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ABSTRACT

We consider a mixed variational problem governed by a nonlinear operator and a set of constraints. Existence, uniqueness and convergence results for this problem have already been obtained in the literature. In this current paper we complete these results by proving the well-posedness of the problem, in the sense of Tykhonov. To this end we introduce a family of approximating problems for which we state and prove various equivalence and convergence results. We illustrate these abstract results in the study of a frictionless contact model with elastic materials. The process is assumed to be static and the contact is with unilateral constraints. We derive a weak formulation of the model which is in the form of a mixed variational problem with unknowns being the displacement field and the Lagrange multiplier. Then, we prove various results on the corresponding mixed problem, including its well-posedness in the sense of Tykhonov, under various assumptions on the data. Finally, we provide mechanical interpretation of our results.

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1. Introduction

Mixed variational problems represent a class of inequality problems which arise in the variational analysis of a large number of nonlinear boundary value problems with unilateral constraints. Their major ingredient consists in introducing a new variable, the Lagrange multiplier, associated to the constraints. Existence and uniqueness results can be found in [1–6]. References on the numerical treatment of mixed variational problems include [7–12]. As it follows from these papers, the numerical treatment of mixed variational problems is efficient and accurate, which explains why they are widely used in Solid and Contact Mechanics as well as in various Engineering Applications.

Besides the unique solvability, the notion of well-posedness for a given mathematical problem represents an important topic which was widely studied in the literature. The concepts of well-posedness vary from problem to problem and from author to author. Some examples are the concept of well-posedness

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in the sense of Hadamard for partial differential equations, the concept of well-posedness in the sense of Tykhonov for a minimization problem, the concept of well-posedness in the sense of Levitin–Polyak for a constrained optimization problem, among others. The literature in the field includes various extensions to mathematical problems like variational and hemivariational inequalities, inclusions, fixed point, equilibrium point and saddle point problems, see [13–21,32], for instance. For an abstract mathematical problem, the concept of well-posedness in the sense of Tykhonov is based on two main ingredients: the existence of a unique solution and the convergence of any approximating sequence to the solution. Details can be found in [22].

In this current paper we study the well-posedness in the sense of Tykhonov (well-posedness, for short) for an elliptic mixed variational problem. The functional framework, which we assume in Sections 2 and 3 of this paper, is the following. First, X and Y represent real Hilbert spaces endowed with the inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$ and associated norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Moreover, $X\times Y$ denotes their product space endowed with the canonical inner product. A typical element of $X\times Y$ will be denoted by (u,λ). In addition, we assume that $A: X\to X$, $b: X\times Y\to\mathbb{R}$, $\Lambda\subset Y$, f, $h\in X$ and, finally, $X\times\Lambda$ represents the product of the sets X and Λ. Then, the mixed variational problem we consider can be formulated as follows.

Problem $\mathcal{P}$: *Find $(u,\lambda)\in X\times\Lambda$ such that*

$$(Au,v)_X + b(v,\lambda) = (f,v)_X \quad \forall\, v\in X, \tag{1}$$

$$b(u,\mu-\lambda)\le b(h,\mu-\lambda) \quad \forall\,\mu\in\Lambda. \tag{2}$$

Consider now a function $\theta:[0,+\infty)\to[0,+\infty)$. Then, for any $\varepsilon>0$ we consider the following perturbed version of Problem $\mathcal{P}$.

Problem $\mathcal{P}_\varepsilon$: *Find $(u,\lambda)\in X\times\Lambda$ such that*

$$(Au,v)_X + b(v,\lambda)\le (f,v)_X+\theta(\varepsilon)\|v\|_X \quad \forall\, v\in X, \tag{3}$$

$$b(u,\mu-\lambda)\le b(h,\mu-\lambda)+\theta(\varepsilon)\|\mu-\lambda\|_Y \quad \forall\,\mu\in\Lambda. \tag{4}$$

We denote in what follows by $\mathcal{S}$ and $\mathcal{S}_\varepsilon$ the set of solutions of Problems $\mathcal{P}$ and $\mathcal{P}_\varepsilon$, respectively, and we recall that a set is said to be a singleton if it has only one element. We proceed with the following definitions.

Definition 1.1: A sequence $\{(u_n,\lambda_n)\}\subset X\times Y$ is called an approximating sequence for the Problem $\mathcal{P}$ if there exists a positive sequence $\{\varepsilon_n\}\subset\mathbb{R}$ with $\varepsilon_n\to 0$ as $n\to\infty$ such that $(u_n,\lambda_n)\in\mathcal{S}_{\varepsilon_n}$ for each $n\in\mathbb{N}$.

Definition 1.2: The Problem $\mathcal{P}$ is said to be well-posed if it has a unique solution (i.e. $\mathcal{S}$ is a singleton) and every approximating sequence for Problem $\mathcal{P}$ converges in $X\times Y$ to its solution.

For simplicity, for any sequence $\{\varepsilon_n\} \subset \mathbb{R}$ which satisfies the conditions of Definition 1.1 we shall write $0 < \varepsilon_n \to 0$. Moreover, assuming that $\mathcal{S}_\varepsilon$ is nonempty, we denote by $\mathrm{diam}(\mathcal{S}_\varepsilon)$ its diameter, which is defined by equality

$$\mathrm{diam}(\mathcal{S}_\varepsilon) = \sup_{a,\, b \in \mathcal{S}_\varepsilon} \|a - b\|_{X \times Y}. \tag{5}$$

We now consider the following statements.

(i) $\mathcal{S} \neq \emptyset$.
(ii) $\mathcal{S}$ is a singleton.
(iii) $\mathcal{S}_\varepsilon \neq \emptyset$, for each $\varepsilon > 0$.
(iv) $\mathrm{diam}(\mathcal{S}_\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0$.
(v) $\begin{cases} \mathcal{S} = \{(u, \lambda)\} \text{ and} \\ \text{any approximating sequence } \{(u_n, \lambda_n)\} \text{ converges to } (u, \lambda). \end{cases}$

Concerning these statements we have the following preliminary comments. First, since θ is a positive function we have that $\mathcal{S} \subset \mathcal{S}_\varepsilon$ for each $\varepsilon > 0$ and, therefore, (i) implies (iii), and (ii) implies (iii), too. Next, the statement (iv) makes sense only when (iii) holds and, in particular, it makes sense if (i) or (ii) holds. Moreover, (v) implies (ii). In addition, since a singleton is a nonempty set, it is clear that (ii) implies (i). Finally, Definition 1.2 shows that Problem $\mathcal{P}$ is well-posed if and only if the statement (v) holds.

Our aim in this paper is threefold. The first one is to establish the link between the statements (i) –(v) above. To this end we indicate sufficient conditions which guarantee various implications and equivalences between these statements. The second aim is to provide sufficient conditions which guarantee the validity of the statements (i)–(v), which, implicitly, guarantees the well-posedness of Problem $\mathcal{P}$. Finally, our third aim is to provide an example of Problems $\mathcal{P}$ and $\mathcal{P}_\varepsilon$ for which our results hold, together with the corresponding interpretations.

The rest of this paper is structured as follows. In Section 2 we list the assumptions on the data and present our main results in the study of Problems $\mathcal{P}_\varepsilon$ and $\mathcal{P}$, Theorems 2.2, 2.4, 2.6, 2.8. Besides their novelty and their mathematical interest, these results are important since they provide tools useful in the variational and numerical analysis of various contact problems with unilateral constraints. To provide an example, in Section 3 we present two mathematical models which describe the contact between an elastic body and a rigid foundation covered by a layer of soft material. We list the assumptions on the data and derive their variational formulation which is in the form of Problems $\mathcal{P}$ and $\mathcal{P}_\varepsilon$, respectively. Finally, in Section 4 we apply our abstract results in the study of these problems and provide the corresponding mechanical interpretations. In this way we illustrate the cross fertilization between models and applications, on one hand, and the nonlinear functional analysis, on the other hand.

2. Main results

Below in this section 0_X and 0_Y will represent the zero elements of the spaces X and Y. Moreover, the symbols '$\to$ ' and '$\rightharpoonup$ ' denote the strong and weak convergence in the space X, respectively. In the study of problems $\mathcal{P}$ and $\mathcal{P}_\varepsilon$ we consider the following assumptions.

$$\left.\begin{array}{l} A : X \to X \text{ is a strongly monotone Lipschitz continuous operator, i.e.} \\ \text{there exist } m > 0 \text{ and } L > 0 \text{ such that} \\ \text{(a) } (Au - Av, u - v)_X \geq m\|u - v\|_X^2 \quad \forall\, u, v \in X; \\ \text{(b) } \|Au - Av\|_X \leq L\,\|u - v\|_X \quad \forall\, u, v \in X. \end{array}\right. \tag{6}$$

$$\left.\begin{array}{l} A : X \to X \text{ is a demicontinuous operator, i.e.} \\ u_n \to u \text{ in } X \implies Au_n \rightharpoonup Au \text{ in } X. \end{array}\right. \tag{7}$$

$$\left.\begin{array}{l} b : X \times Y \to \mathbb{R} \text{ is a bilinear continuous form,} \\ \text{i.e. there exists M >0 such that} \\ |b(v, \mu)| \leq M\|v\|_X\|\mu\|_Y \quad \forall v \in X,\ \mu \in Y. \end{array}\right. \tag{8}$$

$$\left.\begin{array}{l} b : X \times Y \to \mathbb{R}\text{is a bilinear form which satisfies} \\ \text{the inf-sup condition, i.e. there exists} \alpha > 0 \text{such that} \\ \inf\limits_{\mu\in Y,\mu\neq 0_Y}\ \sup\limits_{v\in X, v\neq 0_X} \dfrac{b(v,\mu)}{\|v\|_X\|\mu\|_Y} \geq \alpha. \end{array}\right. \tag{9}$$

$$\theta : [0, +\infty) \to [0, +\infty) \text{ is a continuous function such that } \theta(0) = 0. \tag{10}$$

$$\theta : [0, +\infty) \to [0, +\infty) \text{ is an increasing function.} \tag{11}$$

$$\Lambda \text{is a closed convex subset of} Y \text{ such that } 0_Y \in \Lambda. \tag{12}$$

Moreover, recall that

$$f \in X, \quad h \in X. \tag{13}$$

The following existence and uniqueness result guarantees the unique solvability of Problem $\mathcal{P}$.

Theorem 2.1: *Assume that* (6), (8), (9), (12)), (13) *hold. Then, Problem* $\mathcal{P}$ *has a unique solution* $(u, \lambda) \in X \times \Lambda$.

Theorem 2.1 was obtained in [23] (the case when Λ is an unbounded subset) and [24,31] (the case when Λ is bounded). In both references the proofs are carried out in several steps, based on arguments of saddle points and the Banach fixed point theorem.

Our main results in this section concern the statements (i)–(v) presented in the Introduction and are gathered in the Theorems 2.2, 2.4, 2.6, 2.8 that we state and prove below.

Theorem 2.2: *Assume that* (6), (9), (10), (13) *hold. Then, the statements* (ii) *and* (v) *are equivalent.*

Proof: Let (u, λ) be the solution to Problem $\mathcal{P}$, guaranteed by (ii), and let $\{(u_n, \lambda_n)\}$ be an approximating sequence. Then, there exists a sequence $0 < \varepsilon_n \to 0$ such that, for each $n \in \mathbb{N}$ the following inequalities hold:

$$(Au_n, v)_X + b(v, \lambda_n) \le (f, v)_X + \theta(\varepsilon_n)\|v\|_X \quad \forall\, v \in X, \tag{14}$$

$$b(u_n, \mu - \lambda_n) \le b(h, \mu - \lambda_n) + \theta(\varepsilon_n)\|\mu - \lambda_n\|_Y \quad \forall\, \mu \in \Lambda. \tag{15}$$

We subtract (1) from (14) to see that

$$(Au_n - Au, v)_X + b(v, \lambda_n - \lambda) \le \theta(\varepsilon_n)\|v\|_X \quad \forall\, v \in X, \tag{16}$$

which implies that

$$b(v, \lambda_n - \lambda) \le \theta(\varepsilon_n)\|v\|_X + (Au - Au_n, v)_X \quad \forall\, v \in X.$$

Moreover, using assumption (6)(b) yields

$$b(v, \lambda_n - \lambda) \le \theta(\varepsilon_n)\|v\|_X + L\|u_n - u\|_X\|v\|_X \quad \forall\, v \in X.$$

This inequality combined with assumption (9) implies that

$$\|\lambda_n - \lambda\|_Y \le \frac{\theta(\varepsilon_n)}{\alpha} + \frac{L}{\alpha}\,\|u_n - u\|_X. \tag{17}$$

On the other hand, we take $\mu = \lambda$ in (15), $\mu = \lambda_n$ in (2) and add the resulting inequalities to obtain that

$$b(u_n - u, \lambda - \lambda_n) \le \theta(\varepsilon_n)\|\lambda_n - \lambda\|_Y. \tag{18}$$

Next, we take $v = u_n - u$ in (16) and use (18) to deduce that

$$\begin{aligned}(Au_n - Au, u_n - u)_X &\le b(u_n - u, \lambda - \lambda_n) + \theta(\varepsilon_n)\|u_n - u\|_X \\ &\le \theta(\varepsilon_n)\|\lambda_n - \lambda\|_Y + \theta(\varepsilon_n)\|u_n - u\|_X.\end{aligned}$$

Therefore, using assumption (6)(a) we find that

$$m\|u_n - u\|_X^2 \le \theta(\varepsilon_n)\|\lambda_n - \lambda\|_Y + \theta(\varepsilon_n)\|u_n - u\|_X. \tag{19}$$

We now substitute inequality (17) in (19) to obtain that

$$\|u_n - u\|_X^2 \le \frac{\theta^2(\varepsilon_n)}{\alpha m} + \frac{\theta(\varepsilon_n)}{m}\left(\frac{L}{\alpha} + 1\right)\|u_n - u\|_X. \tag{20}$$

Then, using the elementary inequality

$$x^2 \le ax + b \Longrightarrow x \le a + \sqrt{b} \quad \forall\, x,\ a,\ b \ge 0$$

we find that

$$\|u_n - u\|_X \le \frac{\theta(\varepsilon_n)}{m}\left(\frac{L}{\alpha} + 1\right) + \frac{\theta(\varepsilon_n)}{\sqrt{\alpha m}}.$$

We now combine this inequality with the bound (17) to deduce that there exists a constant $C > 0$, which depends only on m, L, and α rather than n, such that

$$\|u_n - u\|_X + \|\lambda_n - \lambda\|_Y \leq C\theta(\varepsilon_n). \tag{21}$$

Next, we use assumption (10) and inequality (21) to see that $(u_n, \lambda_n) \to (u, \lambda)$ in $X \times Y$ which shows that (v) holds. On the other hand, recall that (v) implies (ii) and this concludes the proof. ■

We complete Theorem 2.2 with the following result.

Corollary 2.3: *Assume that* (6), (9), (10)), (13) *hold. Then, Problem* $\mathcal{P}$ *has a unique solution if and only if it is well-posed.*

Proof: It follows from Definition 1.2 that Problem $\mathcal{P}$ is well-posed if and only if the statement (v) holds. On the other hand, Problem $\mathcal{P}$ has a unique solution if and only if the statement (ii) holds. Corollary 2.3 is now a direct consequence of Theorem 2.2. ■

We now proceed our analysis with a second equivalence result, obtained under different assumptions on the data.

Theorem 2.4: *Assume that* (7), (8), (10), (11), (12), (13) *hold. Then, the statements* (v), (i) *and* (iv), (iii) *and* (iv) *are equivalent, i.e.*

$$\text{(v)} \iff \text{(i) and (iv)} \iff \text{(iii) and (iv).}$$

Proof: The statement of Theorem 2.4 follows from the implications

$$\text{(v)} \implies \text{(i) and (iv)} \implies \text{(iii) and (iv)} \implies \text{(v)}$$

that we prove in what follows.

Assume that (v) holds. Then it is clear that (i) holds. Arguing by contradiction, assume in what follows that (iv) does not hold, i.e. diam $(\mathcal{S}_\varepsilon) \not\to 0$ as $\varepsilon \to 0$. Then, there exist $\delta_0 \geq 0$, a sequence $0 < \varepsilon_n \to 0$ and two sequences $\{(u_n, \lambda_n)\}$, $\{(v_n, \mu_n)\} \subset X \times Y$ with (u_n, λ_n), $(v_n, \mu_n) \in \mathcal{S}_{\varepsilon_n}$ such that

$$\|(u_n, \lambda_n) - (v_n, \mu_n)\|_{X\times Y} \geq \frac{\delta_0}{2} \quad \forall\, n \in \mathbb{N}. \tag{22}$$

Now, Definition 1.1 implies that both $\{(u_n, \lambda_n)\}$ and $\{(v_n, \mu_n)\}$ are approximating sequences for Problem $\mathcal{P}$. Therefore, (v) implies that $(u_n, \lambda_n) \to (u, \lambda)$ and $(v_n, \mu_n) \to (u, \lambda)$ in $X \times Y$ where (u, λ) denotes the unique element of $\mathcal{S}$. This is in contradiction with inequality 22. We conclude from here that diam$(\mathcal{S}_\varepsilon) \to 0$ as $\varepsilon \to 0$ and, therefore, (iv) holds.

Assume now that (i) and (iv) hold. Recall that (i) implies (iii), since $\mathcal{S} \subset \mathcal{S}_\varepsilon$. Therefore, we deduce that (iii) and (iv) hold.

Finally, assume that (iii) and (iv) hold. Consider a sequence $0 < \varepsilon_n \to 0$ and let $\{(u_n, \lambda_n)\}$ be a sequence of $X \times Y$ such that $(u_n, \lambda_n) \in \mathcal{S}_{\varepsilon_n}$ for all $n \in \mathbb{N}$. Since $\mathrm{diam}(\mathcal{S}_\varepsilon) \to 0$, for any $\delta > 0$ there exists a positive integer N_δ such that

$$\mathrm{diam}(\mathcal{S}_{\varepsilon_n}) \le \delta \quad \forall\, n \ge N_\delta. \tag{23}$$

Let $n, m \in \mathbb{N}$ be such $n, m \ge N_\delta$ and assume that $\varepsilon_m \le \varepsilon_n$. Then, using (3), (4) and (11) we have $(u_n, \lambda_n), (u_m, \lambda_m) \in \mathcal{S}_{\varepsilon_n}$ and, therefore, (23) implies that

$$\|(u_n, \lambda_n) - (u_m, \lambda_m)\|_{X\times Y} \le \delta.$$

This inequality holds if $\varepsilon_m > \varepsilon_n$, too, since in this case $(u_n, \lambda_n), (u_m, \lambda_m) \in \mathcal{S}_{\varepsilon_m}$. We conclude from here that $\{(u_n, \lambda_n)\}$ is a Cauchy sequence in $X \times Y$, hence there exists $(u, \lambda) \in X \times Y$ such that

$$(u_n, \lambda_n) \to (u, \lambda) \quad \text{in } X \times Y. \tag{24}$$

Next,we recall that for each $n \in \mathbb{N}$, $\lambda_n \in \Lambda$ and, therefore, assumption (12) implies that $\lambda \in \Lambda$. On the other hand, by the definition of $\mathcal{S}_{\varepsilon_n}$ it follows that inequalities (14) and (15) hold. Therefore, passing to the limit in these inequalities and using the convergence (24) combined with assumptions (7), (8), (10) we deduce that (u, λ) satisfies (1) and (2). This shows that (u, λ) is a solution to Problem $\mathcal{P}$, i.e. $(u, \lambda) \in \mathcal{S}$. We conclude from here that $\mathcal{S} \neq \emptyset$.

Next, we claim that $\mathcal{S}$ is a singleton. Indeed, let $(u, \lambda), (u', \lambda') \in \mathcal{S}$. Then, we deduce that $(u, \lambda), (u', \lambda') \in \mathcal{S}_\varepsilon$, for any $\varepsilon > 0$. Thus,

$$\|(u, \lambda) - (u', \lambda')\|_{X\times Y} \le \mathrm{diam}(\mathcal{S}_\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

which implies that $(u, \lambda) = (u', \lambda')$ and proves the claim.

Let now $\{(u_n, \lambda_n)\} \subset X \times Y$ be an approximating sequence for the Problem $\mathcal{P}$. Then there exists a sequence $0 < \varepsilon_n \to 0$ such that $(u_n, \lambda_n) \in \mathcal{S}_{\varepsilon_n}$ for each $n \in \mathbb{N}$. We use the statement (iv) to see that

$$\|(u_n, \lambda_n) - (u, \lambda)\|_{X\times Y} \le \mathrm{diam}(\mathcal{S}_{\varepsilon_n}) \to 0.$$

This implies that $(u_n, \lambda_n) \to (u, \lambda)$ in $X \times Y$. We deduce from above that (v) holds which concludes the proof. ■

A direct consequence of Theorem 2.4 is the following.

Corollary 2.5: *Under assumptions* (7), (8), (10), (11), (12), (13), *the following statements are equivalent*:

(a) *Problem $\mathcal{P}$ is well-posed.*

(b) $\mathcal{S} \neq \emptyset$ *and* $\operatorname{diam}(\mathcal{S}_\varepsilon) \to 0$ as $\varepsilon \to 0$.
(c) $\mathcal{S}_\varepsilon \neq \emptyset$ *for each* $\varepsilon > 0$ *and* $\operatorname{diam}(\mathcal{S}_\varepsilon) \to 0$ as $\varepsilon \to 0$.

Note that the equivalence of statements (a) and (c) above is important since it characterize the well-posedness of Problem $\mathcal{P}$ in terms of metric properties of the solution of a different problem, Problem $\mathcal{P}_\varepsilon$. Moreover, note that Corollaries 2.3, 2.5 and Theorem 2.4 provide only equivalence results and do not guarantee the well-posedness of the mixed variational problem $\mathcal{P}$. The next theorem provides sufficient conditions which guarantee the well-posedness of this problem.

Theorem 2.6: *Assume that* (6), (8), (9), (10), (11), (12), (13) *hold. Then, the statement* (i)–(v) *hold.*

Proof: We use Theorem 2.1 to see that Problem $\mathcal{P}$ has a unique solution, i.e. the statement (ii) holds. Therefore, by Theorem 2.2 it follows that the statement (v) holds, too. Note also that assumption (6) implies assumption (7). Therefore, we are in a position to apply Theorem 2.4 in order to see that all the statements (i)–(v) hold, which concludes the proof. ∎

A direct consequence of Theorem 2.6 is the following.

Corollary 2.7: *Under assumption* (6), (8)), (9), (10), (11), (12), (13), *the following statements hold.*

(a) *Problem* $\mathcal{P}$ *has a unique solution.*
(b) *Problem* $\mathcal{P}_\varepsilon$ *has at least one solution, for each* $\varepsilon > 0$.
(c) *Every approximating sequence for Problem* $\mathcal{P}$ *converges to its solution.*
(d) *Problem* $\mathcal{P}$ *is well-posed.*

We complete these results with the following one.

Theorem 2.8: *Assume that* (6), (8), (9), (12), (13) *hold. Then, the operator* $(f, h) \mapsto (u, \lambda)$ *which associates to each couple* $(f, h) \in X \times X$ *the unique solution* $(u, \lambda) \in X \times \Lambda$ *of Problem* $\mathcal{P}$ *is Lipschitz continuous.*

Proof: Let (u_i, λ_i) be the solution of Problem $\mathcal{P}$ for the data $(f_i, h_i) \in X \times X$, $i = 1, 2$. We have

$$(Au_1, v)_X + b(v, \lambda_1) = (f_1, v)_X \quad \forall\, v \in X, \tag{25}$$

$$b(u_1, \mu - \lambda_1) \leq b(h_1, \mu - \lambda_1) \quad \forall\, \mu \in \Lambda \tag{26}$$

and, on the other hand,

$$(Au_2, v)_X + b(v, \lambda_2) = (f_2, v)_X \quad \forall\, v \in X, \tag{27}$$

$$b(u_2, \mu - \lambda_2) \leq b(h_2, \mu - \lambda_2) \quad \forall\, \mu \in \Lambda. \tag{28}$$

We now use (27) to write

$$\begin{aligned}(Au_2, v)_X + b(v, \lambda_2) &= (f_1, v)_X + (f_2 - f_1, v)_X \\ &\le (f_1, v)_X + \|f_2 - f_1\|_X \|v\|_X \quad \forall\, v \in X,\end{aligned} \tag{29}$$

then use (28) and assumption (8) to see that

$$\begin{aligned}b(u_2, \mu - \lambda_2) \le b(h_2, \mu - \lambda_2) &= b(h_1, \mu - \lambda_2) + b(h_2 - h_1, \mu - \lambda_2) \\ &\le b(h_1, \mu - \lambda_2) + M\|h_1 - h_2\|_X \|\mu - \lambda_2\|_Y \quad \forall\, \mu \in \Lambda.\end{aligned} \tag{30}$$

Denote

$$\theta = \|f_2 - f_1\|_X + M\|h_1 - h_2\|_X. \tag{31}$$

Then, inequalities (29) and (30) imply that

$$(Au_2, v)_X + b(v, \lambda_2) \le (f_1, v)_X + \theta\|v\|_X \quad \forall\, v \in X, \tag{32}$$

$$b(u_2, \mu - \lambda_2) \le b(h_1, \mu - \lambda_2) + \theta\|\mu - \lambda_2\|_Y \quad \forall\, \mu \in \Lambda. \tag{33}$$

Finally, using (25), (26) on one hand, and (32), (33) on the other hand, arguments similar to those used to obtain inequality (21) imply that

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_Y \le C\theta, \tag{34}$$

where C is a positive constant which depends only on A and b. We now combine (31) and (34) to conclude the proof. ■

3. Two contact models

In this section we present two contact models which lead to mixed variational formulations of the form of Problems $\mathcal{P}$ and $\mathcal{P}_\varepsilon$, respectively. To this end, everywhere in the rest of the paper we assume that Ω is a bounded domain in $\mathbb{R}^d$ ($d = 2, 3$), with a smooth boundary Γ, divided into three measurable disjoint parts Γ_1, Γ_2 and Γ_3 such that *meas* $(\Gamma_1) > 0$. We denote by $\boldsymbol{\nu}$ the outward unit normal to Γ and by $\mathbb{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$. In addition, we denote by '$\cdot$' and '$\|\cdot\|$' the canonical inner products and norms on the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively.

The first contact model we consider is stated as follows.

Problem $\mathcal{Q}$: *Find a displacement field* $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{35}$$

$$\text{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega, \tag{36}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{37}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2, \tag{38}$$

$$\left.\begin{array}{l} u_\nu \le g, \\ \sigma_\nu + p(u_\nu) \le 0, \\ (u_\nu - g)(\sigma_\nu + p(u_\nu)) = 0 \end{array}\right\} \quad \text{on } \Gamma_3, \tag{39}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{on } \Gamma_3. \tag{40}$$

A brief description of the equations and boundary conditions in Problem $\mathcal{Q}$ where, for simplicity, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$, is the following.

First, (35) represents the constitutive law in which $\mathcal{F}$ is the elasticity operator and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ represents the linearized strain tensor. Equation (36) is the equation of equilibrium in which $\boldsymbol{f}_0$ represents the density of body forces. Condition (37) represents the displacement boundary conditions that we use here since we assume that the body is held fixed on its boundary Γ_1. Next, condition (38) is the traction boundary condition in which $\boldsymbol{f}_2$ represents the density of surface tractions which act on Γ_2. Conditions (39) are the contact conditions which model the contact on Γ_3 with rigid foundation covered by a soft material of thickness $g \ge 0$. Here p is a given normal compliance function and u_ν, σ_ν are the normal displacement and the normal stress, respectively. Finally, condition (40) represents the frictionless contact condition in which $\boldsymbol{\sigma}_\tau$ denotes the tangential stress vector on the potential contact surface.

We mention that versions of Problem $\mathcal{Q}$ have been considered in [25,26] and, more recently, in [27,28]. There, more details and explanations on the construction of the model can be find. Here we restrict ourselves to mention that in the case when p vanishes then Problem $\mathcal{Q}$ represents the Signorini frictionless contact with gap and, in the particular case when $g \to \infty$, Problem $\mathcal{Q}$ becomes a frictionless problem with normal compliance. The references mentioned above in this paragraph were oriented to existence, uniqueness, optimal control and optimization results. Also, recall that Problem $\mathcal{Q}$ was already considered in [29]. There, a well-posedness result was obtained for a variational formulation in terms of displacements, different from the mixed variational formulation we present in this paper. The study of the well-posedness in the sense of Tykhonov we provide in the next section, together with the corresponding mechanical interpretations, completes our results in the references above.

The second model of contact is obtained by considering that part of the equations and boundary conditions in Problem $\mathcal{Q}$ are satisfied only approximately and, therefore, some of the equalities are replaced with inequalities. Its statement is the following.

Problem $\mathcal{Q}_\varepsilon$: *Find a displacement field* $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ *such that*

$$\|\boldsymbol{\sigma} - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})\| \le \omega(\varepsilon) \quad \text{in } \Omega, \tag{41}$$

$$\|\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0\| \le \omega_0(\varepsilon) \quad \text{in } \Omega, \tag{42}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{43}$$

$$\|\boldsymbol{\sigma}\boldsymbol{\nu} - \boldsymbol{f}_2\| \le \omega_2(\varepsilon) \quad \text{on } \Gamma_2, \tag{44}$$

$$\left.\begin{array}{l} u_\nu \le \widetilde{g}, \\ \sigma_\nu + p(u_\nu) + \widetilde{\sigma}_\nu \le 0, \\ (u_\nu - \widetilde{g})(\sigma_\nu + p(u_\nu) + \widetilde{\sigma}_\nu) = 0, \\ |\widetilde{g} - g| \le \omega_3(\varepsilon), \quad |\widetilde{\sigma}_\nu| \le \omega_\nu(\varepsilon) \end{array}\right\} \quad \text{on } \Gamma_3, \tag{45}$$

$$\|\boldsymbol{\sigma}_\tau\| \le \omega_\tau(\varepsilon) \quad \text{on } \Gamma_3. \tag{46}$$

Here ε is a positive parameter, $\omega, \omega_0, \omega_2, \omega_3, \omega_\nu, \omega_\tau$ are real-valued positive functions defined on $[0, +\infty)$ which vanish for $\varepsilon = 0$ and $\widetilde{g}, \widetilde{\sigma}_\nu$ are real-valued regular functions defined on Γ_3 which satisfy the inequality prescribed in (45). The statement of Problem $\mathcal{Q}_\varepsilon$ is based on the fact that the data in Problem $\mathcal{Q}$ are obtained by experiments and, therefore, could involve error measurements. As a result, the equations and conditions (35)–(40) are valid only approximately, and have to be replaced by inequalities. For instance, assume that the constitutive equation (35) represents only an idealization and, in fact, the stress field satisfies an equality of the form

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \widetilde{\boldsymbol{\sigma}} \quad \text{in } \Omega$$

where $\widetilde{\boldsymbol{\sigma}}$ represents a small perturbation. Then, in each point $\boldsymbol{x} \in \Omega$, the quantity $\|\boldsymbol{\sigma} - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})\|$ represents the error between the real stress field $\boldsymbol{\sigma}$ and the idealized stress $\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$ and, therefore, inequality (41) provides a bound of this error. Next, if the real density of the body forces which act on the body is $\widetilde{\boldsymbol{f}}_0$ then the equation of equilibrium has to be

$$\mathrm{Div}\,\boldsymbol{\sigma} + \widetilde{\boldsymbol{f}}_0 = \boldsymbol{0} \quad \text{in } \Omega$$

and, therefore, $\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{f}_0 - \widetilde{\boldsymbol{f}}_0$ in Ω. Thus, in each point $\boldsymbol{x} \in \Omega$, the quantity $\|\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0\|$ represents the error made when using the density $\boldsymbol{f}_0$ instead of the real density $\widetilde{\boldsymbol{f}}_0$. We conclude from here that inequality (42) provides a bound of this error. The rest of conditions (43)–(46) can be interpreted in similar way, $\widetilde{g}$ and $\widetilde{\sigma_\nu}$ being a perturbation of the thickness g and the normal stress on the contact surface, respectively.

In the variational analysis of the contact problems $\mathcal{Q}$ and $\mathcal{Q}_\varepsilon$ we use standard notation for Sobolev and Lebesgue spaces associated to $\Omega \subset \mathbb{R}^d$ and Γ and, for an element $\boldsymbol{v} \in H^1(\Omega)^d$, we usually write $\boldsymbol{v}$ for the trace $\gamma\boldsymbol{v} \in L^2(\Gamma)^d$ of $\boldsymbol{v}$ to Γ. In addition, we denote by v_ν and $\boldsymbol{v}_\tau$ its normal and tangential components on Γ given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$, respectively.

Next, we consider the spaces

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^d : \ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \,\},$$
$$Q = \{\, \boldsymbol{\sigma} = (\sigma_{ij}) : \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega), \ i, j = 1, 2, \ldots, d \,\},$$

which are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) \, \mathrm{d}x, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, \mathrm{d}x, \tag{47}$$

where $\boldsymbol{\varepsilon}$ represents the deformation operator, i.e.

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The associated norms on these spaces are denoted by $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Using standard arguments it follows that there exist two constants $c_0 > 0, m_0 > 0$ such that the following inequalities hold:

$$\int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{v})\| \, \mathrm{d}x \le c_0 \|\boldsymbol{v}\|_V, \quad \int_\Omega \|\boldsymbol{v}\| \, \mathrm{d}x \le c_0 \|\boldsymbol{v}\|_V, \quad \int_\Gamma \|\boldsymbol{v}\| \, \mathrm{d}x \le c_0 \|\boldsymbol{v}\|_V, \tag{48}$$

$$\|\gamma\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le m_0 \|\boldsymbol{v}\|_V, \tag{49}$$

for all $\boldsymbol{v} \in V$.

We now follow [23] and recall that the space $\gamma(V)$ is a closed subspace of the Hilbert space $\gamma(H^1(\Omega)^d)$ and, therefore, it is a Hilbert space. Let Y be its dual (which, in turn, can be organized as a real Hilbert space) and denote by $\langle\cdot,\cdot\rangle$ the duality pairing between Y and $\gamma(V)$. Below we shall use the short hand notation $\langle\boldsymbol{\mu}, \boldsymbol{v}\rangle$ instead of $\langle\boldsymbol{\mu}, \gamma\boldsymbol{v}\rangle$, for any $\boldsymbol{\mu} \in Y$ and $\boldsymbol{v} \in V$. Recall also that $\gamma(V)$ is continuously embedded in $L^2(\Gamma)^d$.

Next, we list the assumptions on the data of the contact problem $\mathcal{Q}$. Thus, we assume that the elasticity operator $\mathcal{F}$ satisfies the following conditions.

$$
\begin{aligned}
&\text{(a) } \mathcal{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d.\\
&\text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega,\\
&\qquad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d.\\
&\text{(c) The mapping } \boldsymbol{\varepsilon} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}) : \mathbb{S}^d \to \mathbb{S}^d \text{ is continuous,}\\
&\quad \text{a.e. } \boldsymbol{x} \in \Omega.\\
&\text{(d) There exist } d_0 > 0 \text{ and } d_1 > 0 \text{ such that}\\
&\qquad \|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon})\| \le d_0 + d_1\|\boldsymbol{\varepsilon}\| \ \text{ for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega.
\end{aligned}
\tag{50}
$$

We also assume that the densities of body forces and tractions and the bound of the normal displacement are such that

$$\boldsymbol{f}_0 \in L^2(\Omega)^d, \quad \boldsymbol{f}_2 \in L^2(\Gamma_2)^d. \tag{51}$$

$$g \ge 0. \tag{52}$$

Moreover, the normal compliance function p satisfies the following conditions.

$$
\begin{aligned}
&\text{(a) } p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+.\\
&\text{(b) The mapping } \boldsymbol{x} \mapsto p(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3,\\
&\quad \text{for any } r \in \mathbb{R}.\\
&\text{(c) The mapping } r \mapsto p(\boldsymbol{x}, r) : \mathbb{R} \to \mathbb{R} \text{ is continuous,}\\
&\quad \text{a.e. } \boldsymbol{x} \in \Gamma_3.\\
&\text{(d) There exist } e_0 > 0 \text{ and } e_1 > 0 \text{ such that}\\
&\quad |p(\boldsymbol{x}, r)| \le e_0 + e_1|r| \ \text{ for any } r \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3.\\
&\text{(e) } p(\boldsymbol{x}, r) = 0 \ \text{ for any } r \le 0, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3.
\end{aligned}
\tag{53}
$$

Finally, we assume that

$$\text{there exists } \boldsymbol{\phi} \in V \ \text{ such that } \phi_\nu = 1 \ \text{ a.e. on } \Gamma_3 \tag{54}$$

and, in addition,

$$
\begin{aligned}
&\omega,\ \omega_0,\ \omega_2,\ \omega_3,\ \omega_\nu,\ \omega_\tau : [0, +\infty) \to [0, +\infty)\\
&\text{are continuous increasing functions such that}\\
&\omega(0) = \omega_0(0) = \omega_2(0) = \omega_3(0) = \omega_\nu(0) = \omega_\tau(0) = 0.
\end{aligned}
\tag{55}
$$

Note that under these assumptions all the integrals we use below in this section make sense. Moreover, using assumption (55) it is easy to see that Problem $\mathcal{Q}$ can be obtained from Problem $\mathcal{Q}_\varepsilon$ in the particular case when $\varepsilon = 0$.

Next, we introduce the operator $A : V \to V$, the form $b : V \times Y \to \mathbb{R}$, the sets $U \subset V$, $\Lambda \subset Y$, the element $\boldsymbol{f} \in V$ and the function $\theta : [0, +\infty) \to [0, +\infty)$

defined as follows:

$$(A\boldsymbol{u},\boldsymbol{v})_V = \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x + \int_{\Gamma_3} p(u_\nu)v_\nu\,\mathrm{d}a \quad \forall\,\boldsymbol{u},\,\boldsymbol{v}\in V, \tag{56}$$

$$b(\boldsymbol{v},\boldsymbol{\mu}) = \langle\boldsymbol{\mu},\boldsymbol{v}\rangle, \qquad \forall\,\boldsymbol{v}\in V,\ \boldsymbol{\mu}\in Y, \tag{57}$$

$$U = \{\,\boldsymbol{v}\in V\ :\ v_\nu \le 0 \text{ a.e. on } \Gamma_3\,\}. \tag{58}$$

$$\Lambda = \{\,\boldsymbol{\mu}\in Y\ :\ \langle\boldsymbol{\mu},\,\boldsymbol{v}\rangle \le 0 \quad \forall\,\boldsymbol{v}\in U\,\}, \tag{59}$$

$$(\boldsymbol{f},\boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_2}\boldsymbol{f}_2\cdot\boldsymbol{v}\,\mathrm{d}a \quad \forall\,\boldsymbol{v}\in V, \tag{60}$$

$$\theta(\varepsilon) = c_0\big(\omega(\varepsilon)+\omega_0(\varepsilon)+\omega_2(\varepsilon)+\omega_\nu(\varepsilon)+\omega_\tau(\varepsilon)\big) + m_0\omega_3(\varepsilon)\|\boldsymbol{\phi}\|_V \quad \forall\,\varepsilon\ge 0. \tag{61}$$

Due to the inequalities involved in its statement, the variational formulations of Problem $\mathcal{Q}_\varepsilon$ is obtained by using nonstandard arguments that we present in what follows. Let $\varepsilon \ge 0$, let $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ be regular functions which verify (41)–(46) and let $\boldsymbol{v}\in V$, $\boldsymbol{\mu}\in\Lambda$. Then, using an integration by parts it follows that

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x + \int_\Omega \mathrm{Div}\,\boldsymbol{\sigma}\cdot\boldsymbol{v}\,\mathrm{d}x = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{v}\,\mathrm{d}a$$

and, therefore,

$$\begin{aligned}\int_\Omega (\boldsymbol{\sigma}-\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}))\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x &+ \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x + \int_\Omega (\mathrm{Div}\,\boldsymbol{\sigma}+\boldsymbol{f}_0)\cdot\boldsymbol{v}\,\mathrm{d}x\\ &= \int_\Omega \boldsymbol{f}_0\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_1}\boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{v}\,\mathrm{d}a + \int_{\Gamma_2}(\boldsymbol{\sigma}\boldsymbol{\nu}-\boldsymbol{f}_2)\cdot\boldsymbol{v}\,\mathrm{d}a + \int_{\Gamma_2}\boldsymbol{f}_2\cdot\boldsymbol{v}\,\mathrm{d}a\\ &\quad + \int_{\Gamma_3}\sigma_\nu v_\nu\,\mathrm{d}a + \int_{\Gamma_3}\boldsymbol{\sigma}_\tau\cdot\boldsymbol{v}_\tau\,\mathrm{d}a.\end{aligned} \tag{62}$$

We now introduce the Lagrange multiplier $\boldsymbol{\lambda}\in Y$ defined by

$$\langle\boldsymbol{\lambda},\boldsymbol{w}\rangle = -\int_{\Gamma_3}(\sigma_\nu + p(u_\nu) + \widetilde{\sigma}_\nu)w_\nu\,\mathrm{d}a \quad \forall\,\boldsymbol{w}\in V \tag{63}$$

and note that (63) and (57) imply that

$$\int_{\Gamma_3}\sigma_\nu v_\nu\,\mathrm{d}a = -b(\boldsymbol{v},\boldsymbol{\lambda}) - \int_{\Gamma_3} p(u_\nu)v_\nu\,\mathrm{d}a - \int_{\Gamma_3}\widetilde{\sigma}_\nu v_\nu\,\mathrm{d}a. \tag{64}$$

We now combine the equality (62) with (64), and then we use the definitions (56), (60) and equality $\boldsymbol{v} = \mathbf{0}$ a.e. on Γ_1 to deduce that

$$(A\boldsymbol{u}, \boldsymbol{v})_V + b(\boldsymbol{v}, \boldsymbol{\lambda}) = (\boldsymbol{f}, \boldsymbol{v})_V + \int_\Omega (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\sigma}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x \\ - \int_\Omega (\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0) \cdot \boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_2} (\boldsymbol{\sigma}\boldsymbol{\nu} - \boldsymbol{f}_2) \cdot \boldsymbol{v}\,\mathrm{d}a \\ - \int_{\Gamma_3} \widetilde{\sigma}_\nu v_\nu\,\mathrm{d}a + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau\,\mathrm{d}a. \tag{65}$$

Next, we use (41), (42), (44)–(46) to see that

$$\int_\Omega (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\sigma}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x - \int_\Omega (\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0) \cdot \boldsymbol{v}\,\mathrm{d}x \\ + \int_{\Gamma_2} (\boldsymbol{\sigma}\boldsymbol{\nu} - \boldsymbol{f}_2) \cdot \boldsymbol{v}\,\mathrm{d}a - \int_{\Gamma_3} \widetilde{\sigma}_\nu v_\nu\,\mathrm{d}a + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau\,\mathrm{d}a \\ \le \omega(\varepsilon) \int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|\,\mathrm{d}x + \omega_0(\varepsilon) \int_\Omega \|\boldsymbol{v}\|\,\mathrm{d}x + \omega_2(\varepsilon) \int_{\Gamma_2} \|\boldsymbol{v}\|\,\mathrm{d}a \\ + \omega_\nu(\varepsilon) \int_{\Gamma_3} |v_\nu|\,\mathrm{d}a + \omega_\tau(\varepsilon) \int_{\Gamma_3} \|\boldsymbol{v}_\tau\|\,\mathrm{d}a$$

and, using inequalities (48) combined with notation (61), we find that

$$\int_\Omega (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\sigma}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x - \int_\Omega (\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0) \cdot \boldsymbol{v}\,\mathrm{d}x \\ + \int_{\Gamma_2} (\boldsymbol{\sigma}\boldsymbol{\nu} - \boldsymbol{f}_2) \cdot \boldsymbol{v}\,\mathrm{d}a - \int_{\Gamma_3} \widetilde{\sigma}_\nu v_\nu\,\mathrm{d}a + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau\,\mathrm{d}a \le \theta(\varepsilon)\|\boldsymbol{v}\|_V. \tag{66}$$

Finally, we use (65) and (66) to obtain that

$$(A\boldsymbol{u}, \boldsymbol{v})_V + b(\boldsymbol{v}, \boldsymbol{\lambda}) \le (\boldsymbol{f}, \boldsymbol{v})_V + \theta(\varepsilon)\|\boldsymbol{v}\|_V. \tag{67}$$

On the other hand, using definition (63), condition (45) and notation (58), (59), we deduce that $\boldsymbol{\lambda} \in \Lambda$. Moreover, using assumption (54) and the definition (57) of the bilinear form b it is easy to see that

$$b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) = b(\boldsymbol{u} - \widetilde{g}\boldsymbol{\phi}, \boldsymbol{\mu} - \boldsymbol{\lambda}) + b((\widetilde{g} - g)\boldsymbol{\phi}, \boldsymbol{\mu} - \boldsymbol{\lambda}) + b(g\boldsymbol{\phi}, \boldsymbol{\mu} - \boldsymbol{\lambda}) \\ = \langle \boldsymbol{\mu} - \boldsymbol{\lambda}, \boldsymbol{u} - \widetilde{g}\boldsymbol{\phi} \rangle + \langle \boldsymbol{\mu} - \boldsymbol{\lambda}, (\widetilde{g} - g)\boldsymbol{\phi} \rangle + b(g\boldsymbol{\phi}, \boldsymbol{\mu} - \boldsymbol{\lambda}).$$

Therefore, using inequalities $|\widetilde{g} - g| \le \omega_3(\varepsilon)$ and (49) we find that

$$b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) \le \langle \boldsymbol{\mu}, \boldsymbol{u} - \widetilde{g}\boldsymbol{\phi} \rangle - \langle \boldsymbol{\lambda}, \boldsymbol{u} - \widetilde{g}\boldsymbol{\phi} \rangle \\ + m_0\omega_3(\varepsilon)\|\boldsymbol{\phi}\|_V \|\boldsymbol{\mu} - \boldsymbol{\lambda}\|_Y + b(g\boldsymbol{\phi}, \boldsymbol{\mu} - \boldsymbol{\lambda}). \tag{68}$$

In addition, (45), (52) and (54) imply that

$$\boldsymbol{u} - \widetilde{g}\boldsymbol{\phi} \in U, \quad \langle \boldsymbol{\lambda}, \boldsymbol{u} \rangle = \langle \boldsymbol{\lambda}, \widetilde{g}\boldsymbol{\phi} \rangle.$$

Therefore,

$$\langle \boldsymbol{\mu}, \boldsymbol{u} - \widetilde{g}\boldsymbol{\phi} \rangle \leq 0 \quad \text{and} \quad \langle \boldsymbol{\lambda}, \boldsymbol{u} - \widetilde{g}\boldsymbol{\phi} \rangle = 0. \tag{69}$$

We combine now (68), (69) and use definition (61) to deduce that

$$b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) \leq b(g\boldsymbol{\phi}, \boldsymbol{\mu} - \boldsymbol{\lambda}) + \theta(\varepsilon)\|\boldsymbol{\mu} - \boldsymbol{\lambda}\|_Y. \tag{70}$$

The inequalities (67) and (70) lead us to the following variational formulation of Problem $\mathcal{Q}_\varepsilon$.

Problem $\mathcal{Q}_\varepsilon^V$: *Find a displacement field $\boldsymbol{u} \in V$ and a Lagrange multiplier $\boldsymbol{\lambda} \in \Lambda$ such that*

$$(A\boldsymbol{u}, \boldsymbol{v})_V + b(\boldsymbol{v}, \boldsymbol{\lambda}) \leq (\boldsymbol{f}, \boldsymbol{v})_V + \theta(\varepsilon)\|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V, \tag{71}$$

$$b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) \leq b(g\boldsymbol{\phi}, \boldsymbol{\mu} - \boldsymbol{\lambda}) + \theta(\varepsilon)\|\boldsymbol{\mu} - \boldsymbol{\lambda}\|_Y \quad \forall\, \boldsymbol{\mu} \in \Lambda. \tag{72}$$

We now use (55) to see that Problem $\mathcal{Q}$ can be recovered from Problem $\mathcal{Q}_\varepsilon$, when $\varepsilon = 0$. Therefore, since $\theta(0) = 0$, we deduce from above the following variational formulation of Problem $\mathcal{Q}$.

Problem $\mathcal{Q}^V$: *Find a displacement field $\boldsymbol{u} \in V$ and a Lagrange multiplier $\boldsymbol{\lambda} \in \Lambda$ such that*

$$(A\boldsymbol{u}, \boldsymbol{v})_V + b(\boldsymbol{v}, \boldsymbol{\lambda}) = (\boldsymbol{f}, \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V, \tag{73}$$

$$b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) \leq b(g\boldsymbol{\phi}, \boldsymbol{\mu} - \boldsymbol{\lambda}) \quad \forall\, \boldsymbol{\mu} \in \Lambda. \tag{74}$$

A couple functions $(\boldsymbol{u}, \boldsymbol{\lambda}) \in V \times \Lambda$ which satisfies (73), (74) is called a weak solution to the contact problem $\mathcal{Q}$ and couple of functions $(\boldsymbol{u}, \boldsymbol{\lambda}) \in V \times \Lambda$ which satisfies (71), (72) is called a weak solution to the contact problem $\mathcal{Q}_\varepsilon$.

4. Well-posedness results

In this section we apply our abstract results in Section 2 in the study of Problems $\mathcal{Q}^V$ and $\mathcal{Q}_\varepsilon^V$ and provide the corresponding mechanical interpretations. To this end we restate Definitions 1.1 and 1.2 in the context of the Problems $\mathcal{Q}^V$ and $\mathcal{Q}_\varepsilon^V$.

Definition 4.1: A sequence $\{(u_n, \lambda_n)\} \subset V \times Y$ is called an approximating sequence if there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}$ such that $\varepsilon_n > 0$, (u_n, λ_n) is a solution of Problem $\mathcal{Q}_{\varepsilon_n}^V$ for each $n \in \mathbb{N}$ and, moreover, $\varepsilon_n \to 0$ as $n \to \infty$.

Definition 4.2: The Problem $\mathcal{Q}^V$ is said to be well-posed if it has a unique solution and this solution is the limit in $V \times Y$ to any approximating sequence.

Moreover, we consider the following additional assumptions.

$$\left.\begin{array}{l}\text{There exists } m_{\mathcal{F}} > 0 \text{ such that}\\ \quad (\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2) \geq m_{\mathcal{F}}\,\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2\\ \qquad \text{for any } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\end{array}\right. \tag{75}$$

$$\left.\begin{array}{l}\text{There exists } L_{\mathcal{F}} > 0 \text{ such that}\\ \quad \|\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{F}}\,\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2\\ \qquad \text{for any } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\end{array}\right. \tag{76}$$

$$\left.\begin{array}{l}\text{There exists } L_p > 0 \text{ such that}\\ \quad |p(\boldsymbol{x},r_1) - p(\boldsymbol{x},r_2)| \leq L_p|r_1-r_2|\\ \qquad \text{for any } r_1,\ r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\end{array}\right. \tag{77}$$

$$\left.\begin{array}{l}(p(\boldsymbol{x},r_1) - p(\boldsymbol{x},r_2))(r_1-r_2) \geq 0\\ \qquad \text{for any } r_1,\ r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\end{array}\right. \tag{78}$$

$$m_0^2 L_p < m_{\mathcal{F}}. \tag{79}$$

Our first result in this section is the following.

Theorem 4.3: *Assume* (50)–(55). *Then, the following statements are equivalent*:

(a) *Problem $\mathcal{Q}^V$ is well-posed.*
(b) *Problem $\mathcal{Q}^V$ has at least one solution and the diameter of the set of solutions of Problem $\mathcal{Q}^V_\varepsilon$ converges to zero as $\varepsilon \to 0$.*
(c) *Problem $\mathcal{Q}^V_\varepsilon$ has at least one solution, for each $\varepsilon > 0$, and the diameter of the set of its solutions converges to zero as $\varepsilon \to 0$.*

Proof: First, we prove that, the operator $A : V \to V$ defined by 56 is demicontinuous. Let $\{\boldsymbol{u}_n\} \subset V$ be such that

$$\boldsymbol{u}_n \to \boldsymbol{u} \quad \text{in } V.$$

Then, the following convergences hold:

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_n) \to \boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } Q \quad \text{and} \quad u_{n\nu} \to u_\nu \quad \text{in } L^2(\Gamma_3). \tag{80}$$

Moreover, assumptions (50) and (53) allow us to apply Krasnoselski's Theorem (see [30, p.60]) to see that the operators $\boldsymbol{\sigma} \mapsto \mathcal{F}(\boldsymbol{\sigma}) : Q \to Q$ and $\xi \mapsto p(\xi) : L^2(\Gamma_3) \to L^2(\Gamma_3)$ are continuous. Therefore, the convergences (80) yield

$$\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) \to \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } Q \quad \text{and} \quad p(u_{n\nu}) \to p(u_\nu) \quad \text{in } L^2(\Gamma_3). \tag{81}$$

It follows from here that

$$(A\boldsymbol{u}_n,\boldsymbol{v})_V \to (A\boldsymbol{u},\boldsymbol{v})_V \quad \forall\, \boldsymbol{v}\in V,$$

which shows that A is demicontinuous and, therefore, it satisfies condition (7). Moreover, the form b given by (57) satisfies conditions (8), (9). For the proof of this statement we refer the reader to [23], for instance. On the other hand, assumption (55) guarantees that the function θ defined by (61) verifies conditions (10) and (11). In addition, the set Λ defined by (59) satisfies condition (12) and, finally, assumptions (51), (52), (54), imply (13) for the element $\boldsymbol{f}$ given by (60) and $\boldsymbol{h} = g\boldsymbol{\phi}$. Therefore, Theorem 4.4 is now a direct consequence of Corollary 2.5, applied with $X = V$ an $K = U$. ■

We now proceed with the following result.

Theorem 4.4: *Assume* (50)–(55), (75)–(77) *and either* (78) *or* (79). *Then, the following statements hold*:

(a) *Problem $\mathcal{Q}^V$ has a unique solution.*
(b) *Problem $\mathcal{Q}^V_\varepsilon$ has at least one solution, for each $\varepsilon > 0$.*
(c) *Every approximating sequence converges to the solution of Problem $\mathcal{Q}^V$.*
(d) *Problem $\mathcal{Q}^V$ is well-posed.*
(e) *The solution of Problem $\mathcal{Q}^V$ depends Lipschitz continuously on the data $\boldsymbol{f}_0$,$\boldsymbol{f}_2$ and g.*

Proof: We first prove that the operator A is strongly monotone and Lipschitz continuous. Let $\boldsymbol{u}$, $\boldsymbol{v} \in V$. We use (56) to see that

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_V = \int_\Omega (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{v})) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{v}))\,\mathrm{d}x \\ + \int_{\Gamma_3} (p(u_\nu) - p(v_\nu))(u_\nu - v_\nu)\,\mathrm{d}a. \tag{82}$$

Assume (78) holds. Then (82) and (75) yield

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_V \geq m_{\mathcal{F}} \|\boldsymbol{u} - \boldsymbol{v}\|_V^2 \tag{83}$$

and, therefore, condition (6)(a) holds with $m = m_{\mathcal{F}} > 0$. Assume now (79). Then, using (82) and (75) we find that

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_V \geq m_{\mathcal{F}} \int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{v})\|^2\,\mathrm{d}x \\ - \int_{\Gamma_3} |p(u_\nu) - p(v_\nu)|\,|(u_\nu - v_\nu)|\,\mathrm{d}a \\ \geq m_{\mathcal{F}} \int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{v})\|^2\,\mathrm{d}x - L_p \int_{\Gamma_3} |u_\nu - v_\nu|^2\,\mathrm{d}a$$

and, therefore, (49) yields

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_V \geq (m_{\mathcal{F}} - m_0^2 L_p) \|\boldsymbol{u} - \boldsymbol{v}\|_V^2.$$

We now use inequality (83) and the smallness assumption (79) to see that the operator $A : V \to V$ defined by (56) still satisfies condition (6)(a) with $m =$

$m_{\mathcal{F}} - m_0^2 L_p > 0$. Next, Let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in V$. We use (56) to see that

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{w})_V = \int_\Omega (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{v})) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w})\,\mathrm{d}x + \int_{\Gamma_3} (p(u_\nu) - p(v_\nu)) w_\nu \,\mathrm{d}a \tag{84}$$

and, therefore, assumptions (76), (77) combined with inequality (49) show that

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{w})_V \le (L_{\mathcal{F}} + m_0^2 L_p)\|\boldsymbol{u} - \boldsymbol{v}\|_V \|\boldsymbol{w}\|_V.$$

It follows from here that

$$\|A\boldsymbol{u} - A\boldsymbol{v}\|_V \le (L_{\mathcal{F}} + m_0^2 L_p)\|\boldsymbol{u} - \boldsymbol{v}\|_V \tag{85}$$

which shows that condition (6)(b) holds with $L = L_{\mathcal{F}} + m_0^2 L_p$. Note also that, obviously, condition (8), (9), (10), (12), (13) are satisfied for b, θ, Λ, $\boldsymbol{f}$ and h defined in Section 3. Theorem 4.4 is now a direct consequence of Corollary 2.7 and Theorem 2.8, applied with $X = V$ an $K = U$. ■

Let us remark that Theorem 4.3 provide equivalence results. These statements do not guarantee that Problem $\mathcal{Q}^V$ is well-posed. In contrast, Theorem 4.4 provides sufficient conditions which guarantee the well-posedness of Problem $\mathcal{Q}^V$.

We end this section with some comments and mechanical interpretations. First, we remark that any weak solution of the elastic frictionless contact problem $\mathcal{Q}$ is weak solution to the elastic frictional contact problem $\mathcal{Q}_\varepsilon$. This implies that the weak solvability of Problem $\mathcal{Q}$ implies the weak solvability of Problem $\mathcal{Q}_\varepsilon$.

Next, we underline the importance of Theorem 4.4 which provides existence, uniqueness and convergence results between contact problems which have a different feature and are formulated in terms of different mechanical assumptions. For instance, Problem $\mathcal{Q}$ is frictionless while Problem $\mathcal{Q}_\varepsilon$ could be frictional; the elastic constitutive law in Problem $\mathcal{Q}$ is strongly monotone but the elastic constitutive law in Problem $\mathcal{Q}_\varepsilon$ could fail to be monotone; the normal compliance function law in Problem $\mathcal{Q}$ is monotone (or Lipschitz continuous) but the normal compliance function law in Problem $\mathcal{Q}_\varepsilon$ could fail to be monotone (or Lipschitz continuous). In fact, these results show that small perturbation on the constitutive law, densities of applied forces and contact boundary condition lead to small perturbation to the solutions of the corresponding problems. Moreover, in the case when $\omega = \omega_3 = \omega_\nu = \omega_\tau = 0$ these results provide a continuous dependence result of the weak solution of Problem $\mathcal{Q}$ with respect to the densities of body forces and tractions.

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