + gh(u) = 0 \text{ on } \Gamma_3.$$

If we set $h(u) = \frac{u}{\sqrt{u^2+r^2}}$, then the corresponding nonlinear Robin type boundary condition expresses another regularization of Tresca's law, see [10] (page 174) or [9]. In [10, Section 9.1] the reader can find details related to the weak solvability and convergence results involving these two regularizations of Tresca's law. For other interesting works related to elliptic equations with nonlinear Robin type boundary condition in different contexts we refer to, e.g., [3–5,11].

Our aim is to study an optimal control problem which consists in minimizing the distance between $u$ and a given target $u_d$ by acting with a control force $f_2$ only on the part $\Gamma_2$ of the boundary. At the same time, we want to keep as small as possible the $L^2$–norm of the control $f_2$. Consequently, our objective will be to minimize the functional $J : L^2(\Gamma_2) \to \mathbb{R}$ defined by

$$J(f_2) = \frac{\alpha}{2}\|\nabla(u - u_d)\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|f_2\|^2_{L^2(\Gamma_2)}, \tag{5}$$

where $\alpha$ and $\beta$ are two positive real numbers and $u$ is the solution of (1)-(4). Thus, we are dealing with a control problem in which we want to keep the deformation of the body as close as possible to a reference target by acting on a small part of the boundary with a minimal cost.

To study this problem we consider an equivalent formulation consisting in minimizing the bifunctional

$$L(u, f_2) = \frac{\alpha}{2}\|\nabla(u - u_d)\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|f_2\|^2_{L^2(\Gamma_2)},$$

defined on the set $\mathcal{V}_{ad}$ of pairs $[u, f_2]$ verifying (1)-(4). A minimizer of $L$ will be called optimal pair and its second component optimal control. By using the direct method in the calculus of variations, we justify the existence of at least one optimal pair. Notice that we cannot use the classical convex minimization results because the admissible set $\mathcal{V}_{ad}$ is not convex. Studying the case $0 < r < 1$, we are able to write an optimality condition only for a regularized problem by means of a parameter $\rho$. Convergence results as $\rho$ tends to 0 are proved for the state problem as well as for the control problem. On the other hand, for the case $r \geq 1$, an optimality condition is written for the original problem, without any regularization. Then we focus on the computation of the optimal control. To this end in view, we linearize the problem and, by means of the saddle point theory, we characterize the corresponding optimal control. We go back to the original nonlinear problem by using a fixed point technique. In each of the cases $r \in (0, 1)$ and $r \geq 1$ we show that, under appropriate smallness assumptions on the data, it is possible to define a contraction map and to obtain an optimal pair from its unique fixed point. In the case $r \geq 1$ for the original problem and $r \in (0, 1)$ for the regularized one, this technique allows to prove the uniqueness of the optimal control, a result which is not so common in this non convex context.

Our results represent a contribution to the optimal control theory of contact problems; see also, e.g., [6,13,14] for other recent results on the existence of optimal controls for problems governed by variational and hemivariational inequalities.

In the last part of the article, we present some numerical experiments based on linearization and the fixed point method described above. For implementation reasons, a slightly modified optimal control problem is considered. As predicted by the theory, conclusive numerical results are obtained in a limited framework in which the problem data are small enough, in the case $r \geq 1$, or the regularizing parameter $\rho$ is sufficiently large, in the case $r \in (0, 1)$.

The paper has the following structure. In Section 2 we describe the functional setting and we introduce the state and the optimal control problems. Section 3 is devoted to the optimality conditions, treating separately the problem in the case $r \geq 1$ and the regularized version of it when $0 < r < 1$. In Section 4 we present convergence results of the regularized problem to the original one. In Section 5 we study the fixed point method approach to our optimal control problem and in Section 6 we present the numerical experiments. The relation between the original optimization problem and the modified one used in Section 6 is described in the final Appendix.

## 2. The optimal control problem

Firstly, let us describe the functional setting and fix the appropriate hypothesis allowing to study problem (1)-(4). We assume that

$$f_0 \in L^2(\Omega), \quad f_2 \in L^2(\Gamma_2), \tag{6}$$

$$\mu \in L^\infty(\Omega) \text{ and there exists } \mu^* > 0 \text{ such that } \mu(x) \geq \mu^* \text{ a.e. } x \in \Omega, \tag{7}$$

$$g \in L^\infty(\Gamma_3) \text{ and } g(x) \geq 0 \text{ a.e. } x \in \Gamma_3, \tag{8}$$

and we introduce the following Hilbert space

$$V = \{v \in H^1(\Omega) : \gamma v = 0 \text{ a.e. on } \Gamma_1\},$$

where $\gamma$ denotes the Sobolev trace operator. We consider the inner product on $V$, defined by

$$(\cdot,\cdot)_V : V \times V \to \mathbb{R}, \quad (u, v)_V = \int_\Omega \nabla u(x) \cdot \nabla v(x)\, \mathrm{d}x,$$

and denote by $\|\cdot\|_V$ the associated norm. We recall that, according to, e. g., [12, Theorem 2.21], for each $s \geq 1$, the trace operator $\gamma\colon H^1(\Omega) \to L^s(\Gamma)$ is linear, continuous and compact. So, there exists $c_0 = c_0(s) > 0$ such that

$$\|\gamma v\|_{L^s(\Gamma)} \leq c_0 \|v\|_V \qquad (v \in V). \tag{9}$$

In the sequel, although $c_0$ in (9) depends on the exponent $s$ of the space $L^s(\Gamma)$, to simplify the writing, we shall replace it by an absolute upper bound $c_0$ independent of $s$. This is justified by the fact that we apply this inequality only for a finite number of values $s$. Moreover, we recall the following Poincaré type inequality

$$\|v\|_{L^2(\Omega)} \leq c_P \|v\|_V \qquad (v \in V), \tag{10}$$

where $c_P$ is a positive constant depending on $\Omega$ and $\Gamma_1$.

Let $r > 0$. We define the operator $A$: $V \to V$ and the functional $j : V \to \mathbb{R}$ by the equalities

$$(Au, v)_V = \int_\Omega \mu(x) \nabla u(x) \cdot \nabla v(x)\, \mathrm{d}x \qquad (u, v \in V), \tag{11}$$

$$j(v) = \frac{1}{r+1} \int_{\Gamma_3} g(x) |\gamma v(x)|^{r+1}\, \mathrm{d}\Gamma \qquad (v \in V), \tag{12}$$

and note that by assumptions (7)-(8) and since $\Omega \subseteq \mathbb{R}^2$ the integrals in (11) and (12) are well defined. We consider the following *state problem* representing the weak formulation of (1)-(4):

$$\left.\begin{array}{c} \textit{Let } f_2 \in L^2(\Gamma_2) \textit{ (called control). Find } u \in V \textit{ such that} \\ (Au, v-u)_V + j(v) - j(u) \geq \displaystyle\int_\Omega f_0(x)(v(x) - u(x))\, \mathrm{d}x + \int_{\Gamma_2} f_2(x)(\gamma v(x) - \gamma u(x))\, \mathrm{d}\Gamma \qquad (v \in V). \end{array}\right\} \tag{SP}$$

According to Matei and Sofonea [10, Theorem 3.1], for every control $f_2 \in L^2(\Gamma_2)$, problem (SP) has a unique solution $u = u(f_2) \in V$ verifying

$$\|u\|_V \leq \frac{1}{\mu^*}\left(\|f_0\|_{L^2(\Omega)} + c_0 \|f_2\|_{L^2(\Gamma_2)}\right), \tag{13}$$

where $\mu^*$ is the constant in (7) and $c_0$ is the constant in (9).

Now, let us introduce the optimal control problem we want to study. Let $\alpha$, $\beta > 0$ be two positive constants and we define the following functional

$$L : V \times L^2(\Gamma_2) \to \mathbb{R}, \quad L(u, f_2) = \frac{\alpha}{2}\|u - u_d\|_V^2 + \frac{\beta}{2}\|f_2\|_{L^2(\Gamma_2)}^2. \tag{14}$$

Furthermore, we denote

$$\mathcal{V}_{\mathrm{ad}} = \left\{[u, f_2] \,\middle|\, [u, f_2] \in V \times L^2(\Gamma_2), \text{ such that (SP) is verified}\right\},$$

and we introduce the following *optimal control problem:*

$$\textit{Find } [u^*, f_2^*] \in \mathcal{V}_{\mathrm{ad}} \textit{ such that } L(u^*, f_2^*) = \min_{[u, f_2] \in \mathcal{V}_{\mathrm{ad}}} L(u, f_2). \qquad (OCP)$$

Notice that, problem (OCP) is equivalent to the minimization of the functional $J$ introduced in (5). Unlike $J$, the functional $L$ given by (14) is convex. However, the minimization domain $\mathcal{V}_{\mathrm{ad}}$ is not convex. By using the same type of arguments as in [9, Theorem 3.7] we can deduce the following result.

**Theorem 1.** *Let $r > 0$ and assume that (6), (7) and (8) hold. Then, (OCP) has at least one solution $[u^*, f_2^*]$.*

A solution of (OCP) will be called an *optimal pair*. The second component of the optimal pair is called an *optimal control*.

**Remark 1.** Under the previous hypothesis we cannot ensure the uniqueness of the optimal pair. In Sections 5.2 and 5.3 we shall give some uniqueness results under more restrictive conditions (see Theorems 7 and 9 below).

**Remark 2.** When the power $r$ in (1)-(4) tends to zero we obtain in the limit that the boundary condition (4) is replaced by the Tresca friction law

$$|\mu(x)\partial_\nu u(x)| \le g(x), \quad \mu(x)\partial_\nu u(x) = -g(x)\frac{u(x)}{|u(x)|} \text{ if } u(x) \neq 0 \text{ on } \Gamma_3. \tag{15}$$

Therefore, we can see (4) as a regularized version of (15). Finally, let us notice that a boundary control problem for a model with Tresca friction law has been studied in [9].

## 3. Optimality conditions

In this section we study the optimality conditions corresponding to our minimization problem (OCP). We shall analyze separately the cases $r \ge 1$ and $r \in (0, 1)$. In the former case, the regularity properties of the functional $j$ allows to deduce an optimality condition using a classical approach due to J.-L. Lions (see, for instance [8] and also [1, Lemma 3.11, p. 1127]). When $r \in (0, 1)$ we cannot use directly this argument and we need to introduce a regularized version of the problem depending on a small parameter $\rho$. In the sequel, $D^kG$ will denote the $k$−th derivative of the function $G$.

### 3.1. The case r ≥ 1

The following result gives the optimality condition in this case.

**Theorem 2.** *Let $r \ge 1$. Any optimal control $f_2^*$ of problem (OCP) verifies*

$$f_2^* = -\frac{1}{\beta}\gamma(\eta(f_2^*)), \tag{16}$$

*where, for each $f_2 \in L^2(\Gamma_2)$, $\eta(f_2)$ is the unique solution of equation*

$$\alpha(u - u_d, w)_V = \big(\eta(f_2), Aw + D^2 j(u)w\big)_V \qquad (w \in V), \tag{17}$$

*and $u = u(f_2)$ is the solution of (SP).*

**Proof.** Let us define $F$: $V \times L^2(\Gamma_2) \to V$,

$$F(u, f_2) = Au + D^1 j(u) - y(f_0) - y(f_2), \qquad \big(u \in V,\ f_2 \in L^2(\Gamma_2)\big), \tag{18}$$

where $y(f_0)$ and $y(f_2)$ are two elements from $V$ given by

$$(y(f_0), v)_V = \int_\Omega f_0(x)v(x)\,\mathrm{d}x \qquad (v \in V), \tag{19}$$

$$(y(f_2), v)_V = \int_{\Gamma_2} f_2(x)\gamma v(x)\,\mathrm{d}\Gamma \qquad (v \in V). \tag{20}$$

Notice that $F(u, f_2) = 0$ is equivalent to the fact that $u$ is a solution of (SP).

According to Lemma 3.11 from [1], we obtain that the derivative of the functional $J$ defined by (5) is given by

$$\big(D^1 J(f_2), \xi\big)_{L^2(\Gamma_2)} = (\partial_2 L(u(f_2), f_2), \xi)_{L^2(\Gamma_2)} - (\eta(f_2), \partial_2 F(u(f_2), f_2)\xi)_V \qquad \big(\xi \in L^2(\Gamma_2)\big), \tag{21}$$

if $\partial_1 F(u(f_2), f_2)$ is a homeomorphism in $V$ and $\eta(f_2) \in V$ verifies (17). Since $\partial_1 F(u, f_2)v = Av + D^2 j(u)v$, the homeomorphism property of $\partial_1 F(u(f_2), f_2)$ follows from the fact that, according to Lax Milgram's lemma, for each $h \in V$, there exists a unique $v^* \in V$ such that

$$(Av^*, w)_V + (D^2 j(u(f_2))v^*, w)_V = (h, w)_V \qquad (w \in V). \tag{22}$$

From (21) we deduce that

$$(D^1 J(f_2), \xi)_{L^2(\Gamma_2)} = \beta(f_2, \xi)_{L^2(\Gamma_2)} + (\eta(f_2), y(\xi))_V \qquad (\xi \in L^2(\Gamma_2)), \tag{23}$$

where $y(\xi)$ is defined by (20). Since $f_2^*$ is a minimizer of $J$, from (23) we obtain the following optimality condition

$$\beta(f_2^*, \xi)_{L^2(\Gamma_2)} + (\eta(f_2^*), y(\xi))_V = 0 \qquad (\xi \in L^2(\Gamma_2)), \tag{24}$$

where $\eta(f_2^*)$ is the unique solution of (17) with $f_2 = f_2^*$. By taking into account (20), relation (24) is equivalent to (16) which concludes the proof. □

**Remark 3.** It is easy to see that, for each $u \in V$, the second derivative of $j$ in $u$ is given by

$$\big(D^2 j(u)w, v\big)_V = r\int_{\Gamma_3} g(x)|\gamma u(x)|^{r-1}\gamma w(x)\gamma v(x)\,\mathrm{d}\Gamma \qquad (v, w \in V). \tag{25}$$

Notice that, if $r \in (0, 1)$, the right hand side expression in (25) is not well-defined. This indicates that $j$ given by (12) is not sufficiently smooth to derive it two times and no optimality condition can be obtained. In the next section we regularize the function $j$ and we deduce an optimality condition for the corresponding problem.

*3.2. The case* r $\in$ *(0, 1)*

Let us fix $\rho > 0$. We introduce the following regularized version of the state problem (SP):

$$\left.\begin{array}{c}\text{Let } f_2 \in L^2(\Gamma_2) \text{ (called control). Find } u_\rho \in V \text{ such that} \\ (Au_\rho, v - u_\rho)_V + j_\rho(v) - j_\rho(u_\rho) \geq \int_\Omega f_0(x)(v(x) - u_\rho(x))\,dx + \int_{\Gamma_2} f_2(x)(\gamma v(x) - \gamma u_\rho(x))\,d\Gamma \quad (v \in V),\end{array}\right\} \quad (\mathrm{SP}^\rho)$$

where the functional $j_\rho : V \to \mathbb{R}$ is defined as follows

$$j_\rho(v) = \frac{1}{r+1}\int_{\Gamma_3} g(x)\left(\sqrt{|\gamma v(x)|^{2r+2} + \rho^2} - \rho\right) d\Gamma \qquad (v \in V). \tag{26}$$

As in the case of problem (SP), we can deduce that, for every $f_2 \in L^2(\Gamma_2)$, the problem $(\mathrm{SP}^\rho)$ has a unique solution $u_\rho = u_\rho(f_2) \in V$, verifying (13).

By defining the new admissible set,

$$\mathcal{V}_{\mathrm{ad}}^\rho = \left\{[u, f_2] \,\middle|\, [u, f_2] \in V \times L^2(\Gamma_2), \text{ such that} (SP^\rho) \text{ is verified}\right\},$$

and by using the functional $L$ from (14), we introduce a new *optimal control problem*:

$$\mathit{Find}[u_\rho^*, f_{2,\rho}^*] \in \mathcal{V}_{\mathrm{ad}}^\rho \text{ such that } L(u_\rho^*, f_{2,\rho}^*) = \min_{[u, f_2] \in \mathcal{V}_{\mathrm{ad}}^\rho} L(u, f_2). \qquad (OCP^\rho)$$

As in the case of Theorem 1, the next result can be proved following [9, Theorem 3.7].

**Theorem 3.** *Assume that (6), (7) and (8) hold. Then, $(OCP^\rho)$ has at least one solution $[u_\rho^*, f_{2,\rho}^*]$.*

Moreover, we can give an optimality condition for problem $(\mathrm{OCP}^\rho)$.

**Theorem 4.** *Let $\rho > 0$ and $r \in (0, 1)$. Any optimal control $f_{2,\rho}^*$ of problem $(OCP^\rho)$ verifies*

$$f_{2,\rho}^* = -\frac{1}{\beta}\gamma\left(\eta(f_{2,\rho}^*)\right), \tag{27}$$

*where, for each $f_2 \in L^2(\Gamma_2)$, $\eta(f_2)$ is the unique solution of equation*

$$\alpha(u_\rho - u_d, w)_V = \left(\eta(f_2), Aw + D^2 j_\rho(u_\rho)w\right)_V \qquad (w \in V), \tag{28}$$

*and, for all $v \in V$*

$$\left(D^2 j_\rho(u_\rho)w, v\right)_V = \int_{\Gamma_3} g(x)\frac{|u_\rho(x)|^{2r}\left[r|u_\rho(x)|^{2r+2} + (2r+1)\rho^2\right]}{\left(|u_\rho(x)|^{2r+2} + \rho^2\right)^{3/2}} w(x)v(x)\,d\Gamma,$$

*$u_\rho = u_\rho(f_2)$ being the solution of $(SP^\rho)$.*

**Proof.** It is similar to the proof of Theorem 2 and we omit it. □

**Remark 4.** The replacement of the functional $j$ from (SP) by its regularized version $j_\rho$ in $(\mathrm{SP}^\rho)$ has enabled us to deduce the optimality condition (27)-(28) for the corresponding minimization problem $(\mathrm{OCP}^\rho)$. Indeed, as $j$ in Theorem 2, $j_\rho$ is a convex, lower semi-continuous and two times Gâteaux differentiable functional. The question that an optimality condition can be obtained for the initial minimization problem (OCP) in the case $r \in (0, 1)$ remains open.

Finally, let us mention the following result which will allow us to evaluate the accuracy of our computational results in Section 6 for a particular choice of the parameters of the problem.

**Lemma 1.** *Given $f_0$, $f_2 := f_2^{\mathrm{ex}}$ and $g$, let $u_{\mathrm{ex}}$ be the unique solution of $(SP^\rho)$. If we choose $u_d = u_{\mathrm{ex}}$ in definition (14) of the functional L, then the following estimates hold*

$$\left\|u_\rho^* - u_{\mathrm{ex}}\right\|_V \leq \sqrt{\frac{\beta}{\alpha}}\|f_2^{\mathrm{ex}}\|_{L^2(\Gamma_2)}, \tag{29}$$

*where $[u_\rho^*, f_{2,\rho}^*]$ is any optimal pair for $(OCP^\rho)$.*

**Proof.** We have that

$$L(u_{\mathrm{ex}}, f_2^{\mathrm{ex}}) = \frac{\alpha}{2}\|u_{\mathrm{ex}} - u_{\mathrm{ex}}\|_V^2 + \frac{\beta}{2}\|f_2^{\mathrm{ex}}\|_{L^2(\Gamma_2)}^2 = \frac{\beta}{2}\|f_2^{\mathrm{ex}}\|_{L^2(\Gamma_2)}^2. \tag{30}$$

Since $[u_\rho^*, f_{2,\rho}^*]$ is an optimal pair for (OCP$^\rho$), it follows that

$$L(u_\rho^*, f_{2,\rho}^*) = \frac{\alpha}{2}\left\|u_\rho^* - u_{\rm ex}\right\|_V^2 + \frac{\beta}{2}\left\|f_{2,\rho}^*\right\|_{L^2(\Gamma_2)}^2 \le L(u_{\rm ex}, f_2^{\rm ex}).$$

The last relation, combined with (30), implies that (29) holds. □

**Remark 5.** The previous lemma shows that $u_{\rm ex}$ represents a good approximation of the first component of the optimal pair $[u_\rho^*, f_{2,\rho}^*]$ in the $V-$norm, if the quantity $\sqrt{\frac{\beta}{\alpha}}$ is small enough.

## 4. Convergence properties

In the first part of this section we prove the convergence of the unique solution of problem (SP$^\rho$) to the solution of (SP).

**Proposition 1.** *Let $r \in (0, 1)$, $f_0 \in L^2(\Omega)$ and $f_2 \in L^2(\Gamma_2)$ be given and let $u$ be the corresponding solution of problem (SP). For each $\rho > 0$, let $u_\rho$ be the solution of problem (SP$^\rho$). Then $u_\rho \to u$ in $V$ as $\rho \to 0$.*

**Proof.** Since $g$ verifies (8), the following inequality holds for all $v \in V$,

$$\int_{\Gamma_3} g(x)\left(\rho + |\gamma v(x)|^{r+1} - \sqrt{|\gamma v(x)|^{2r+2} + \rho^2}\right) \mathrm{d}\Gamma \le \rho \int_{\Gamma_3} g(x)\,\mathrm{d}\Gamma. \tag{31}$$

On the other hand, by definition of the functionals $j_\rho$ and $j$ we obtain

$$|j_\rho(v) - j(v)| \le \frac{1}{r+1}\int_{\Gamma_3} g(x)\left|\sqrt{|\gamma v(x)|^{2r+2} + \rho^2} - \rho - |\gamma v(x)|^{r+1}\right| \mathrm{d}\Gamma.$$

From the last inequality and (31) we are led to

$$|j_\rho(v) - j(v)| \le \rho \int_{\Gamma_3} g(x)\,\mathrm{d}\Gamma \qquad (v \in V). \tag{32}$$

The conclusion follows from [10, Theorem 3.6]. □

Next, we prove that a solution of the minimization problem (OCP) can be obtained as limit of solutions $[u_\rho^*, f_{2,\rho}^*]$ to the regularized minimization problems (OCP$^\rho$) when $\rho$ tends to zero.

**Theorem 5.** *For each $\rho > 0$, let $[u_\rho^*, f_{2,\rho}^*]$ be a solution of problem (OCP$^\rho$). Then, there exists a subsequence of the family $\left([u_\rho^*, f_{2,\rho}^*]\right)_{\rho>0}$, denoted in the same way, and a solution $[u^*, f_2^*]$ of problem (OCP) such that*

$$u_\rho^* \to u^* \text{ in } V \text{ and } f_{2,\rho}^* \rightharpoonup f_2^* \text{ in } L^2(\Gamma_2) \text{ as } \rho \to 0. \tag{33}$$

**Proof.** Let $u_\rho^0$ be the unique solution of (SP$^\rho$) with $f_2 = 0$. From (13) we deduce that

$$L(u_\rho^*, f_{2,\rho}^*) \le L(u_\rho^0, 0) \le \alpha\left(\frac{\|f_0\|_{L^2(\Omega)}^2}{(\mu^*)^2} + \|u_d\|_V^2\right).$$

Thus, we deduce that $\left([u_\rho^*, f_{2,\rho}^*]\right)_{\rho>0}$ is a bounded sequence in $V \times L^2(\Gamma_2)$. Consequently, there exists $[u^*, f_2^*] \in V \times L^2(\Gamma_2)$ such that, passing eventually to a subsequence, but keeping the notation to simplify the writing, we have

$$u_\rho^* \rightharpoonup u^* \text{ in } V \text{ and } f_{2,\rho}^* \rightharpoonup f_2^* \text{ in } L^2(\Gamma_2) \text{ as } \rho \to 0.$$

In fact,

$$u_\rho^* \to u^* \text{ in } V \text{ as } \rho \to 0. \tag{34}$$

Indeed, since the operator $A$ is strongly monotone, by (SP$^\rho$), we have

$$\begin{aligned}\mu^*\left\|u_\rho^* - u^*\right\|_V^2 &\le (Au^*, u^* - u_\rho^*)_V + (Au_\rho^*, u_\rho^* - u^*)_V \\ &\le (Au^*, u^* - u_\rho^*)_V + j_\rho(u^*) - j_\rho(u_\rho^*) \\ &\quad - (f_0, u^* - u_\rho^*)_{L^2(\Omega)} - (f_{2,\rho}^*, \gamma u^* - \gamma u_\rho^*)_{L^2(\Gamma_2)}.\end{aligned}$$

Now, (34) follows immediately from the above inequality if we prove that

$$\lim_{\rho\to 0}(j_\rho(u^*) - j_\rho(u_\rho^*)) = 0. \tag{35}$$

To prove (35) we remark that

$$\left|j_\rho(u^*) - j_\rho(u_\rho^*)\right| \le \frac{\|g\|_\infty}{r+1}\int_{\Gamma_3}\left||\gamma u^*(x)|^{r+1} - \left|\gamma u_\rho^*(x)\right|^{r+1}\right| \mathrm{d}\Gamma.$$

From the above inequality, since $u^*_\rho \rightharpoonup u^*$ as $\rho$ tends to zero, we deduce from [12, Theorem 2.21] (with $s = r + 1$) that (35) holds true.

On the other hand we have that $[u^*, f^*_2] \in \mathcal{V}_{\rm ad}$. Indeed, since $[u^*_\rho, f^*_{2,\rho}] \in \mathcal{V}^\rho_{\rm ad}$, $u^*_\rho \to u^*$ in $V$ and $f^*_{2,\rho} \rightharpoonup f^*_2$ in $L^2(\Gamma_2)$ as $\rho$ tends to zero, by passing to the limit in (SP$^\rho$) we deduce that $[u^*, f^*_2]$ verifies (SP).

Let $[\widehat{u}, \widehat{f_2}] \in \mathcal{V}_{\rm ad}$ be a solution of (OCP). For each $\rho > 0$, let $u_\rho$ be the unique solution of the problem (SP$^\rho$) with $f_2 = \widehat{f_2}$. Obviously, $[u_\rho, \widehat{f_2}] \in \mathcal{V}^\rho_{\rm ad}$ for each $\rho > 0$. From Proposition 1 we deduce that the sequence $(u_\rho)_{\rho > 0}$ converges to $\widehat{u}$ in $V$ as $\rho \to 0$. Since the functional $L$ is convex and continuous, we have

$$L(u^*, f^*_2) \le \liminf_{\rho\to 0} L\big(u^*_\rho, f^*_{2,\rho}\big). \tag{36}$$

Moreover, since $[u^*_\rho, f^*_{2,\rho}]$ is a solution of (OCP$^\rho$), it follows that

$$\limsup_{\rho\to 0} L\big(u^*_\rho, f^*_{2,\rho}\big) \le \limsup_{\rho\to 0} L\Big(u_\rho, \hat{f}_2\Big) = L\Big(\hat{u}, \hat{f}_2\Big), \tag{37}$$

and, consequently,

$$L(u^*, f^*_2) \le L(\widehat{u}, \widehat{f_2}). \tag{38}$$

On the other hand, since $[\widehat{u}, \widehat{f_2}]$ is a solution of (OCP), we have that

$$L(\widehat{u}, \widehat{f_2}) \le L(u^*, f^*_2). \tag{39}$$

Thus, from (38)-(39) we deduce that $L(\widehat{u}, \widehat{f_2}) = L(u^*, f^*_2)$, and the proof of the theorem is complete. $\square$

**Remark 6.** Theorem 5 shows that the regularized problem (OCP$^\rho$), for which we dispose of the optimality condition (27)-(28), may be used to approximate a solution of (OCP).

## 5. A fixed point method

### 5.1. An auxiliary linear problem

Given $z \in L^2(\Gamma_3)$, we consider the linear problem

$$\begin{cases} \mathrm{div}(\mu(x)\nabla u(x)) + f_0(x) = 0 & (x \in \Omega) \\ u(x) = 0 & (x \in \Gamma_1) \\ \mu(x)\partial_\nu u(x) = f_2(x) & (x \in \Gamma_2) \\ \mu(x)\partial_\nu u(x) = z(x) & (x \in \Gamma_3). \end{cases} \tag{40}$$

It is easy to see that for every $f_0 \in L^2(\Omega)$ and $f_2 \in L^2(\Gamma_2)$ problem (40) has a unique weak solution $u \in V$. For every $z \in L^2(\Gamma_3)$, we consider the following optimal control problem:

$$\textit{Find } [u^*, f^*_2] \in \vartheta_{\rm ad} \textit{ such that } L(u^*, f^*_2) = \inf_{[u, f_2]\in\vartheta_{\rm ad}} L(u, f_2), \tag{41}$$

where $L$ is given by (14) and the admissible set $\vartheta_{\rm ad}$ is defined as follows

$$\vartheta_{\rm ad} = \Big\{[u, f_2] \in V \times L^2(\Gamma_2) \text{ such that (40) is verified in a weak sense}\Big\}.$$

Notice that (40) is similar to (1)-(4), with the nonlinear boundary condition on $\Gamma_3$ replaced by a simpler, non homogeneous one. Moreover, unlike (OCP), the minimization problem (41) is convex and much easier to study.

Since (40) is a linear problem, we can decompose its solutions as a sum of two functions $\overline{v}$ and $w$, the solutions of (40) with $f_2 = 0$ and (40) with $z = f_0 = 0$, respectively. If $w_d = \overline{v} - u_d$, the optimal control problem (41) becomes equivalent to the following one:

$$\textit{Find } [w^*, f^*_2] \in \mathcal{W}_{\rm ad} \textit{ such that } L_{\mathcal{W}}(w^*, f^*_2) = \inf_{[w, f_2]\in\mathcal{W}_{\rm ad}} L_{\mathcal{W}}(w, f_2), \qquad (OCP^{\rm lin})$$

where $\mathcal{W}_{\rm ad}$ is the linear space of all pairs $[w, f_2] \in V \times L^2(\Gamma_2)$ verifying the variational formulation of (40) with $z = f_0 = 0$ and $L_{\mathcal{W}}$ is defined by

$$L_{\mathcal{W}}(w, f_2) = \frac{\alpha}{2}\|w + w_d\|^2_V + \frac{\beta}{2}\|f_2\|^2_{L^2(\Gamma_2)} \qquad ([w, f_2] \in \mathcal{W}_{\rm ad}). \tag{42}$$

Notice that $L_{\mathcal{W}}$ is a strictly convex functional in $\mathcal{W}_{\rm ad}$. By using the saddle point theory we can give an optimality condition for the minimization problem (OCP$^{\rm lin}$).

**Proposition 2.** *There exists a unique solution $[w^*, f^*_2, \lambda^*] \in V \times L^2(\Gamma_2) \times V$ of the following variational problem*

$$\begin{cases} a\Big([w^*, f^*_2], [\widetilde{w}, \widetilde{f_2}]\Big) + b\Big([\widetilde{w}, \widetilde{f_2}], \lambda^*\Big) & = \quad \ell\Big([\widetilde{w}, \widetilde{f_2}]\Big) \\ b\Big([w^*, f^*_2], \widetilde{\lambda}\Big) & = \quad 0, \end{cases} \tag{43}$$

*for any* $[\widetilde{w}, \widetilde{f_2}] \in V \times L^2(\Gamma_2)$ *and* $\widetilde{\lambda} \in V$, *where*

$$a : \big[V \times L^2(\Gamma_2)\big] \times \big[V \times L^2(\Gamma_2)\big] \to \mathbb{R},\ \ a\Big([w, f_2], \big[\widetilde{w}, \widetilde{f_2}\big]\Big) = \alpha(w, \widetilde{w})_V + \beta\big(f_2, \widetilde{f_2}\big)_{L^2(\Gamma_2)}, \tag{44}$$

$$b : \big[V \times L^2(\Gamma_2)\big] \times V \to \mathbb{R},\ \ b\big([w, f_2], \tilde{\lambda}\big) = \big(f_2, \gamma\tilde{\lambda}\big)_{L^2(\Gamma_2)} - \big(Aw, \tilde{\lambda}\big)_V, \tag{45}$$

$$\ell : V \times L^2(\Gamma_2) \to \mathbb{R}, \quad \ell([w, f_2]) = -\alpha(w_d, w)_V. \tag{46}$$

*Moreover,* $[w^*, f_2^*]$ *is the unique solution of the constrained minimization problem* ($OCP^{lin}$).

**Proof.** Firstly, we show that the variational problem (43) has a unique solution. We remark that the bilinear form $a$ is continuous and coercive, the bilinear form $b$ is continuous and the linear functional $\ell$ is continuous. Moreover, the following *inf-sup* property is verified for some constant $\varrho > 0$:

$$\inf_{\lambda \in V} \sup_{[w, f_2] \in V \times L^2(\Gamma_2)} \frac{b([w, f_2], \lambda)}{\|[w, f_2]\|_{V \times L^2(\Gamma_2)} \|\lambda\|_V} \ge \varrho. \tag{47}$$

Indeed, to prove (47), for each $\lambda \in V$, let $w_\lambda \in V$ be the unique solution of the variational problem

$$b([w_\lambda, \gamma\lambda], \varphi) = (\lambda, \varphi)_V \qquad (\varphi \in V). \tag{48}$$

From (48) and (45) we deduce that $b([w_\lambda, \gamma\lambda], \lambda) = \|\lambda\|_V^2$ and $\|w_\lambda\|_V \le \frac{c_0^2+1}{\mu^*}\|\lambda\|_V$, where $\mu^*$ and $c_0$ are the constants in (7) and (9), respectively. The last two relations imply that, for each $\lambda \in V$, we have

$$\frac{b([w_\lambda, \gamma\lambda], \lambda)}{\|[w_\lambda, \gamma\lambda]\|_{V \times L^2(\Gamma_2)} \|\lambda\|_V} \ge \frac{\mu^*}{\sqrt{(c_0^2 + 1)^2 + (\mu^* c_0)^2}}.$$

The last inequality implies that (47) is verified with $\varrho = \frac{\mu^*}{\sqrt{(c_0^2+1)^2+(\mu^* c_0)^2}}$.

We conclude that the variational problem (43) has a unique solution $[w^*, f_2^*, \lambda^*] \in V \times L^2(\Gamma_2) \times V$ (see, for instance, [2, Theorem 4.2.3]).

Notice that, the second relation in (43) implies that $[w^*, f_2^*] \in \mathcal{W}_{ad}$. Moreover, the first relation in (43) ensures that $[w^*, f_2^*]$ is a critical point of the functional $L_{\mathcal{W}}(\cdot, \cdot) + b([\cdot, \cdot], \lambda^*)$ defined in $V \times L^2(\Gamma_2)$. Since this functional is convex, it follows $[w^*, f_2^*]$ is a minimizer. Hence, $[w^*, f_2^*]$ is a solution of the constrained minimization problem ($OCP^{lin}$). The strict convexity of $L_{\mathcal{W}}$ yields the uniqueness of the minimizer and the proof is complete. □

**Remark 7.** The solution $[w^*, f_2^*, \lambda^*] \in V \times L^2(\Gamma_2) \times V$ of (43) is in fact a saddle point for the functional $L_{\mathcal{W}} + b$.

Now we define the operator $\mathcal{C} : V \times L^2(\Omega) \times L^2(\Gamma_3) \to L^2(\Gamma_2)$ as follows

$$\mathcal{C}(u_d, f_0, z) = f_2^*, \tag{49}$$

where $f_2^*$ is given by Proposition 2.

The following result gives some useful estimates for the operator $\mathcal{C}$.

**Proposition 3.** *If* $\mathcal{C}$ *is the operator defined by (49), then:*

- *There exists* $M_1 > 0$ *such that, for any* $z \in L^2(\Gamma_3)$:
$$\|\mathcal{C}(u_d, f_0, z)\|_{L^2(\Gamma_2)} \le M_1\big(\|u_d\|_V + \|f_0\|_{L^2(\Omega)} + \|z\|_{L^2(\Gamma_3)}\big). \tag{50}$$
- *There exists* $M_2 > 0$ *such that, for any* $z_1, z_2 \in L^2(\Gamma_3)$:
$$\|\mathcal{C}(u_d, f_0, z_1) - \mathcal{C}(u_d, f_0, z_2)\|_{L^2(\Gamma_2)} \le M_2\|z_1 - z_2\|_{L^2(\Gamma_3)}. \tag{51}$$
- *There exists* $M_3 > 0$ *such that, for any* $z_1, z_2 \in L^2(\Gamma_3)$:
$$\|u_1 - u_2\|_V \le M_3\|z_1 - z_2\|_{L^2(\Gamma_3)}, \tag{52}$$
*where* $u_i$ *is the weak solution of the linear elliptic problem (40) with* $z = z_i$ *and* $f_2 = \mathcal{C}(u_d, f_0, z_i)$, $i \in \{1, 2\}$.

**Proof.** Since $[w^*, f_2^*]$ is the minimizer of ($OCP^{lin}$), we have that

$$\frac{\beta}{2}\|f_2^*\|_{L^2(\Gamma_2)}^2 \le L_{\mathcal{W}}(w^*, f_2^*) \le L_{\mathcal{W}}(0, 0) = \frac{\alpha}{2}\|w_d\|_V^2,$$

from which we deduce that

$$\|f_2^*\|_{L^2(\Gamma_2)} \le \sqrt{\frac{\alpha}{\beta}}(\|\overline{v}\|_V + \|u_d\|_V). \tag{53}$$

Since $\overline{v}$ verifies (40) with $f_2 = 0$, we deduce that

$$\|\overline{v}\|_V \leq \frac{\max\{c_0,\, c_P\}}{\mu^*}\big(\|f_0\|_{L^2(\Omega)} + \|z\|_{L^2(\Gamma_3)}\big), \tag{54}$$

where $c_0$ and $c_P$ are the constants from (9) and (10), respectively. From (53)-(54) we deduce that (50) holds with $M_1 = \sqrt{\frac{\alpha}{\beta}}\max\left\{\frac{\max\{c_0,c_P\}}{\mu^*},\, 1\right\}$.

Let us now pass to prove (51). Let $\mathcal{C}(u_d, f_0, z_i) = f^*_{2,i}$, $i = 1, 2$. From the optimality condition in (43), we deduce that

$$a\big([\theta^*, \tau^*], [\widetilde{w}, \widetilde{f_2}]\big) + b\big([\widetilde{w}, \widetilde{f_2}], \lambda_1^* - \lambda_2^*\big) = -\alpha(\overline{v}(z_1) - \overline{v}(z_2), \widetilde{w})_V,$$

where $\theta^* = w_1^* - w_2^*$, $\tau^* = f^*_{2,1} - f^*_{2,2}$ and $\overline{v}(z)$ is the weak solution of (40) with $f_2 = 0$.

Now, if we chose $[\widetilde{w}, \widetilde{f_2}] = [\theta^*, \tau^*]$ and we take into account the second relation in (43), we obtain that

$$a([\theta^*, \tau^*], [\theta^*, \tau^*]) = -\alpha(\overline{v}(z_1) - \overline{v}(z_2), \theta^*)_V. \tag{55}$$

Relation (55), combined with the definition of $a$, the fact that $\overline{v}(z_1) - \overline{v}(z_2)$ verifies (40) with $f_2 = f_0 = 0$ and estimate (54), implies that

$$\|\theta^*\|_V \leq \frac{\max\{c_0,\, c_P\}}{\mu^*}\|z_1 - z_2\|_{L^2(\Gamma_3)}, \tag{56}$$

and

$$\beta\|\tau^*\|^2_{L^2(\Gamma_2)} \leq \alpha\left(\frac{\max\{c_0,\, c_P\}}{\mu^*}\right)^2\|z_1 - z_2\|^2_{L^2(\Gamma_3)}, \tag{57}$$

The last relation gives us precisely (51) with $M_2 = \sqrt{\frac{\alpha}{\beta}}\frac{\max\{c_0,c_P\}}{\mu^*}$.

To prove (52) we remark that

$$\int_{\Gamma_2}(\mathcal{C}(u_d, f_0, z_1)(x) - \mathcal{C}(u_d, f_0, z_2)(x))(u_1 - u_2)(x)\,\mathrm{d}\Gamma$$
$$+ \int_{\Gamma_3}(z_1(x) - z_2(x))(u_1 - u_2)(x)\,\mathrm{d}\Gamma - \int_\Omega \mu(x)|\nabla(u_1 - u_2)(x)|^2\,\mathrm{d}x = 0.$$

Consequently, we obtain that

$$\|u_1 - u_2\|_V \leq \frac{c_0}{\mu^*}\big(\|z_1 - z_2\|_{L^2(\Gamma_3)} + \|\mathcal{C}(u_d, f_0, z_1) - \mathcal{C}(u_d, f_0, z_2)\|_{L^2(\Gamma_2)}\big).$$

From the last estimation, combined with (51), we infer that (52) holds with $M_3 = \frac{c_0}{\mu^*}(1 + M_2)$ and the proof of the proposition is complete. □

Finally, we have the following lemma which will be used in the next two sections.

**Lemma 2.** *If $\rho > 0$ and $r \in [0, 1]$ then, for any $a, b \in \mathbb{R}$,*

$$\left|\frac{|a|^{2r}a}{\sqrt{|a|^{2r+2} + \rho^2}} - \frac{|b|^{2r}b}{\sqrt{|b|^{2r+2} + \rho^2}}\right| \leq (3r + 1)\left(\frac{1}{\rho}\right)^{\frac{1-r}{1+r}}|a - b|. \tag{58}$$

*On the other hand, if $r > 1$ then, for any $a, b \in \mathbb{R}$,*

$$\left||a|^{r-1}a - |b|^{r-1}b\right| \leq r|a - b|(|a| + |b|)^{r-1}. \tag{59}$$

**Proof.** For inequality (58), we use Mean Value Theorem applied to the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) = |t|^{2r}t/\sqrt{|t|^{2r+2} + \rho^2}$, and take into account that $\sup_{t\in\mathbb{R}}|h'(t)| \leq (3r + 1)\rho^{\frac{r-1}{1+r}}$. Concerning (59), the same idea may be used by considering $h(t) = |t|^{r-1}t$. □

*5.2. The case* r $\in$ *(0, 1)*

This section is devoted to present and to study a fixed point method for the approximation of the optimal controls of problem (OCP$^\rho$). Let $r \in (0, 1)$ and $\rho > 0$ be two real numbers. Now, given $g \in L^\infty(\Gamma_3)$, we define the operator $\mathcal{N} : L^2(\Gamma_3) \to L^2(\Gamma_3)$ by

$$\mathcal{N}(z)(x) = -g(x)\frac{|\gamma(u)(x)|^{2r}\gamma(u)(x)}{\sqrt{|\gamma(u)(x)|^{2r+2} + \rho^2}} \qquad (z \in L^2(\Gamma_3),\ x \in \Gamma_3), \tag{60}$$

where $u$ is the solution of (40) associated to $z$ and $f_2 = \mathcal{C}(u_d, f_0, z)$. Clearly, $\mathcal{N}$ is well defined.

**Remark 8.** If $z$ is a fixed point of the operator $\mathcal{N}$, then $u$ is a solution of the nonlinear problem (SP$^\rho$) and $f_2^* = \mathcal{C}(u_d, f_0, z)$ gives an optimal control for the nonlinear minimization problem (OCP$^\rho$).

The remaining part of this section is devoted to present some sufficient conditions for the existence of a fixed point of $\mathcal{N}$. The most important properties of the map $\mathcal{N}$ are the following.

**Theorem 6.** *Let $\rho > 0$ be given. There exists a positive constant $\delta$ such that the operator $\mathcal{N}$ defined by (60) is a contraction in $L^2(\Gamma_3)$ for every $g \in L^\infty(\Gamma_3)$ satisfying*

$$\left(\frac{1}{\rho}\right)^{\frac{1-r}{1+r}} \|g\|_{L^\infty(\Gamma_3)} < \delta. \tag{61}$$

*Thus $\mathcal{N}$ has a unique fixed point $z_\rho^* \in L^2(\Gamma_3)$. Furthermore, $[u_\rho^*, f_{2,\rho}^*]$ is a solution of (OCP$^\rho$), where $f_{2,\rho}^* = \mathcal{C}(u_d, f_0, z_\rho^*)$ and $u_\rho^*$ is the solution of (SP$^\rho$) with $f_2 = f_{2,\rho}^*$.*

**Proof.** For each $z_1, z_2 \in L^2(\Gamma_3)$ the following estimate holds

$$\|\mathcal{N}(z_1) - \mathcal{N}(z_2)\|^2_{L^2(\Gamma_3)} \le \|g\|^2_{L^\infty(\Gamma_3)} \int_{\Gamma_3} \left| \frac{|\gamma(u_1)(x)|^{2r}\gamma(u_1)(x)}{\sqrt{|\gamma(u_1)(x)|^{2r+2} + \rho^2}} - \frac{|\gamma(u_2)(x)|^{2r}\gamma(u_2)(x)}{\sqrt{|\gamma(u_2)(x)|^{2r+2} + \rho^2}} \right|^2 \mathrm{d}\Gamma,$$

where $u_i$ is the weak solution of the linear elliptic problem (40) with $z = z_i$ and $f_2 = \mathcal{C}(u_d, f_0, z_i)$, $i \in \{1, 2\}$. By applying (58) from Lemma 2, we deduce that

$$\|\mathcal{N}(z_1) - \mathcal{N}(z_2)\|_{L^2(\Gamma_3)} \le \frac{3r+1}{\rho^{\frac{1-r}{1+r}}} \|g\|_{L^\infty(\Gamma_3)} \|\gamma(u_1) - \gamma(u_2)\|_{L^2(\Gamma_3)}. \tag{62}$$

From the trace theorem and (52) we get

$$\|\mathcal{N}(z_1) - \mathcal{N}(z_2)\|_{L^2(\Gamma_3)} \le c_0 M_3 \frac{3r+1}{\rho^{\frac{1-r}{1+r}}} \|g\|_{L^\infty(\Gamma_3)} \|z_1 - z_2\|_{L^2(\Gamma_3)}. \tag{63}$$

Thus, for $\left(\frac{1}{\rho}\right)^{\frac{1-r}{1+r}} \|g\|_{L^\infty(\Gamma_3)}$ small enough, $\mathcal{N}$ is a contraction on $L^2(\Gamma_3)$.

The last part of the theorem follows from the definition of the operator $\mathcal{C}$ in (49) and Remark 8. □

In Section 6 we shall approximate the control $\mathcal{C}(u_d, f_0, z_\rho^*)$ by using the classical fixed point iteration method. Let us finish this section by proving that, in the hypothesis of Theorem 6, the optimal control is unique.

**Theorem 7.** *Suppose that condition (61) holds. Then the optimization problem (OCP$^\rho$) has exactly one optimal pair $[u_\rho^*, f_{2,\rho}^*]$.*

**Proof.** Let $[\widetilde{u}_\rho^*, \widetilde{f}_{2,\rho}^*]$ and $[u_\rho^*, f_{2,\rho}^*]$ be two optimal pairs. Since $[\widetilde{u}_\rho^*, \widetilde{f}_{2,\rho}^*]$ and $[u_\rho^*, f_{2,\rho}^*] \in \mathcal{V}_{\mathrm{ad}}^\rho$ it follows that if $f_{2,\rho}^* = \widetilde{f}_{2,\rho}^*$ then $u_\rho^* = \widetilde{u}_\rho^*$.

Suppose that $f_{2,\rho}^* \neq \widetilde{f}_{2,\rho}^*$. Let $z_\rho^*$ and $\widetilde{z}_\rho^*$ be given by

$$z_\rho^* = -\frac{g\left|\gamma(u_\rho^*)\right|^{2r}\gamma(u_\rho^*)}{\sqrt{\left|\gamma(u_\rho^*)\right|^{2r+2} + \rho^2}}, \quad \widetilde{z}_\rho^* = -\frac{g\left|\gamma(\widetilde{u}_\rho^*)\right|^{2r}\gamma(\widetilde{u}_\rho^*)}{\sqrt{\left|\gamma(\widetilde{u}_\rho^*)\right|^{2r+2} + \rho^2}}.$$

We recall that $f_{2,\rho}^* = \mathcal{C}(u_d, f_0, z_\rho^*)$ and $\widetilde{f}_{2,\rho}^* = \mathcal{C}(u_d, f_0, \widetilde{z}_\rho^*)$. Arguing by contradiction and taking into account (51), we conclude that $z_\rho^* \neq \widetilde{z}_\rho^*$.

On the other hand, from definition (60) of the operator $\mathcal{N}$, we have that $z_\rho^*$ and $\widetilde{z}_\rho^*$ are both fixed points of $\mathcal{N}$. Therefore, $\mathcal{N}$ has two different fixed points which is a contradiction. Hence $f_{2,\rho}^* = \widetilde{f}_{2,\rho}^*$ and the proof is complete. □

### *5.3. The case* $r \geq 1$

In this case we can use a similar fixed point method to study the existence of the optimal controls for problem (OCP). Let $B(0, R)$ denote the ball in $L^2(\Gamma_3)$ centered in 0 and of radius $R$. Given $g \in L^\infty(\Gamma_3)$, we define the operator $\mathcal{N} : B(0, R) \to B(0, R)$ by

$$\mathcal{N}(z)(x) = -g(x)|\gamma(u)(x)|^{r-1}\gamma(u)(x) \qquad (z \in B(0, R), x \in \Gamma_3), \tag{64}$$

where $u$ is the solution of (40) with $f_2 = \mathcal{C}(u_d, f_0, z)$ given by (49).

The following result gives the most important properties of the map $\mathcal{N}$.

**Theorem 8.** *Let $R > 0$. There exists $\delta = \delta(R) > 0$ such that $\mathcal{N}$ is a contraction on $B(0, R)$ if*

$$\|g\|_{L^\infty(\Gamma_3)} < \delta. \tag{65}$$

*Thus $\mathcal{N}$ has a unique fixed point $z^* \in B(0, R)$. Furthermore, $[u^*, f_2^*]$ is a solution of (OCP), where $f_2^* = \mathcal{C}(u_d, f_0, z^*)$ and $u^*$ is the solution of (SP) with $f_2 = f_2^*$.*

**Proof.** Since the case $r = 1$ is obvious, we consider that $r > 1$. Firstly, we show that $\mathcal{N}$ is well defined on $B(0, R)$. From (50), it is easy to prove that

$$\|u\|_V \le \frac{\max\{c_0,\ c_P\}}{\mu^*}(M_1+1)\big(\|f_0\|_{L^2(\Omega)} + \|u_d\|_V + \|z\|_{L^2(\Gamma_3)}\big). \tag{66}$$

Next, by using (9), we have that

$$\|\mathcal{N}(z)\|^2_{L^2(\Gamma_3)} = \int_{\Gamma_3} |g(x)|^2 |\gamma(u)(x)|^{2r}\,\mathrm{d}\Gamma \le \|g\|^2_{L^\infty(\Gamma_3)} \|\gamma(u)\|^{2r}_{L^{2r}(\Gamma_3)} \le c_0^{2r}\|g\|^2_{L^\infty(\Gamma_3)}\|u\|_V^{2r}.$$

Hence, $\mathcal{N}(B(0,R)) \subset B(0,R)$ if

$$\|g\|_{L^\infty(\Gamma_3)} < \frac{R}{\Big(c_0\frac{\max\{c_0,\,c_P\}}{\mu^*}(M_1+1)\Big)^r\Big(\|f_0\|_{L^2(\Omega)} + \|u_d\|_V + R\Big)^r}. \tag{67}$$

Now, we show that $\mathcal{N}$ is a contraction on $B(0, R)$. Let $z_1, z_2 \in B(0, R)$. From (59) in Lemma 2 we deduce that

$$\|\mathcal{N}(z_1) - \mathcal{N}(z_2)\|^2_{L^2(\Gamma_3)} \le r^2\|g\|^2_{L^\infty(\Gamma_3)} \int_{\Gamma_3} |\gamma(u_1)(x) - \gamma(u_2)(x)|^2 (|\gamma(u_1)(x)| + |\gamma(u_2)(x)|)^{2(r-1)}\,\mathrm{d}\Gamma$$
$$\le r^2\|g\|^2_{L^\infty(\Gamma_3)}\|\gamma(u_1) - \gamma(u_2)\|^2_{L^p}\Big(\|\gamma(u_1)\|_{L^{\frac{2(r-1)p}{p-2}}} + \|\gamma(u_2)\|_{L^{\frac{2(r-1)p}{p-2}}}\Big)^{2(r-1)},$$

where $p = \frac{2}{3-2r}$ if $r \in (1, \frac{3}{2})$ and $p = 3$ if $r \ge \frac{3}{2}$. From the last estimate, (9), (52) and (66) we obtain that

$$\|\mathcal{N}(z_1) - \mathcal{N}(z_2)\|^2_{L^2(\Gamma_3)} \le r^2 c_0^{2r}\|g\|^2_{L^\infty(\Gamma_3)}\|u_1 - u_2\|_V^2(\|u_1\|_V + \|u_2\|_V)^{2(r-1)}$$
$$\le r^2 c_0^{2r} M_3^2\left(2\frac{\max\{c_0,\ c_P\}}{\mu^*}(M_1+1)\right)^{2(r-1)}\|g\|^2_{L^\infty(\Gamma_3)}\|z_1 - z_2\|^2_{L^2(\Gamma_3)}$$
$$\big(\|z_1\|_{L^2(\Gamma_3)} + \|z_2\|_{L^2(\Gamma_3)} + \|u_d\|_V + \|f_0\|_{L^2(\Omega)}\big)^{2(r-1)}.$$

Thus, if $g$ verifies (65) and $\delta$ is sufficiently small, the application $\mathcal{N}$ is a contraction on $B(0, R)$. The last part of the theorem follows from the definition of the operator $\mathcal{C}$ in (49) and the fact that, if $z$ is a fixed point of the operator $\mathcal{N}$, then $u$ is a solution of the nonlinear problem (SP). □

**Remark 9.** In the case $r > 1$ the arguments used in the proof of Theorem 8 show that $\mathcal{N}$ is a contraction on $B(0, R)$ even if (65) is not verified, by assuming that $R$, $\|u_d\|_V$ and $\|f_0\|_{L^2(\Omega)}$ are small enough. Indeed, given any $g \in L^\infty(\Gamma_3)$, we have that, for sufficiently small $R$, $\|u_d\|_V$ and $\|f_0\|_{L^2(\Omega)}$, both relation (67) and the contraction property of $\mathcal{N}$ are verified.

As in Theorem 7 for the case $r \in (0, 1)$, we can show that problem (OCP) has a unique solution under appropriate hypothesis.

**Theorem 9.** *Under the hypothesis of Theorem 8, the optimization problem (OCP) with $r \ge 1$ has exactly one optimal pair $[u^*, f_2^*]$.*

## 6. Numerical results

In this section we use the characterization of the optimal pair in Theorems 6 and 8 to propose a numerical approximation scheme. Indeed, under appropriate conditions (61) and (65), respectively, given $z^0 \in L^2(\Gamma_3)$, the sequence of successive approximations $(z_n)_{n\ge 0}$ defined by $z^{n+1} = \mathcal{N}(z^n)$ for every $n \ge 0$, converges in $L^2(\Gamma_3)$ to the fixed point $z^*$ of $\mathcal{N}$ which gives the optimal pair. We propose the following algorithm:

At each step of the Algorithm 1 , an optimal control $\mathcal{C}(u_d, f_0, z^n)$ for (OCP$^{\text{lin}}$) is computed by (43). From a numerical point of view it is convenient to consider a minimization problem with respect to a pair $[w, f_2] \in V \times H^1(\Omega)$. This permits to avoid the simultaneous manipulation of distributed and boundary terms. More precisely, for a given parameter $\zeta > 0$ we solve, instead of (OCP$^{\text{lin}}$), the following minimization problem:

$$\textit{Find } [w_\zeta^*, f_{2,\zeta}^*] \in \mathcal{W}_{\text{ad}}^\zeta \textit{ such that } L^\zeta(w_\zeta^*, f_{2,\zeta}^*) = \inf_{[w,f_2]\in\mathcal{W}_{\text{ad}}^\zeta} L^\zeta(w, f_2) \qquad (OCP_\zeta^{\text{lin}})$$

with $\mathcal{W}_{\text{ad}}^\zeta = \big\{[w, f_2] \in V \times H^1(\Omega) \text{ such that } [w, \gamma f_2] \in \mathcal{W}_{\text{ad}}\big\}$ and

$$L^\zeta(w, f_2) = \frac{\alpha}{2}\|w + w_d\|_V^2 + \frac{\beta}{2}\|\gamma f_2\|^2_{L^2(\Gamma_2)} + \frac{\zeta}{2}\|f_2\|^2_{H^1(\Omega)}.$$

**let** $u_d \in V,\ f_0 \in L^2(\Omega),\ g \in L^\infty(\Gamma_3),\ \epsilon > 0, n_{\max} \in \mathbb{N}^*$;
  **let** $z^0 \in L^2(\Gamma_3)$;
  **let** $f_2^0 = \mathcal{C}(u_d, f_0, z^0)$;
  **let** $n = 0$;
  **do**
    **let** $z^{n+1} = \mathcal{N}(z^n)$;
    **let** $n = n + 1$;
    **let** $f_2^n = \mathcal{C}(u_d, f_0, z^n)$;
    **let** $E^n = \|z^{n+1} - z^n\|_{L^2(\Gamma_3)}$;
  **until** $(E^n \le \epsilon)$ **or** $(n > n_{\max})$.
  **compute** $\overline{v}^n$ solution of~(**??**) with $f_2 \equiv 0$ and $z = z^n$;
  **compute** $w^n$ solution of~(**??**) with $f_2 \equiv f_2^n$, $z \equiv 0$ and $f_0 \equiv 0$;
  **let** $u_h^* = \overline{v}^n + w^n$;
  **let** $f_{2,h}^* = f_2^n$.

**Algorithm 1:** A fixed point algorithm to compute the optimal control.

In the sequel, in order to simplify the notation, the subscript $\zeta$ will be omitted in the notation of the optimal pair. We notice that all the results from Section 5 can be easily adapted to this new functional framework. In practice, we numerically approach the unique solution of the following mixed formulation:

$$\begin{cases} a^\zeta\left([w, f_2], [\widetilde{w}, \widetilde{f}_2]\right) + b^\zeta\left([\widetilde{w}, \widetilde{f}_2], \lambda\right) & = \ell^\zeta\left([\widetilde{w}, \widetilde{f}_2]\right) \\ b^\zeta\left([w, f_2], \widetilde{\lambda}\right) & = 0, \end{cases} \tag{68}$$

for any $[\widetilde{w}, \widetilde{f}_2] \in V \times H^1(\Omega)$ and $\widetilde{\lambda} \in V$. For every $[w, f_2]$, $[\widetilde{w}, \widetilde{f}_2]$ in $V \times H^1$ and $\lambda \in V$, the bilinear form $a^\zeta$ is given by

$$a^\zeta\left([w, f_2], [\widetilde{w}, \widetilde{f}_2]\right) = a\left([w, \gamma f_2], [\widetilde{w}, \gamma \widetilde{f}_2]\right) + \zeta\,(f_2, \widetilde{f}_2)_{H^1(\Omega)},$$

and $b^\zeta([w, f_2], \lambda) = b([w, \gamma f_2], \lambda)$, $\ell^\zeta([w, f_2]) = \ell([w, \gamma f_2])$. Remark that the term $\zeta(f_2, \widetilde{f}_2)_V$ appearing in $a^\zeta$ could be seen as a stabilization term. Replacing the mixed formulation (43) by (68) is also very convenient from the numerical implementation point of view, this avoiding the use of boundary finite elements. Moreover, Remark 10 in the Appendix shows that the solution $[w_\zeta^*, f_{2,\zeta}^*]$ to $(\mathrm{OCP}_\zeta^{\mathrm{lin}})$ converges to the solution $[w^*, f_2^*]$ to $(\mathrm{OCP}^{\mathrm{lin}})$ when $\zeta \to 0$.

To approximate the solutions of (68), we consider a triangulation $\mathcal{T}_h$ of $\Omega$, where $h$ is the largest diameter of triangles forming $\mathcal{T}_h$. For every $h > 0$, we define the finite dimensional spaces $V_h \subset V$ and $W_h \subset H^1(\Omega)$ as the Lagrange $P_1$ finite elements spaces: $V_h = \{\varphi \in V \mid \varphi|_T$ is an affine function, for every $T \in \mathcal{T}_h\}$ and $W_h = \{\varphi \in H^1(\Omega) \mid \varphi|_T$ is an affine function, for every $T \in \mathcal{T}_h\}$. The corresponding discrete version of (68) is the following finite dimensional system: find $[w^h, f_2^h, \lambda^h] \in V_h \times W_h \times V_h$ solution to

$$\begin{cases} a^\zeta\left([w^h, f_2^h], [\widetilde{w}^h, \widetilde{f}_2^h]\right) + b^\zeta\left([\widetilde{w}^h, \widetilde{f}_2^h], \lambda^h\right) & = \ell^\zeta\left([\widetilde{w}^h, \widetilde{f}_2^h]\right) \\ b^\zeta\left([w^h, f_2^h], \widetilde{\lambda}^h\right) & = 0, \end{cases} \tag{69}$$

for any $[\widetilde{w}^h, \widetilde{f}_2^h] \in V_h \times W_h$ and $\widetilde{\lambda}^h \in V_h$. If the inf-sup condition

$$\inf_{\lambda^h \in V_h} \sup_{[w^h, f_2^h] \in V_h \times W_h} \frac{b^\zeta\left([w^h, f_2^h], \lambda^h\right)}{\|[w^h, f_2^h]\|_{V_h \times W_h} \|\lambda^h\|_{V_h}} \ge \varrho_h > 0, \tag{70}$$

holds uniformly with respect to $h$, then for every $h$ there exists an unique solution of (69), and the sequence of solutions converges to the solution of (68) when $h \to 0$. Indeed, in this particular situation, the uniformity with respect to $h$ of the discrete inf-sup constant $\varrho_h$ is a consequence of the fact that $V_h \subset V$ (and, implicitly, $\|\cdot\|_{V_h} = \|\cdot\|_V$) and the following estimates:

$$\sup_{[w^h, f_2^h] \in V_h \times W_h} \frac{b^\zeta\left([w^h, f_2^h], \lambda^h\right)}{\|[w^h, f_2^h]\|_{V_h \times W_h}} \ge \frac{b^\zeta\left([-\lambda^h, 0], \lambda^h\right)}{\|\lambda^h\|_{V_h}} = \frac{\left(A\lambda^h, \lambda^h\right)_V}{\|\lambda^h\|_{V_h}} \ge \mu^* \|\lambda^h\|_{V_h}.$$

In all the numerical experiments the domain $\Omega$ is either the disk $\Omega_d \subset \mathbb{R}^2$ centered in the origin and of radius 1, with its boundary formed by the three arcs of circle represented in Fig. 1 (left), either the unit square $\Omega_s = (0, 1) \times (0, 1)$ with the boundary $\Gamma = \bigcup_{i=1}^3 \Gamma_i$ as depicted in Fig. 2 and $\Gamma_3 = \Gamma_3^l \cup \Gamma_3^r$.

We set $\mu \equiv 1$ in $\Omega$, which corresponds to a homogeneous material, and $f_0$ and $g$ are two functions to be chosen later. In order to numerically validate the optimal control strategy proposed in Section 5 and used in Algorithm 1, we compute firstly a reference solution to the nonlinear problem $(\mathrm{SP}^\rho)$ for a given value of $f_2$, which, for the seek of clarity, we denote $f_2^{\mathrm{ex}}$, on the boundary $\Gamma_2$. The reference solution $u_{\mathrm{ex}}$ of $(\mathrm{SP}^\rho)$ was approached by numerically solving its associated variational formulation on a very fine mesh of the domain $\Omega$ (mesh $\mathcal{T}_{\mathrm{d},h}^{\mathrm{ref}}$ and mesh $\mathcal{T}_{\mathrm{s},h}^{\mathrm{ref}}$ described in Table 1). Then we choose $u_d = u_{\mathrm{ex}}$ and we apply Algorithm 1 on other five coarser meshes $(\mathcal{T}_{\mathrm{d},h}^i)_{1 \le i \le 5}$ and $(\mathcal{T}_{\mathrm{s},h}^i)_{1 \le i \le 5}$. For every triangulation $\mathcal{T}_h^i$, the notation

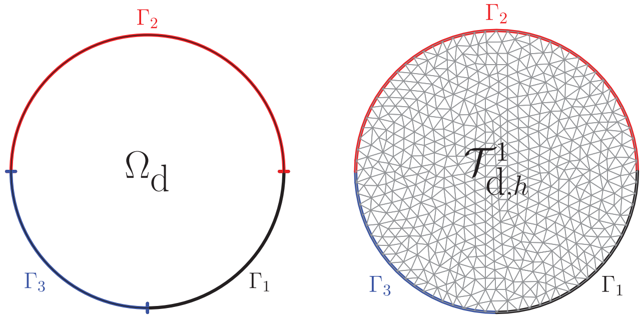



**Fig. 1.** Unit disk $\Omega_d$ with the partition of its boundary (left) and the associated mesh $\mathcal{T}^1_{d,h}$ (right).

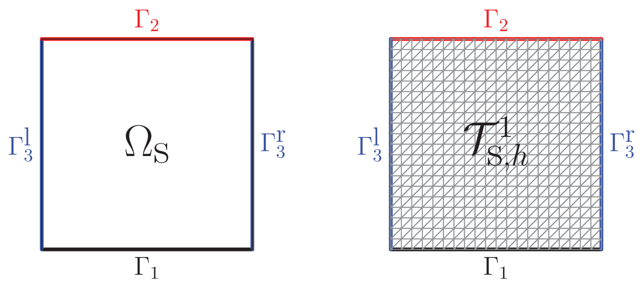



**Fig. 2.** Unit square $\Omega_s$ with the partition of its boundary (left) and the associated mesh $\mathcal{T}^1_{s,h}$ (right).

**Table 1**
Description of six meshes of the unit disk $\Omega_d$ and of the square $\Omega_s$, respectively.

| $\Omega = \Omega_d$ | $\mathcal{T}^1_{d,h}$ | $\mathcal{T}^2_{d,h}$ | $\mathcal{T}^3_{d,h}$ | $\mathcal{T}^4_{d,h}$ | $\mathcal{T}^5_{d,h}$ | $\mathcal{T}^{ref}_{d,h}$ |
|---|---|---|---|---|---|---|
| ♯ points | 592 | 2 349 | 8 984 | 35 878 | 143 489 | 344 148 |
| ♯ triangles | 1 102 | 4 536 | 17 646 | 71 114 | 285 396 | 686 294 |
| ♯ segments in $\Gamma$ | 80 | 160 | 320 | 640 | 1280 | 2 000 |
| $\Omega = \Omega_s$ | $\mathcal{T}^1_{s,h}$ | $\mathcal{T}^2_{s,h}$ | $\mathcal{T}^3_{s,h}$ | $\mathcal{T}^4_{s,h}$ | $\mathcal{T}^5_{s,h}$ | $\mathcal{T}^{ref}_{s,h}$ |
| ♯ points | 441 | 1 681 | 6 561 | 25 921 | 103 041 | 251 001 |
| ♯ triangles | 800 | 3 200 | 12 800 | 51 200 | 204 800 | 500 000 |
| ♯ segments in $\Gamma$ | 80 | 160 | 320 | 640 | 1280 | 2 000 |

**Table 2**
Number of iterations needed for convergence for mesh $\mathcal{T}^2_h$ and different values of $\rho$ for the Numerical experiment 1.

| $\rho$ | 10 | 1 | 0.25 | 0.15 | 0.14 | 0.13 |
|---|---|---|---|---|---|---|
| ♯ iterations | 6 | 9 | 15 | 37 | 212 | Non-convergence |

$h$ stands for the maximum diameter of the triangles composing the triangulation. On each of these meshes we compute an optimal pair $[u_h^*, f_{2,h}^*]$ which aims to approach $[u_d, f_2^{ex}]$. In Table 1 we list the number of points and elements associated to these meshes and in Fig. 1 and Fig. 2 we display the domain $\Omega$ and the coarsest mesh we consider for our study.

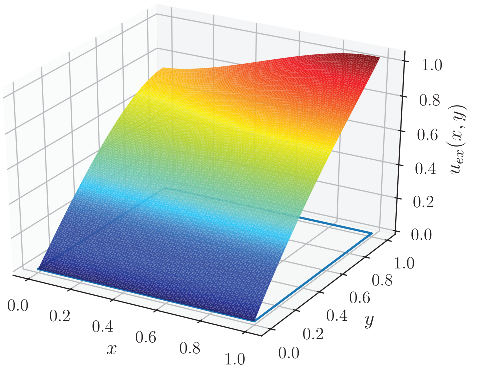
Algorithm 1 was implemented in `FreeFem++` [7]. The choice of parameters $\alpha$, $\beta$ and $\zeta$ appearing in the definition of the functional $L^\zeta$ is, in general, a difficult issue. In our case, since we want that the reconstructed solution $u^*$ to be as close as possible to the target $u_d$, the coefficient $\alpha$ should be much larger than $\beta$. More precisely, if not specified otherwise, for the numerical experiments which follow, we choose $\alpha = 10^8$. The parameter $\beta$ is chosen equal to 1 and $\zeta = \beta h$. Remark that when $h \to 0$ the parameter $\zeta$ goes to zero and, therefore, the solution of $(\mathrm{OCP}^{\mathrm{lin}}_\zeta)$ is close to the solution of $(\mathrm{OCP}^{\mathrm{lin}})$.

In what follows we present several numerical experiments for one or both domains $\Omega_d$ and $\Omega_s$ and different choices of parameters.

**Numerical experiment 1** (***r* = 0.5**): For this numerical experiment we choose $\Omega = \Omega_d$ and let $f_0 \equiv 0$ in $\Omega$, $f_2^{ex} \equiv 0$ on $\Gamma_2$ and $g \equiv 1$ on $\Gamma_3$. For this choice of parameters, the solution of problem $(\mathrm{OCP}^{\mathrm{lin}}_\zeta)$ is evidently the pair $[0, 0]$ and, therefore, the solution of $(\mathrm{OCP}^\rho)$ is $[w^*, f_2^*] = [0, 0]$. For an initial guess of $z_0 = 10$, the norms of solutions $[u_h^*, f_{2,h}^*]$ computed by the Algorithm 1 are close to zero ($< 10^{-15}$) for values of $\rho > 0$ large enough. In Table 2 we gather the number of iterations

**Table 3**
Reconstruction errors and number of iterations needed for convergence for meshes $\mathcal{T}^i_{d,h}$ and $\mathcal{T}^i_{s,h}$ of domains $\Omega_d$ and $\Omega_s$, respectively and for $(f_0, f_2^{ex})$ given in the numerical experiment 2.

| $\Omega = \Omega_d$ | $\mathcal{T}^1_{d,h}$ | $\mathcal{T}^2_{d,h}$ | $\mathcal{T}^3_{d,h}$ | $\mathcal{T}^4_{d,h}$ | $\mathcal{T}^5_{d,h}$ |
|---|---|---|---|---|---|
| $\Vert u_h^* - u_{ex}\Vert_V$ | 0.27 | 0.17 | 0.12 | 0.071 | 0.026 |
| $\Vert f_{2,h}^* - f_2^{ex}\Vert_{L^2(\Gamma_2)}$ | 0.048 | 0.036 | 0.066 | 0.071 | 0.086 |
| $\sharp$ iterations | 5 | 5 | 5 | 5 | 5 |
| $\Omega = \Omega_s$ | $\mathcal{T}^1_{s,h}$ | $\mathcal{T}^2_{s,h}$ | $\mathcal{T}^3_{s,h}$ | $\mathcal{T}^4_{s,h}$ | $\mathcal{T}^5_{s,h}$ |
| $\Vert u_h^* - u_{ex}\Vert_V$ | 0.014 | 0.0067 | 0.003 | 0.0012 | 0.00032 |
| $\Vert f_{2,h}^* - f_2^{ex}\Vert_{L^2(\Gamma_2)}$ | 0.000087 | 0.000073 | 0.00025 | 0.00032 | 0.00025 |
| $\sharp$ iterations | 5 | 5 | 5 | 5 | 5 |



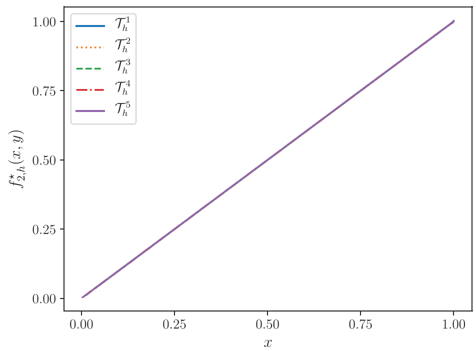




**Fig. 3.** Optimal solution $u_h^*$ (left) on $\mathcal{T}^3_{s,h}$ and the optimal control $f_{2,h}^*$ computed on different meshes for $\rho = 10$ and data from Numerical experiment 2.

**Table 4**
Reconstruction errors and number of iterations needed for convergence for mesh $\mathcal{T}^5_{d,h}$, $(f_0, f_2^{ex})$ given in Numerical experiment 2, $\beta = 1$, $\zeta = \beta h$ and $\alpha = 10^{2i}$.

| | $i=0$ | $i=1$ | $i=2$ | $i=3$ | $i=4$ |
|---|---|---|---|---|---|
| $\Vert u_h^* - u_{ex}\Vert_V$ | 0.27 | 0.007 | 0.00033 | 0.00032 | 0.00032 |
| $\Vert f_{2,h}^* - f_2^{ex}\Vert_{L^2(\Gamma_2)}$ | 0.34 | 0.011 | 0.00021 | 0.00025 | 0.00025 |
| $\sharp$ iterations | 5 | 5 | 5 | 5 | 5 |

needed for the convergence in Algorithm 1 for the mesh $\mathcal{T}^2_{d,h}$ and several values of $\rho$. For all the numerical simulations we choose the parameter $\epsilon$ appearing in the stopping criteria of Algorithm 1 equal to $10^{-6}$. We remark that the number of iterations needed for convergence rapidly grows when $\rho$ becomes very small. Thus, for values of $\rho$ close to zero we lose the convergence. This is due to the fact that, if $\rho$ is too small, the condition (61) is no more fulfilled.

**Numerical experiment 2 ($r = 0.5$)**: A less trivial example is the following one: $f_0(x,y) = \sqrt{x^2+y^2}$ for $(x,y) \in \Omega$, $f_2^{ex}(x,y) = x$ for $(x,y) \in \Gamma_2$ and $g \equiv 1$ on $\Gamma_3$. The results obtained in this case, for $\rho = 10$ and $\Omega$ being $\Omega_d$ and $\Omega_s$, respectively, are summarized in Table 3.

The results for Experiment 2 obtained for $\Omega = \Omega_s$ are illustrated in Fig. 3. More exactly, at left we display the optimal solution $u_h^*$ computed on the mesh $\mathcal{T}^3_{s,h}$ and, at right, we plot the optimal controls $f_{2,h}^*$ obtained for different meshes. Since $\Gamma_2$ can be represented as the graph of continuous function defined for $x \in (-1,1)$, we can represent the optimal controls as functions of $x$. Considering different values of $\rho$, we observe again that the number of iterations needed for the convergence explodes when $\rho \to 0$, for small values of $\rho$ the converge being lost.

In order to illustrate the behaviour of the optimal solution with respect to the fraction $\beta\alpha^{-1}$ we consider $\beta = 1$ and $\alpha = 10^{2i}$ for $i \in \{0, 1, 2, 3, 4\}$. The values of the reconstruction errors and the number of iterations needed for the convergence for $\Omega = \Omega_s$ are gathered in Table 4. The results are in agreement with Lemma 1. Moreover, we observe that, regardless the size of $\beta\alpha^{-1}$, the errors cannot be smaller than the precision of the numerical resolution. The number of iterations needed for convergence is not influenced by the choice of $\alpha$ and $\beta$.

In the same context of the Numerical experiment 2, we fix $\alpha = 10^8$, $\beta = 1$ and we consider several values of $\zeta$. More precisely, we consider the mesh $\mathcal{T}^5_{s,h}$ of the domain $\Omega_s$. The reconstruction errors are given in Table 5. We observe that the reconstruction is not sensitive to the choice of $\zeta$ of the form $\zeta = \beta h^i$ with positive values of $h$. For $i = -1$ we observe a small amelioration in the reconstruction of $f_2$. This can be explained that larger values of $\zeta$ force the $H^1$ norm of the reconstructed

**Table 5**
Reconstruction errors and number of iterations needed for convergence for mesh $\mathcal{T}^5_{d,h}$, $(f_0, f_2^{ex})$ given in Numerical experiment 2, $\alpha = 10^8$, $\beta = 1$, $\zeta = \beta h^i$.

| | $i=2$ | $i=1$ | $i=0$ | $i=-1$ | $i=-2$ |
|---|---|---|---|---|---|
| $\Vert u_h^* - u_{ex}\Vert_V$ | 0.00032 | 0.00032 | 0.00032 | 0.00032 | 0.0021 |
| $\Vert f_{2,h}^* - f_2^{ex}\Vert_{L^2(\Gamma_2)}$ | 0.00025 | 0.00025 | 0.00025 | 0.00015 | 0.006 |
| ♯ iterations | 5 | 5 | 5 | 5 | 5 |

**Table 6**
Reconstruction errors and number of iterations needed for convergence for meshes $\mathcal{T}^i_h$ with $i \in \{1, 2, 3, 4, 5\}$ for $r = 2$, $(f_0, f_2^{ex})$ given in Numerical experiment 3. The domain is $\Omega = \Omega_s$, $g \equiv 1$ and $\rho = 0$.

| | $\mathcal{T}^1_{s,h}$ | $\mathcal{T}^2_{s,h}$ | $\mathcal{T}^3_{s,h}$ | $\mathcal{T}^4_{s,h}$ | $\mathcal{T}^5_{s,h}$ |
|---|---|---|---|---|---|
| $\Vert u_h^* - u_{ex}\Vert_V$ | 0.015 | 0.0073 | 0.0033 | 0.0013 | 0.00035 |
| $\Vert f_{2,h}^* - f_2^{ex}\Vert_{L^2(\Gamma_2)}$ | 0.00091 | 0.00023 | 0.00025 | 0.00035 | 0.00031 |
| ♯ iterations | 16 | 16 | 16 | 16 | 16 |

**Table 7**
Number of iterations needed for convergence for data in Numerical experiment 3 on the mesh $\mathcal{T}^1_{s,h}$ and different values of $g$.

| | $g \equiv 0.5$ | $g \equiv 1$ | $g \equiv 2$ | $g \equiv 3$ | $g \equiv 4$ | $g \equiv 5$ |
|---|---|---|---|---|---|---|
| ♯ iterations | 10 | 16 | 29 | 57 | 158 | Non-convergence |

$f_2$ to be smaller, and, therefore, enhance the regularity of $f_2$. If $i < -1$, or in other words, for even larger values of $\zeta$, the reconstruction seems to deteriorate.

**Numerical experiment 3 ($r = 2$)**: Finally we consider an example with the parameter $r = 2$. We choose here $\Omega = \Omega_s$ and the same data as for Experiment 2. As in Section 5.3, we choose $\rho = 0$ and the operator $\mathcal{N}$ given by (64). Table 6 shows that, in oposition to the Experiments 1 and 2, the number of iterations needed for the convergence of the fixed point algorithm does not explode for $\rho = 0$, in agreement with the results in Section 5.3. As expected, the value of $\Vert g\Vert_{L^\infty(\Gamma_3)}$ affects the convergence of the algorithm. In Table 7 we gather the number of iterations needed for convergence with the same data, the mesh $\mathcal{T}^1_{s,h}$ and different values of $g$. We observe that the convergence is lost for large values of $g$. As mentioned in Remark 9, the values of $\Vert g\Vert_{L^\infty(\Gamma_3)}$ for which the fixed point algorithm converges can be larger, if the other parameters, typically $f_0$ and $u_d$, are small. In order to illustrate such a situation, we consider an example with $g \equiv 5$, $f_0(x) = \frac{\sqrt{x^2+y^2}}{2}$ and $f_2^{ex}(x) = \frac{x}{2}$. With the rest of the parameters as above and the mesh $\mathcal{T}^1_{s,h}$, the fixed point algorithm converges in 671 iterations. Hence, this is an example in which smaller data $f_0$ and $u_d$ ensure the convergence of the scheme for larger value of $g$.

## Acknowledgments

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In Section 6, the functional $L$ defined by (14) has been replaced by a slightly modified version of it, $L^\zeta$. In this Appendix we establish the relation between the optimal pairs corresponding to $L^\zeta$ and those corresponding to $L$. Let us first describe the context in which we work. If $\alpha$, $\beta$ and $\zeta$ are three positive constants, let us define the following functional $L^\zeta : V \times H^1(\Omega) \to \mathbb{R}$,

$$L^\zeta(u, f_2) = \frac{\alpha}{2}\Vert u - u_d\Vert_V^2 + \frac{\beta}{2}\Vert \gamma f_2\Vert^2_{L^2(\Gamma_2)} + \frac{\zeta}{2}\Vert f_2\Vert^2_{H^1(\Omega)}. \tag{71}$$

We introduce the following *optimal control problem* associated to $L^\zeta$:

$$\textit{Find } [u_\zeta^*, f_{2,\zeta}^*] \in \mathcal{V}_{ad}^\zeta \textit{ such that } L^\zeta(u_\zeta^*, f_{2,\zeta}^*) = \min_{[u, f_2] \in \mathcal{V}_{ad}^\zeta} L^\zeta(u, f_2), \qquad (OCP_\zeta)$$

with $\mathcal{V}_{ad}^\zeta = \left\{[u, f_2] \in V \times H^1(\Omega) \text{ such that } [u, \gamma f_2] \in \mathcal{V}_{ad}\right\}$. As in the case of Theorem 1, following [9, Theorem 3.7], we obtain that $(OCP_\zeta)$ has at least one optimal pair $[u_\zeta^*, f_{2,\zeta}^*]$. Now we pass to study the behavior of the optimal pairs of the functional $L^\zeta$ when $\zeta$ tends to zero. We have the following result.

**Theorem 10.** *For each* $\zeta > 0$, *let* $[u^*_\zeta, f^*_{2,\zeta}]$ *be a solution of problem* (OCP$_\zeta$)*. Then, there exists a subsequence of the family* $\left([u^*_\zeta, f^*_{2,\zeta}]\right)_{\zeta>0}$*, denoted in the same way, and a solution* $[u^*, f^*_2]$ *of problem (OCP) such that*

$$u^*_\zeta \to u^* \text{ in } V \text{ and } \gamma f^*_{2,\zeta} \rightharpoonup f^*_2 \text{ in } L^2(\Gamma_2) \text{ as } \zeta \to 0. \tag{72}$$

**Proof.** Like in Theorem 5, we deduce that there exists $[u^*, f^*_2] \in V \times L^2(\Gamma_2)$ such that, passing eventually to a subsequence, but keeping the notation to simplify the writing, we have

$$u^*_\zeta \rightharpoonup u^* \text{ in } V \text{ and } \gamma f^*_{2,\zeta} \rightharpoonup f^*_2 \text{ in } L^2(\Gamma_2) \text{ as } \zeta \to 0.$$

Moreover, as in the proof of Theorem 5, it follows that

$$u^*_\zeta \to u^* \text{ in } V \text{ as } \zeta \to 0.$$

On the other hand we have that $[u^*, f^*_2] \in \mathcal{V}_{\text{ad}}$. Indeed, since $[u^*_\zeta, \gamma f^*_{2,\zeta}] \in \mathcal{V}_{\text{ad}}$, by passing to the limit in (SP) we deduce that $[u^*, f^*_2]$ verifies (SP).

Let $[\widehat{u}, \widehat{f}_2] \in \mathcal{V}_{\text{ad}}$ be a solution of (OCP). For each $\widetilde{f}_2 \in H^1(\Omega)$ with the property that $\gamma \widetilde{f}_2 = \widehat{f}_2$, we have that

$$L(u^*, f^*_2) \le \liminf_{\zeta\to 0} L^\zeta\left(u^*_\zeta, f^*_{2,\zeta}\right) \le \limsup_{\zeta\to 0} L^\zeta\left(\hat{u}, \tilde{f}_2\right) = L\left(\hat{u}, \hat{f}_2\right). \tag{73}$$

On the other hand, since $[\widehat{u}, \widehat{f}_2]$ is a solution of (OCP), we have that

$$L(\widehat{u}, \widehat{f}_2) \le L(u^*, f^*_2). \tag{74}$$

Thus, from (73) and (74) we deduce that $L(\widehat{u}, \widehat{f}_2) = L(u^*, f^*_2)$. Hence, $[u^*, f^*_2]$ is an optimal pair of (OCP) and the proof of the theorem is complete. □

**Remark 10.** The same type of arguments allow to show that the convergence property (72) holds for the minimization problems (OCP$^{\text{lin}}_\zeta$) and (OCP$^{\text{lin}}$) from Sections 6 and 5, respectively.

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