Nonlinear Analysis: Real World Applications 50 (2019) 86–103

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Optimization problems for a viscoelastic frictional contact problem with unilateral constraints

Mircea Sofonea [a,b], Yi-bin Xiao [a,*], Maxime Couderc [b]

[a] School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan, 611731, PR China
[b] Laboratoire de Mathématiques et Physique, University of Perpignan Via Domitia, 52 Avenue Paul Alduy, 66860 Perpignan, France

A R T I C L E I N F O

Article history:
Received 23 October 2018
Accepted 10 April 2019
Available online 6 May 2019

Keywords:
Viscoelastic material
Frictional contact
Unilateral constraint
Weak solution
Convergence results
Optimization problems

A B S T R A C T

We consider a mathematical model which describes the contact between a viscoelastic body and a rigid-deformable foundation with memory effects. We derive a variational formulation of the model which is in the form of a history-dependent variational inequality for the displacement field. Then we prove the existence of a unique weak solution to the problem. We also study the continuous dependence of the solution with respect to the data and prove two convergence results, under different assumptions on the data. The proofs are based on arguments of lower semicontinuity, pseudomonotonicity, and compactness. Finally, we use our convergence results in the study of several optimization problems associated to the viscoelastic contact model.

1. Introduction

Processes of contact between deformable bodies are very frequent in industry and daily life. A few simple examples are brake pads in contact with wheels, pistons with skirts, shoes with floor. Because of their importance in mechanical systems, industrial process and various real word application, a large effort has been put into the modeling, analysis and numerical simulations of contact processes. The literature on this field is extensive, both the engineering and the mathematical one. The publications in mathematical literature deal with the variational analysis of various models of contact, which are expressed in terms of strongly elliptic, time dependent or evolutionary nonlinear boundary value problems. References in the field include [1–5]. There, various existence and uniqueness results have been proved, by using the tools of variational and hemivariational inequalities which have been studied extensively in recent years (see, for example, [6–11]). Once existence, uniqueness or nonuniqueness, and stability of solutions have been established, related important questions arise, such as the optimal control of contact problems, the

* Corresponding author.
E-mail address: xiaoyb9999@hotmail.com (Y.-b. Xiao).

https://doi.org/10.1016/j.nonrwa.2019.04.005
1468-1218/

numerical analysis of the solutions and how to construct reliable and efficient algorithms for their numerical approximations with guaranteed convergence. Results on optimal control for various contact problems with elastic materials could be found in [6,12–21]. References on numerical analysis of contact problems within the framework of linearized strain theory include [22–26].

The optimal shape design of contact processes represents a topic of considerable theoretical and applied interest. Indeed, in most applications this is the main interest of the design engineer. Related issues are the observability properties of the contact models and parameter identification. The need to study the continuous dependence of the solution to contact problems with respect to the data and parameters is currently widely recognized, since it plays a crucial role in solving control and optimal design problems related to various mechanical structures. Unfortunately, there are quite few results in the literature related to this topic. The reason arises from the intrinsic structure of the contact models, which usually involve both nonlinear operators, unilateral constraints and nondifferentiable functionals which depend on those parameters.

Our aim in this paper is to study some optimization problems related to a quasistatic frictional contact model with viscoelastic materials. The model describes the contact with a rigid-deformable foundation and takes into account its memory effects, which represents the first trait of novelty of this paper. Besides the densities of body forces and tractions, denoted by $\boldsymbol{f}_0$ and $\boldsymbol{f}_2$, the model is governed by two positive parameters, g and μ, which represent the thickness of the deformable layer of the foundation and the coefficient of friction, respectively. We state the contact problem and provide its variational formulation, which is in a form of a history-dependent variational inequality for the displacement field. Then we prove its unique weak solvability. Next, we state and prove a convergence result of the solution with respect to the data $\boldsymbol{f}_0$, $\boldsymbol{f}_2$ and parameters g and μ, which represents the second trait of novelty of this paper. Finally, we use this convergence result in order to prove the solvability of some optimization problems related to our contact model, which represents the third novelty of our paper. The theoretical results we present here illustrate the cross fertilization between models and applications, on one hand, and the nonlinear functional analysis, on the other hand. Moreover, they could be used in the optimal design for contact processes in real word, as well as in the study of their optimal control.

The rest of the paper is structured as follows. In Section 2 we introduce the contact model. Then, in Section 3 we list the assumptions on the data, derive its variational formulation and prove its unique weak solvability. The proof is based on arguments of history-dependent variational inequalities presented in [27]. In Section 4 we study the continuous dependence of the solution with respect to the data and prove a convergence result. The proof is based on arguments of monotonicity, pseudomonotonicity and compactness. Finally, in Section 5 we introduce two classes of optimization problems related to the contact model and provide their solvability. The proofs are based on the convergence result in Section 4, combined with a Weierstrass-type argument.

2. The contact model

We consider a viscoelastic body which occupies the domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, with smooth boundary Γ. The boundary Γ is divided into three measurable disjoint parts Γ_1, Γ_2 and Γ_3 such that $meas\,(\Gamma_1) > 0$. The body is fixed on Γ_1, is acted upon by time-dependent surface traction on Γ_2 and is in frictional contact on Γ_3 with a rigid-deformable obstacle, the so-called foundation. The contact process is quasistatic and the time interval of interest is $\mathbb{R}_+ = [0, +\infty)$. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The inner product, norm and zero element of the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$ will be denoted by "$\cdot$", $\|\cdot\|$ and $\mathbf{0}$, respectively. Moreover, we use the index ν and

τ for the normal and tangential components of vectors and tensors, respectively. Then, the mathematical model we use to describe the equilibrium of the body in this physical setting is the following.

Problem $\mathcal{P}$. Find a displacement field $\boldsymbol{u} = (u_i) \colon \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} = (\sigma_{ij}) \colon \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ and two interface functions $\eta_\nu \colon \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}$, $\xi_\nu \colon \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}$ such that, for all $t \in \mathbb{R}_+$,

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds \qquad \text{in } \Omega, \tag{2.1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \qquad \text{in } \Omega, \tag{2.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{2.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \qquad \text{on } \Gamma_2, \tag{2.4}$$

$$\left.\begin{array}{l} u_\nu(t) \le g, \\ \sigma_\nu(t) + \eta_\nu(t) + \xi_\nu(t) \le 0, \\ (u_\nu(t) - g)(\sigma_\nu(t) + \eta_\nu(t) + \xi_\nu(t)) = 0, \\ 0 \le \eta_\nu(t) \le F\Big(\displaystyle\int_0^t u_\nu^+(s)\,ds\Big), \\ \eta_\nu(t) = \begin{cases} 0 & \text{if} \quad u_\nu(t) < 0, \\ F\Big(\displaystyle\int_0^t u_\nu^+(s)\,ds\Big) & \text{if} \quad u_\nu(t) > 0, \end{cases} \\ \xi_\nu(t) = p(u_\nu(t)) \end{array}\right\} \text{on } \Gamma_3, \tag{2.5}$$

$$\left.\begin{array}{l} \|\boldsymbol{\sigma}_\tau(t)\| \le \mu\, p(u_\nu(t)), \\ -\boldsymbol{\sigma}_\tau(t) = \mu\, p(u_\nu(t))\, \frac{\boldsymbol{u}_\tau(t)}{\|\boldsymbol{u}_\tau(t)\|} \quad \text{if} \quad \boldsymbol{u}_\tau(t) \ne \boldsymbol{0} \end{array}\right\} \text{on } \Gamma_3. \tag{2.6}$$

We now provide a description of the equations and boundary conditions in Problem $\mathcal{P}$ where, for simplicity, we do not mention the dependence of various functions with respect to the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$.

First, Eq. (2.1) represents the viscoelastic constitutive law of the material in which $\mathcal{A}$ is the elasticity operator and $\mathcal{B}$ is the relaxation tensor. Eq. (2.2) is the equation of equilibrium in which $\boldsymbol{f}_0$ denotes the density of body forces, assumed to be time-dependent. We use it here since we neglect the inertial term in the equation of motion. Conditions (2.3), (2.4) represent the displacement and traction boundary conditions, respectively, where $\boldsymbol{\nu}$ denotes the unit outward normal to Γ and $\boldsymbol{f}_2$ represents the density of surface tractions which, again, are assumed to be time-dependent.

Next, condition (2.5) represents the contact condition in which $g > 0$, p and F are given positive functions which will be described below and r^+ represents the positive part of r, i.e., $r^+ = \max\{r, 0\}$. This condition was introduced in [9, p. 247] and used in the study of a time-dependent frictionless contact problem with elastic materials. There, various mechanical interpretations can be found and, for this reason, we do not provide here the physical assumptions which lead to this boundary condition. We restrict ourselves to mention that it models the contact with a foundation made of a rigid body covered by a rigid-deformable layer of thickness g which involves memory effects. The function p represents the normal compliance function (which depends on the current penetration) and F represents the yield limit (assumed to depend on the accumulated penetration).

Finally, condition (2.6) represents a version of the Coulomb's law of dry friction, in which $\mu \ge 0$ denotes the coefficient of friction. Details and mechanical interpretation concerning this law can be found in [5,28] and the references therein. Here we restrict ourselves to mention that this law describes a transition between the classical version of Coulomb's law (which governs the contact with the rigid-deformable layer) and the Tresca friction law (which governs the contact with the rigid body).

In the analysis of Problem $\mathcal{P}$, besides the standard Lebesgue and Sobolev spaces associated to Ω and Γ, we use the spaces

$$V = \{\, \boldsymbol{v} = (v_i) \in H^1(\Omega)^d : \ \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma_1 \,\},$$
$$Q = \{\, \boldsymbol{\sigma} = (\sigma_{ij}) : \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \,\}.$$

These are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx, \tag{2.7}$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Here, $\boldsymbol{\varepsilon} : V \to Q$ represents the linearized deformation operator, that is

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = \frac{1}{2}\Big(\nabla \boldsymbol{v} + \nabla^T \boldsymbol{v}\Big) \qquad \forall\, \boldsymbol{v} \in V.$$

For an element $\boldsymbol{v} \in V$ we still write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ to Γ. Moreover, the normal and tangential components of $\boldsymbol{v}$ on Γ are given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$, respectively. In addition, we denote by $\|\gamma\|$ the norm of the trace operator $\gamma : V \to L^2(\Gamma_3)^d$ and recall that the following inequality holds:

$$\|\gamma \boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le \|\gamma\| \|\boldsymbol{v}\|_V \qquad \forall\, \boldsymbol{v} \in V. \tag{2.8}$$

For a regular function $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ we have $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$ and, moreover, the following Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \int_\Omega \mathrm{Div}\, \boldsymbol{\sigma} \cdot \boldsymbol{v}\, dx = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}\, da \quad \text{ for all } \boldsymbol{v} \in H^1(\Omega)^d. \tag{2.9}$$

Finally, we need the space of fourth order symmetric tensors $\mathbf{Q}_\infty$ given by

$$\mathbf{Q}_\infty = \{\, \mathcal{E} = (e_{ijkl}) \mid e_{ijkl} = e_{jikl} = e_{klij} \in L^\infty(\Omega), \ 1 \le i,j,k,l \le d \,\}\,. \tag{2.10}$$

It is easy to see that $\mathbf{Q}_\infty$ is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \max_{0 \le i,j,k,l \le d} \|e_{ijkl}\|_{L^\infty(\Omega)}$$

and, moreover,

$$\|\mathcal{E}\boldsymbol{\tau}\|_Q \le d\, \|\mathcal{E}\|_{\mathbf{Q}_\infty} \|\boldsymbol{\tau}\|_Q \quad \text{for all } \ \mathcal{E} \in \mathbf{Q}_\infty, \ \boldsymbol{\tau} \in Q. \tag{2.11}$$

Inequalities (2.8) and (2.11) will be used in various places in the next sections.

3. Existence and uniqueness

We start this section with an abstract existence and uniqueness result that we need in the study of Problem $\mathcal{P}$. For each normed space X we use the notation $\|\cdot\|_X$ and 0_X for the norm and the zero element of X, respectively. Also, $C(\mathbb{R}_+; X)$ will represent the space of continuous functions defined on $\mathbb{R}_+$ with values in X. For a subset $K \subset X$ we still use the symbol $C(\mathbb{R}_+; K)$ for the set of continuous functions defined on $\mathbb{R}_+$ with values in K. It is well known that, if X is a Banach space, then $C(\mathbb{R}_+; X)$ can be organized in a canonical way as a Fréchet space, i.e. as a complete metric space in which the corresponding topology is induced by a countable family of seminorms. We also recall that the convergence of a sequence $\{x_n\}$ to the element x, in the space $C(\mathbb{R}_+; X)$, can be described as follows:

$$\begin{cases} x_n \to x \quad \text{in } C(\mathbb{R}_+; X) \ \text{ as } \ n \to \infty \ \text{ if and only if} \\ \displaystyle\max_{r \in [0,m]} \|x_n(r) - x(r)\|_X \to 0 \ \text{ as } n \to \infty, \ \text{ for all } \ m \in \mathbb{N}. \end{cases} \tag{3.1}$$

Consider now a real Hilbert space X with inner product $(\cdot,\cdot)_X$ and associated norm $\|\cdot\|_X$ and let Z be a normed space. Also, let K be a subset of X, let $A : X \to X$, $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Z)$ be two operators, and let $j : Z \times X \times X \to \mathbb{R}$, $f : \mathbb{R}_+ \to X$ be two functions. With these data we consider the following problem.

Problem $\mathcal{Q}$. Find a function $u : \mathbb{R}_+ \to X$ such that, for all $t \in \mathbb{R}_+$, the inequality below holds:

$$u(t) \in K, \qquad (Au(t), v - u(t))_X + j(\mathcal{S}u(t), u(t), v) \\ -j(\mathcal{S}u(t), u(t), u(t)) \ge (f(t), v - u(t))_X \quad \forall\, v \in K. \tag{3.2}$$

In the study of Problem $\mathcal{P}$ we consider the following assumptions.

$$K \text{ is a nonempty closed convex subset of } X, \tag{3.3}$$

$$\begin{cases} \text{(a) There exists } m_A > 0 \text{ such that} \\ \qquad (Au_1 - Au_2, u_1 - u_2)_X \ge m_A \|u_1 - u_2\|_X^2 \\ \qquad\quad \forall\, u_1,\, u_2 \in X. \\ \text{(b) There exists } L_A > 0 \text{ such that} \\ \qquad \|Au_1 - Au_2\|_X \le L_A \|u_1 - u_2\|_X \quad \forall\, u_1,\, u_2 \in X. \end{cases} \tag{3.4}$$

$$\begin{cases} \text{(a) For all } z \in Z, u \in X,\ j(z, u, \cdot) : X \to \mathbb{R} \text{is convex} \\ \qquad \text{and lower semicontinuous.} \\ \text{(b) There exist } \alpha_j \ge 0 \text{ and } \beta_j \ge 0 \text{ such that} \\ \quad j(z_1, u_1, v_2) - j(z_1, u_1, v_1) + j(z_2, u_2, v_1) - j(z_2, u_2, v_2) \\ \qquad \le \alpha_j \|z_1 - z_2\|_Z \|v_1 - v_2\|_X + \beta_j \|u_1 - u_2\|_X \|v_1 - v_2\|_X \\ \qquad\quad \forall\, z_1,\, z_2 \in Z, \quad \forall\, u_1,\, u_2,\, v_1,\, v_2 \in X. \end{cases} \tag{3.5}$$

$$\begin{cases} \text{For every } m \in \mathbb{N} \text{ there exists } d_m > 0 \text{ such that} \\ \quad \|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_Z \le d_m \displaystyle\int_0^t \|u_1(s) - u_2(s)\|_X \, ds \\ \qquad \forall\, u_1,\, u_2 \in C(\mathbb{R}_+; X),\ \forall\, t \in [0, m]. \end{cases} \tag{3.6}$$

$$m_A > \beta_j. \tag{3.7}$$

$$f \in C(\mathbb{R}_+; X). \tag{3.8}$$

On these assumptions we have the following comments. First, condition (3.4) shows that the operator A is a strongly monotone and Lipschitz continuous operator (see, [29–31]). Next, following the terminology introduced in [32,33], condition (3.6) shows that the operator $\mathcal{S}$ is a history-dependent operator and, therefore, we refer to (3.2) as a history-dependent variational inequality. Finally, we note that (3.7) represents a smallness assumption where, recall, m_A and β_j are the constants in (3.4) and (3.5), respectively.

We now recall the following existence and uniqueness result in the study of the history-dependent variational inequality (3.2).

Theorem 3.1. *Assume that (3.3)–(3.8) hold. Then, Problem $\mathcal{Q}$ has a unique solution $u \in C(\mathbb{R}_+; K)$.*

A proof of Theorem 3.1 can be found in [27], based on arguments of time-dependent variational inequalities, monotonicity and fixed-point.

We now turn to the analysis of Problem $\mathcal{P}$ and, to this end, we introduce the assumptions on the data. We assume that the elasticity operator and the relaxation tensor satisfy the following conditions.

$$\left\{\begin{array}{l}\mathcal{A}\colon \Omega\times\mathbb{S}^d\to\mathbb{S}^d \text{ is such that}\\ \text{(a) there exists } m_{\mathcal{F}}>0 \text{ such that}\\ \qquad (\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\geq m_{\mathcal{A}}\,\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2\\ \qquad\quad \text{for all } \ \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d, \text{ a.e. } \boldsymbol{x}\in\Omega,\\ \text{(b) there exists } L_{\mathcal{A}}>0 \text{ such that}\\ \qquad \|\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\|\leq L_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|\\ \qquad\quad \text{for all } \ \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d, \text{ a.e. } \boldsymbol{x}\in\Omega,\\ \text{(c) } \mathcal{A}(\cdot,\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon}\in\mathbb{S}^d,\\ \text{(d) } \mathcal{A}(\boldsymbol{x},\boldsymbol{0})=\boldsymbol{0} \ \text{ for a.e. } \boldsymbol{x}\in\Omega.\end{array}\right. \tag{3.9}$$

$$\mathcal{B}\in C(\mathbb{R}_+,\mathbf{Q}_\infty). \tag{3.10}$$

We also assume the densities of body forces and surface tractions have the regularity

$$\boldsymbol{f}_0\in C(\mathbb{R}_+;L^2(\Omega)^d),\quad \boldsymbol{f}_2\in C(\mathbb{R}_+;L^2(\Gamma_2)^d). \tag{3.11}$$

The yield function and the normal compliance function are such that

$$\left\{\begin{array}{l}F\colon \Gamma_3\times\mathbb{R}\to\mathbb{R}_+ \text{ is such that}\\ \text{(a) there exists } L_F>0 \text{ such that}\\ \qquad |F(\boldsymbol{x},r_1)-F(\boldsymbol{x},r_2)|\leq L_F|r_1-r_2|\\ \qquad\quad \text{for all } \ r_1,r_2\in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Gamma_3,\\ \text{(b) } F(\cdot,r) \text{ is measurable on } \Gamma_3 \text{ for all } r\in\mathbb{R},\\ \text{(c) } F(\boldsymbol{x},0)=0, \text{ a.e. } \boldsymbol{x}\in\Gamma_3.\end{array}\right. \tag{3.12}$$

$$\left\{\begin{array}{l}p\colon \Gamma_3\times\mathbb{R}\to\mathbb{R}_+ \text{ is such that}\\ \text{(a) there exists } L_p>0 \text{ such that}\\ \qquad |p(\boldsymbol{x},r_1)-p(\boldsymbol{x},r_2)|\leq L_p|r_1-r_2|\\ \qquad\quad \text{for all}\, r_1,\ r_2\in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Gamma_3,\\ \text{(b) } (p(\boldsymbol{x},r_1)-p(\boldsymbol{x},r_2))\,(r_1-r_2)\geq 0\\ \qquad\quad \text{for all } \ r_1,\ r_2\in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Gamma_3,\\ \text{(d) } p(\cdot,r) \text{ is measurable on } \Gamma_3 \text{ for all } r\in\mathbb{R},\\ \text{(c) } p(\boldsymbol{x},r)=0 \ \text{ for all } \ r\leq 0, \text{ a.e. } \boldsymbol{x}\in\Gamma_3.\end{array}\right. \tag{3.13}$$

We also recall that

$$g>0,\qquad \mu>0 \tag{3.14}$$

and, finally, we assume that the following condition holds:

$$\mu L_p\|\gamma\|^2<m_{\mathcal{A}}. \tag{3.15}$$

We now introduce the set of admissible displacement fields U defined by

$$U=\{\,\boldsymbol{v}\in V\mid v_\nu\leq g\ \text{ a.e. on } \Gamma_3\,\}. \tag{3.16}$$

Then, following a standard approach based on the Green formula (2.9), we can derive the following variational formulation of Problem $\mathcal{P}$, in terms of displacements.

Problem $\mathcal{P}^V$. Find a displacement field $\boldsymbol{u}\colon \mathbb{R}_+ \to U$ such that, for all $t \in \mathbb{R}_+$, the inequality below holds:

$$\begin{aligned}&\int_\Omega \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\,dx + \int_{\Gamma_3} p(u_\nu(t))(v_\nu - u_\nu(t))\,da \\ &\quad + \int_\Omega \Big(\int_0^t \mathcal{B}(t-s)\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds\Big) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\,dx \\ &\quad + \int_{\Gamma_3} F\Big(\int_0^t u_\nu^+(s)\,ds\Big)(v_\nu^+ - u_\nu^+(t))\,da + \int_{\Gamma_3} \mu p(u_\nu(t))(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau(t)\|)\,da \\ &\quad\geq \int_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\,da \quad \forall\, \boldsymbol{v} \in U.\end{aligned} \tag{3.17}$$

In the study of this problem, we have the following existence and uniqueness result.

Theorem 3.2. *Assume (3.9)–(3.15). Then, Problem $\mathcal{P}^V$ has a unique solution with regularity $\boldsymbol{u} \in C(\mathbb{R}_+; U)$.*

The proof of Theorem 3.2 will be done in several steps, based on the abstract existence and uniqueness provided by Theorem 3.1. To present it, we assume in what follows that (3.9)–(3.15) hold. We introduce the space $Z = Q \times L^2(\Gamma_3)$ together with the norm

$$\|\boldsymbol{z}\|_Z = \|\boldsymbol{\sigma}\|_Q + \|\xi\|_{L^2(\Gamma_3)} \qquad \forall\, \boldsymbol{z} = (\boldsymbol{\sigma}, \boldsymbol{\xi}) \in Z. \tag{3.18}$$

Moreover, we consider the operators $A\colon V \to V$, $\mathcal{S}\colon C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Z)$ and the functions $j\colon Z \times V \times V \to \mathbb{R}$, $\boldsymbol{f}\colon \mathbb{R}_+ \to V$ defined by

$$(A\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx + \int_{\Gamma_3} p(u_\nu) v_\nu\,da \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V, \tag{3.19}$$

$$\mathcal{S}\boldsymbol{u}(t) = \Big(\int_0^t \mathcal{B}(t-s)\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds, F\Big(\int_0^t u_\nu^+(s)\,ds\Big)\Big) \quad \forall\, \boldsymbol{u} \in C(\mathbb{R}_+; V), \tag{3.20}$$

$$\begin{aligned}j(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx + \int_{\Gamma_3} \xi \cdot v_\nu^+\,da + \int_{\Gamma_3} \mu p(u_\nu)\|\boldsymbol{v}_\tau\|\,da \\ \forall\, \boldsymbol{z} = (\boldsymbol{\sigma}, \xi) \in Z,\ \ \boldsymbol{u},\ \boldsymbol{v} \in V,\end{aligned} \tag{3.21}$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\,da \qquad \forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+. \tag{3.22}$$

With these notation, it is easy to obtain the following preliminary result.

Lemma 3.3. *A displacement field $\boldsymbol{u} \in C(\mathbb{R}_+; U)$ is a solution of Problem $\mathcal{P}^V$ if any only if, for all $t \in \mathbb{R}_+$, the inequality below holds:*

$$\begin{aligned}\boldsymbol{u}(t) \in U, \qquad (A\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + j(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{v}) \\ -j(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{u}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U.\end{aligned} \tag{3.23}$$

The next step is provided by the following intermediate result.

Lemma 3.4. *There exists a unique displacement field $\boldsymbol{u} \in C(\mathbb{R}_+; U)$ which satisfies inequality (3.23), for all $t \in \mathbb{R}_+$.*

Proof. We use Theorem 3.1 with $X = V$, $Z = Q \times L^2(\Gamma_3)$, $K = U$, the operators A, $\mathcal{S}$, the functional j and the function $\boldsymbol{f}$ defined by (3.19)–(3.22). To this end, we check in what follows the validity of conditions

(3.3)–(3.8). First, definition (3.16) shows that condition (3.3) is obviously satisfied. Next, we use assumptions (3.9) and (3.13) and inequality (2.8) to see that

$$(A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)_V \geq m_{\mathcal{A}}\, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V^2, \qquad\qquad (3.24)$$
$$\|A\boldsymbol{u}_1 - A\boldsymbol{u}_2\|_V \leq (L_{\mathcal{A}} + L_p\, \|\gamma\|^2)\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V$$

for all $\boldsymbol{u}_1$, $\boldsymbol{u}_2 \in V$. It follows from here that the operator A satisfies condition (3.4) with $m_A = m_{\mathcal{A}}$ and $L_A = L_{\mathcal{A}} + L_p\, \|\gamma\|^2$. It is easy to see that condition (3.5)(a) is satisfied, too, and an elementary calculus based on assumption (3.13) and inequality (2.8) shows that

$$\begin{aligned} &j(\boldsymbol{z}_1, \boldsymbol{u}_1, \boldsymbol{v}_2) - j(\boldsymbol{z}_1, \boldsymbol{u}_1, \boldsymbol{v}_1) + j(\boldsymbol{z}_2, \boldsymbol{u}_2, \boldsymbol{v}_1) - j(\boldsymbol{z}_2, \boldsymbol{u}_2, \boldsymbol{v}_2) \\ &\quad \leq (1 + \|\gamma\|)\|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_Z \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V + \mu L_p \|\gamma\|^2\, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V \end{aligned} \qquad (3.25)$$

for all $\boldsymbol{z}_1, \boldsymbol{z}_2 \in Z$, $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V$. This shows that condition (3.5)(b) holds with $\alpha_j = 1 + \|\gamma\|$ and $\beta_j = \mu L_p \|\gamma\|^2$.

We now use assumptions (3.10) and (3.12) and inequality (2.11) to see that for each $m \in \mathbb{N}$ the following inequality holds:

$$\|\mathcal{S}\boldsymbol{u}_1(t) - \mathcal{S}\boldsymbol{u}_2(t)\|_Z \leq (d \max_{r\in[0,m]} \|\mathcal{B}(r)\|_{\mathbf{Q}_\infty} + L_F \|\gamma\|) \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_X\, ds \qquad (3.26)$$

for all $\boldsymbol{u}_1, \boldsymbol{u}_2 \in C(\mathbb{R}_+; V)$, $t \in [0, m]$. This inequality shows that condition (3.6) is satisfied. Finally, inequality (3.15) shows that condition (3.7) holds and assumption (3.11) on the external forces guarantees that the function $\boldsymbol{f}$ defined by (3.22) has the regularity (3.8).

It follows from above that assumptions (3.3)–(3.8) hold and, therefore, we are in a position to use Theorem 3.1 in order to conclude the proof. □

We end this section with the remark that Theorem 3.2 is now a direct consequence of Lemmas 3.3 and 3.4. Moreover, the solution of Problem $\mathcal{P}^V$, provided by this theorem, represents a weak solution of the viscoelastic contact Problem $\mathcal{P}$.

4. Convergence results

In this section we study the dependence of the solution to Problem $\mathcal{P}$ with respect the body forces $\boldsymbol{f}_0$, the tractions $\boldsymbol{f}_2$, the thickness g and the coefficient of friction μ. To this end, we assume in what follows that (3.9)–(3.15) hold and we denote by $\boldsymbol{u}$ the solution of Problem $\mathcal{P}^V$ obtained in Theorem 3.1. Next, for each $n \in \mathbb{N}$ we consider a perturbation $\boldsymbol{f}_{0n}$, $\boldsymbol{f}_{2n}$, g_n, μ_n of the data $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, g, μ, respectively, such that

$$\boldsymbol{f}_{0n} \in C(\mathbb{R}_+; L^2(\Omega)^d), \quad \boldsymbol{f}_{2n} \in C(\mathbb{R}_+; L^2(\Gamma_2)^d), \qquad (4.1)$$

$$g_n > 0, \qquad \mu_n > 0, \qquad (4.2)$$

$$\mu_n L_p \|\gamma\|^2 < m_{\mathcal{A}}. \qquad (4.3)$$

Now, with the set of admissible displacement fields U_n defined by

$$U_n = \{\, \boldsymbol{v} \in V \mid v_\nu \leq g_n \ \text{ a.e. on } \Gamma_3 \,\}, \qquad (4.4)$$

we introduce the perturbation of Problem $\mathcal{P}^V$ as follows:

Problem $\mathcal{P}_n^V$. Find a displacement field $\boldsymbol{u}_n\colon \mathbb{R}_+ \to U_n$ such that, for all $t \in \mathbb{R}_+$, the inequality below holds:

$$\int_\Omega \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))\, dx + \int_{\Gamma_3} p(u_{n\nu}(t))(v_\nu - u_{n\nu}(t))\, da \qquad (4.5)$$

$$
\begin{aligned}
&+\int_\Omega\Big(\int_0^t \mathcal{B}(t-s)\,\boldsymbol{\varepsilon}(\boldsymbol{u}_n(s))\,ds\Big)\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)))\,dx\\
&\quad+\int_{\Gamma_3} F\Big(\int_0^t u_{n\nu}^+(s)\,ds\Big)(v_\nu^+-u_{n\nu}^+(t))\,da+\int_{\Gamma_3}\mu_n p(u_{n\nu}(t))(\|\boldsymbol{v}_\tau\|-\|\boldsymbol{u}_{n\tau}(t)\|)\,da\\
&\quad\geq\int_\Omega \boldsymbol{f}_{0n}(t)\cdot(\boldsymbol{v}-\boldsymbol{u}_n(t))\,dx+\int_{\Gamma_2}\boldsymbol{f}_{2n}(t)\cdot(\boldsymbol{v}-\boldsymbol{u}_n(t))\,da\quad\forall\,\boldsymbol{v}\in U_n.
\end{aligned}
$$

It follows from Theorem 3.2 Problem $\mathcal{P}_n^V$ has a unique solution $\boldsymbol{u}_n\in C(\mathbb{R}_+;U_n)$. Moreover, for all $t\in\mathbb{R}_+$, the solution satisfies the inequality

$$
\begin{aligned}
&\boldsymbol{u}_n(t)\in U_n,\qquad (A\boldsymbol{u}_n(t),\boldsymbol{v}-\boldsymbol{u}_n(t))_V+j_n(\mathcal{S}\boldsymbol{u}_n(t),\boldsymbol{u}_n(t),\boldsymbol{v})\\
&\qquad -j_n(\mathcal{S}\boldsymbol{u}_n(t),\boldsymbol{u}_n(t),\boldsymbol{u}_n(t))\geq(\boldsymbol{f}_n(t),\boldsymbol{v}-\boldsymbol{u}_n(t))_V\quad\forall\,\boldsymbol{v}\in U_n.
\end{aligned}\tag{4.6}
$$

Here and below in this section the functions $j_n\colon Z\times V\times V\to\mathbb{R}$ and $\boldsymbol{f}_n\colon\mathbb{R}_+\to V$ are defined by equalities

$$
\begin{aligned}
j_n(\boldsymbol{z},\boldsymbol{u},\boldsymbol{v})&=\int_\Omega\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx+\int_{\Gamma_3}\xi\cdot v_\nu^+\,da+\int_{\Gamma_3}\mu_n p(u_\nu)\|\boldsymbol{v}_\tau\|\,da\\
&\qquad\forall\,\boldsymbol{z}=(\boldsymbol{\sigma},\xi)\in Z,\ \ \boldsymbol{u},\ \boldsymbol{v}\in V,
\end{aligned}\tag{4.7}
$$

$$
(\boldsymbol{f}_n(t),\boldsymbol{v})_V=\int_\Omega\boldsymbol{f}_{0n}(t)\cdot\boldsymbol{v}\,dx+\int_{\Gamma_2}\boldsymbol{f}_{2n}(t)\cdot\boldsymbol{v}\,da\qquad\forall\,\boldsymbol{v}\in V,\ t\in\mathbb{R}_+.\tag{4.8}
$$

Moreover, we denote by "$\rightharpoonup$" and "$\to$" the weak and strong convergence in various normed spaces to be specified and we recall that all the limit, upper limit and lower limit are considered when $n\to\infty$, even if we do not mention it explicitly.

Our main result in this section is the following.

Theorem 4.1. *Assume that (3.9)–(3.15) and (4.1)–(4.3) hold. Moreover, assume that*

$$
\begin{cases}\boldsymbol{f}_{0n}(t)\rightharpoonup\boldsymbol{f}_0(t)\quad\text{in}\quad L^2(\Omega)^d,\qquad \boldsymbol{f}_{2n}(t)\rightharpoonup\boldsymbol{f}_2(t)\quad\text{in}\quad L^2(\Gamma_2)^d,\\ \text{as } n\to\infty,\ \ \forall\,t\in\mathbb{R}_+,\end{cases}\tag{4.9}
$$

$$
\begin{cases}\textit{For each } m\in\mathbb{N} \textit{ there exists } \widetilde{N}_m\in\mathbb{N} \textit{ and } \delta_m>0 \textit{ such that}\\ \|\boldsymbol{f}_{0n}(t)\|_{L^2(\Omega)^d}\leq\delta_m,\ \ \|\boldsymbol{f}_{2n}(t)\|_{L^2(\Gamma_3)^d}\leq\delta_m\quad\forall\,t\in[0,m],\ \ n\geq\widetilde{N}_m,\end{cases}\tag{4.10}
$$

$$
g_n\to g,\qquad \mu_n\to\mu\quad\text{in}\quad\mathbb{R}.\tag{4.11}
$$

Then,

$$
\boldsymbol{u}_n(t)\to\boldsymbol{u}(t)\quad\text{in}\quad V\quad\text{as}\quad n\to\infty,\tag{4.12}
$$

for all $t\in\mathbb{R}$.

The proof of Theorem 4.1 will be carried out in several steps that we present in what follows. Everywhere in the rest of this section we assume that (3.9)–(3.15), (4.1)–(4.3) and (4.9)–(4.11) hold. Let $n\in\mathbb{N}$. We start by considering the intermediate problem of finding a function $\overline{\boldsymbol{u}}_n:\mathbb{R}_+\to V$ such that, for each $t\in\mathbb{R}_+$, the inequality below holds:

$$
\begin{aligned}
&\overline{\boldsymbol{u}}_n(t)\in U_n,\qquad (A\overline{\boldsymbol{u}}_n(t),\boldsymbol{v}-\overline{\boldsymbol{u}}_n(t))_V+j_n(\mathcal{S}\boldsymbol{u}(t),\overline{\boldsymbol{u}}_n(t),\boldsymbol{v})\\
&\qquad -j_n(\mathcal{S}\boldsymbol{u}(t),\overline{\boldsymbol{u}}_n(t),\overline{\boldsymbol{u}}_n(t))\geq(\boldsymbol{f}_n(t),\boldsymbol{v}-\overline{\boldsymbol{u}}_n(t))_V\quad\forall\,\boldsymbol{v}\in U_n.
\end{aligned}\tag{4.13}
$$

Note that the operator $\boldsymbol{v}\mapsto\mathcal{S}(\boldsymbol{u}):C(\mathbb{R}_+;V)\to C(\mathbb{R}_+;Z)$ satisfies condition (3.6). Therefore, it follows from Theorem 3.1 that this problem has a unique solution with regularity $\overline{\boldsymbol{u}}_n\in C(\mathbb{R}_+;U_n)$.

We proceed with the following result.

Lemma 4.2. *For each $m \in \mathbb{N}$ there exists $\lambda_m > 0$ and $N_m \in \mathbb{N}$ such that*

$$\|\overline{\boldsymbol{u}}_n(t)\|_V \le \lambda_m \qquad \forall\, t \in [0, m],\ \ n \ge N_m. \tag{4.14}$$

Proof. Let $m \in \mathbb{N}$ be fixed and let $t \in [0, m]$. We test in (4.13) with $\boldsymbol{v} = \boldsymbol{0}_V$ to obtain

$$(A\overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t))_V \le (\boldsymbol{f}_n(t), \overline{\boldsymbol{u}}_n(t))_V - j_n(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t)),$$

and then we use the strong monotonicity of the operator A and equality $A\boldsymbol{0}_V = \boldsymbol{0}_V$ to see that

$$m_{\mathcal{A}}\|\overline{\boldsymbol{u}}_n(t)\|_V^2 \le \|\boldsymbol{f}_n(t)\|_V\|\overline{\boldsymbol{u}}_n(t)\|_V - j_n(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t)). \tag{4.15}$$

On the other hand, using (4.7) we write

$$\begin{aligned}-j_n(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t)) &= j_n(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}_n(t), \boldsymbol{0}_V) - j_n(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t))\\ &\quad + j_n(\boldsymbol{0}_V, \boldsymbol{0}_V, \overline{\boldsymbol{u}}_n(t)) - j_n(\boldsymbol{0}_V, \boldsymbol{0}_V, \boldsymbol{0}_V),\end{aligned}$$

and then we use inequality (3.25) for the function j_n to obtain

$$\begin{aligned}&-j_n(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t))\\ &\quad\le (1 + \|\gamma\|)\|\mathcal{S}\boldsymbol{u}(t)\|_Z\|\overline{\boldsymbol{u}}_n(t))\|_V + \mu_n L_p\|\gamma\|^2\, \|\overline{\boldsymbol{u}}_n(t))\|_V^2.\end{aligned} \tag{4.16}$$

We now combine the inequalities (4.15) and (4.16) to see that

$$(m_{\mathcal{A}} - \mu_n L_p\|\gamma\|^2)\|\overline{\boldsymbol{u}}_n(t)\|_V \le \|\boldsymbol{f}_n(t)\|_V + (1 + \|\gamma\|)\|\mathcal{S}\boldsymbol{u}(t)\|_Z. \tag{4.17}$$

Note that assumptions (4.11) and (3.15) imply that there exists $\overline{N}_m \in \mathbb{N}$ such that

$$m_{\mathcal{A}} - \mu_n L_p\|\gamma\|^2 \ge \frac{1}{2}\,(m_{\mathcal{A}} - \mu L_p\|\gamma\|^2) \qquad \forall\, n \ge \overline{N}_m. \tag{4.18}$$

Therefore, (4.17) allows us to write

$$\|\overline{\boldsymbol{u}}_n(t)\|_V \le c_0(\|\boldsymbol{f}_n(t)\|_V + \|\mathcal{S}\boldsymbol{u}(t)\|_Z) \qquad \forall\, n \ge \overline{N}_m, \tag{4.19}$$

where c_0 is a positive constant which does not depend on n and t. We now use assumption (4.10) and define $N_m = max\,\{\overline{N}_m, \widetilde{N}_m\}$. Then, (4.19) implies that

$$\|\overline{\boldsymbol{u}}_n(t)\|_V \le \widetilde{c}_0(\,\delta_m + \|\mathcal{S}\boldsymbol{u}(t)\|_Z) \qquad \forall\, n \ge N_m, \tag{4.20}$$

where, again, $\widetilde{c}_0$ is a positive constant which does not depend on n and t. Thus, the inequality (4.14) holds with

$$\lambda_m = \widetilde{c}_0(\delta_m + \max_{t\in[0,m]} \|\mathcal{S}\boldsymbol{u}(t)\|_Z),$$

which concludes the proof. □

We proceed with the following weak convergence result.

Lemma 4.3. *For all $t \in \mathbb{R}_+$ the following convergence holds:*

$$\overline{\boldsymbol{u}}_n(t) \rightharpoonup \boldsymbol{u}(t) \quad \text{in} \quad V. \tag{4.21}$$

Proof. We fix $t \in \mathbb{R}_+$ and let $m \in \mathbb{N}$ be such that $t \in [0, m]$. It follows from Lemma 4.2 that the sequence $\{\overline{\boldsymbol{u}}_n(t)\}$ is bounded in V and, therefore, there exists an element $\overline{\boldsymbol{u}}(t) \in V$ such that, passing to a subsequence still denoted $\{\overline{\boldsymbol{u}}_n(t)\}$,

$$\overline{\boldsymbol{u}}_n(t) \rightharpoonup \overline{\boldsymbol{u}}(t) \quad \text{in} \quad V. \tag{4.22}$$

Note that this convergence implies that $\overline{\boldsymbol{u}}_n(t) \to \overline{\boldsymbol{u}}(t)$ a.e. on Γ_3 and, therefore, definitions (4.4), (3.16) and the convergence (4.11) yield

$$\overline{\boldsymbol{u}}(t) \in U. \tag{4.23}$$

Let $\boldsymbol{v} \in U$. Then $\frac{g_n}{g}\boldsymbol{v} \in U_n$ and, using (4.13) we deduce that

$$\begin{aligned}(A\overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t) - \frac{g_n}{g}\boldsymbol{v})_V &\le (\boldsymbol{f}_n(t), \overline{\boldsymbol{u}}_n(t) - \frac{g_n}{g}\boldsymbol{v})_V \\ &+ j_n(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}_n(t), \frac{g_n}{g}\boldsymbol{v}) - j_n(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t)).\end{aligned}$$

We now write $\overline{\boldsymbol{u}}_n(t) - \frac{g_n}{g}\boldsymbol{v} = \overline{\boldsymbol{u}}_n(t) - \boldsymbol{v} + (1 - \frac{g_n}{g})\boldsymbol{v}$ to find that

$$\begin{aligned}&(A\overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t) - \boldsymbol{v})_V + (A\overline{\boldsymbol{u}}_n(t), (1 - \frac{g_n}{g})\boldsymbol{v})_V \\ &\quad\le (\boldsymbol{f}_n(t), \overline{\boldsymbol{u}}_n(t) - \boldsymbol{v})_V + (\boldsymbol{f}_n(t), (1 - \frac{g_n}{g})\boldsymbol{v})_V \\ &\qquad + j_n(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}_n(t), \frac{g_n}{g}\boldsymbol{v}) - j_n(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t)).\end{aligned} \tag{4.24}$$

Next, we use assumptions (4.9)–(4.11), convergence (4.22) and standard compactness arguments to see that the following convergences hold:

$$\begin{aligned}&(A\overline{\boldsymbol{u}}_n(t), (1 - \frac{g_n}{g})\boldsymbol{v})_V \to 0, \\ &(\boldsymbol{f}_n(t), \overline{\boldsymbol{u}}_n(t) - \boldsymbol{v})_V \to (\boldsymbol{f}(t), \overline{\boldsymbol{u}}(t) - \boldsymbol{v})_V, \\ &(\boldsymbol{f}_n(t), (1 - \frac{g_n}{g})\boldsymbol{v})_V \to 0, \\ &j_n(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}_n(t), \frac{g_n}{g}\boldsymbol{v}) \to j(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}(t), \boldsymbol{v}), \\ &j_n(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t)) \to j(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}(t), \overline{\boldsymbol{u}}(t)).\end{aligned}$$

Therefore, taking the upper limit in (4.24) and using these convergences we deduce that

$$\begin{aligned}&\limsup\, (A\overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t) - \boldsymbol{v})_V \\ &\quad\le (\boldsymbol{f}(t), \overline{\boldsymbol{u}}(t) - \boldsymbol{v})_V + j(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}(t), \boldsymbol{v}) - j(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}(t), \overline{\boldsymbol{u}}(t)).\end{aligned} \tag{4.25}$$

The regularity (4.23) allows us to take $\boldsymbol{v} = \overline{\boldsymbol{u}}(t)$ in the previous inequality. As a result we deduce that

$$\limsup\, (A\overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t) - \overline{\boldsymbol{u}}(t))_V \le 0. \tag{4.26}$$

We now use (4.22), (4.26) and a standard pseudomonotonicity argument to see that

$$(A\overline{\boldsymbol{u}}(t), \overline{\boldsymbol{u}}(t) - \boldsymbol{v})_V \le \liminf\, (A\overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t) - \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V. \tag{4.27}$$

We now combine inequalities (4.25) and (4.27) to deduce that

$$\begin{aligned}&(A\overline{\boldsymbol{u}}(t), \overline{\boldsymbol{u}}(t) - \boldsymbol{v})_V \\ &\quad\le (\boldsymbol{f}(t), \overline{\boldsymbol{u}}(t) - \boldsymbol{v})_V + j(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}(t), \boldsymbol{v}) - j(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}(t), \overline{\boldsymbol{u}}(t)) \quad \forall\, \boldsymbol{v} \in V.\end{aligned} \tag{4.28}$$

Inequality (4.28) combined with regularity (4.23) and the uniqueness of the solution of the variational inequality (3.23) shows that $\overline{\boldsymbol{u}}(t) = \boldsymbol{u}(t)$. A careful analysis of the previous results reveals that the sequence $\{\overline{\boldsymbol{u}}_n(t)\}$ is bounded in V and every weakly convergent subsequence of this sequence converges to $\boldsymbol{u}(t)$. This implies the convergence (4.21) and concludes the proof. □

The next step is given by the following strong convergence result.

Lemma 4.4. *For all $t \in \mathbb{R}_+$ the following convergence holds:*

$$\overline{\boldsymbol{u}}_n(t) \to \boldsymbol{u}(t) \quad \text{in} \quad V. \tag{4.29}$$

Proof. Let $t \in \mathbb{R}_+$. We successively test with $\boldsymbol{v} = \overline{\boldsymbol{u}}(t)$ in (4.25) and (4.27) to deduce that

$$0 \le \liminf\, (A\overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t) - \overline{\boldsymbol{u}}(t))_V \le \limsup\, (A\overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t) - \overline{\boldsymbol{u}}(t))_V \le 0,$$

which implies that $\lim\, (A\overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t) - \overline{\boldsymbol{u}}(t))_V = 0$. This result combined with equality $\overline{\boldsymbol{u}}(t) = \boldsymbol{u}(t)$, obtained in the proof of Lemma 4.2, yields

$$(A\overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t) - \boldsymbol{u}(t))_V \to 0. \tag{4.30}$$

Next, we use the strong monotonicity of the operator A to see that

$$\begin{aligned} m_{\mathcal{A}}\|\overline{\boldsymbol{u}}_n(t) - \boldsymbol{u}(t)\|_V^2 &\le (A\overline{\boldsymbol{u}}_n(t) - A\boldsymbol{u}(t), \overline{\boldsymbol{u}}_n(t) - \boldsymbol{u}(t))_V \\ &= (A\overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t) - \boldsymbol{u}(t))_V - (A\boldsymbol{u}(t), \overline{\boldsymbol{u}}_n(t) - \boldsymbol{u}(t))_V. \end{aligned}$$

This inequality combined with the convergences (4.30) and (4.21) imply that

$$\limsup\, \|\overline{\boldsymbol{u}}_n(t) - \boldsymbol{u}(t)\|_V^2 \le 0,$$

which shows that (4.29) holds. □

We now have all the ingredients to provide the proof of Theorem 4.1.

Proof. Let $n \in \mathbb{N}$, $t \in \mathbb{R}_+$ and let $m \in \mathbb{N}$ be such that $t \in [0, m]$. We take $\boldsymbol{v} = \overline{\boldsymbol{u}}_n(t)$ in (4.6) and $\boldsymbol{v} = \boldsymbol{u}_n(t)$ in (4.13), then we add the resulting inequalities to obtain that

$$\begin{aligned} &(A\boldsymbol{u}_n(t) - A\overline{\boldsymbol{u}}_n(t), \boldsymbol{u}_n(t) - \overline{\boldsymbol{u}}_n(t))_V \\ &\quad\le j_n(\mathcal{S}\boldsymbol{u}_n(t), \boldsymbol{u}_n(t), \overline{\boldsymbol{u}}_n(t)) - j_n(\mathcal{S}\boldsymbol{u}_n(t), \boldsymbol{u}_n(t), \boldsymbol{u}_n(t)) \\ &\qquad + j_n(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}_n(t), \boldsymbol{u}_n(t)) - j_n(\mathcal{S}\boldsymbol{u}(t), \overline{\boldsymbol{u}}_n(t), \overline{\boldsymbol{u}}_n(t)). \end{aligned}$$

Next, we use the strong monotonicity of the operator A, as well as inequality (3.25) for the function j_n to see that

$$\begin{aligned} m_{\mathcal{A}}\|\overline{\boldsymbol{u}}_n(t) - \boldsymbol{u}_n(t)\|_V^2 &\le (1 + \|\gamma\|)\|\mathcal{S}\boldsymbol{u}_n(t) - \mathcal{S}\boldsymbol{u}(t)\|_V \|\overline{\boldsymbol{u}}_n(t) - \boldsymbol{u}_n(t)\|_V \\ &\quad + \mu_n L_p \|\gamma\|^2\, \|\overline{\boldsymbol{u}}_n(t) - \boldsymbol{u}_n(t)\|_V^2, \end{aligned}$$

which implies that

$$(m_{\mathcal{A}} - \mu_n L_p\|\gamma\|^2)\|\overline{\boldsymbol{u}}_n(t) - \boldsymbol{u}_n(t)\|_V \le (1 + \|\gamma\|)\|\mathcal{S}\boldsymbol{u}_n(t) - \mathcal{S}\boldsymbol{u}(t)\|_V.$$

Let $n \ge \overline{N}_m$ where, recall, $\overline{N}_m \in \mathbb{N}$ is defined in (4.18) Then, there exists a constant $c_0 > 0$ such that, for $n \ge \overline{N}_m$ we have

$$\|\overline{\boldsymbol{u}}_n(t) - \boldsymbol{u}_n(t)\|_V \le c_0\|\mathcal{S}\boldsymbol{u}_n(t) - \mathcal{S}\boldsymbol{u}(t)\|_V.$$

Therefore, using inequality (3.26) we find that

$$\|\overline{\boldsymbol{u}}_n(t) - \boldsymbol{u}_n(t)\|_V \le c_m \int_0^t \|\boldsymbol{u}_n(s) - \boldsymbol{u}(s)\|_V\, ds, \tag{4.31}$$

where $c_m > 0$ depends on m but does not depend on n and t. It follows from here that

$$\begin{aligned}\|\boldsymbol{u}_n(t)-\boldsymbol{u}(t)\|_V &\le \|\boldsymbol{u}_n(t)-\overline{\boldsymbol{u}}_n(t)\|_V+\|\overline{\boldsymbol{u}}_n(t)-\boldsymbol{u}(t)\|_V\\ &\le c_m\int_0^t \|\boldsymbol{u}_n(s)-\boldsymbol{u}(s)\|_V\,ds+\|\overline{\boldsymbol{u}}_n(t)-\boldsymbol{u}(t)\|_V\end{aligned}$$

and, therefore, using the Gronwall inequality yields

$$\|\boldsymbol{u}_n(t)-\boldsymbol{u}(t)\|_V \le c_m\int_0^t e^{c_m(t-s)}\|\overline{\boldsymbol{u}}_n(s)-\boldsymbol{u}(s)\|_V\,ds+\|\overline{\boldsymbol{u}}_n(t)-\boldsymbol{u}(t)\|_V. \tag{4.32}$$

Note that Lemma 4.2 implies that

$$\|\overline{\boldsymbol{u}}_n(s)-\boldsymbol{u}(s)\|_V \le \|\overline{\boldsymbol{u}}_n(s)\|_V+\|\boldsymbol{u}(s)\|_V \le \lambda_m+\max_{s\in[0,m]}\|\boldsymbol{u}(s)\|_V$$

and, moreover,

$$e^{c_m(t-s)} \le e^{c_m t} \le m e^{c_m} \qquad \forall\, s\in[0,t].$$

These inequalities show that the integrand in (4.32) is bounded. Then, Lemma 4.4 allows us to use the Lebesgue theorem in order to see that

$$\int_0^t e^{c_m(t-s)}\|\overline{\boldsymbol{u}}_n(s)-\boldsymbol{u}(s)\|_V\,ds\to 0, \qquad \|\overline{\boldsymbol{u}}_n(t)-\boldsymbol{u}(t)\|_V\to 0.$$

We now use these convergences in (4.32) to find that (4.12) holds, which concludes the proof. □

Besides its mathematical interest in the convergence result (4.12), it is important from mechanical point of view since it shows that, at any moment t, the weak solution of the contact problem $\mathcal{P}$ depends continuously on the thickness of rigid-deformable layer, the coefficient of friction, and the densities of body forces and tractions.

Theorem 4.1 provides the pointwise convergence of the solution $\boldsymbol{u}_n$ to the solution $\boldsymbol{u}$. It seems that the assumptions of this theorem are not enough to guarantee this convergence in the space $C(\mathbb{R}_+;V)$. For this reason we reinforce in what follows these assumptions to obtain the following result.

Theorem 4.5. *In addition to conditions (3.9)–(3.15), (4.2)–(4.3), (4.11), we assume further that $\boldsymbol{f}_{0n}=\boldsymbol{f}_0$ and $\boldsymbol{f}_{2n}=\boldsymbol{f}_2$ for all $n\in\mathbb{N}$. Then*

$$\boldsymbol{u}_n\to\boldsymbol{u} \quad \text{in} \quad C(\mathbb{R}_+;V) \quad \text{as} \quad n\to\infty, \tag{4.33}$$

i.e., $\boldsymbol{u}_n$ converges uniformly to $\boldsymbol{u}$ on any compact interval $I\subset\mathbb{R}_+$.

The proof is based on estimates already obtained in the proof of Theorem 4.1. Thus, we present only its sketch.

Proof. Let $n\in\mathbb{N}$, $m\in\mathbb{N}$ and let $t\in[0,m]$. We test in (3.23) with $\boldsymbol{v}=\frac{g}{g_n}\boldsymbol{u}_n(t)\in U$, then we multiply the resulting inequality with $\frac{g_n}{g}$ and add it to the inequality obtained from (4.6) with $\boldsymbol{v}=\frac{g_n}{g}\boldsymbol{u}(t)\in U_n$. In this way we deduce that

$$\begin{aligned}&(A\boldsymbol{u}_n(t)-A\boldsymbol{u}(t),\frac{g_n}{g}\boldsymbol{u}(t)-\boldsymbol{u}_n(t))_V+j_n(\mathcal{S}\boldsymbol{u}_n(t),\boldsymbol{u}_n(t),\frac{g_n}{g}\boldsymbol{u}(t))\\ &-j_n(\mathcal{S}\boldsymbol{u}_n(t),\boldsymbol{u}_n(t),\boldsymbol{u}_n(t))+j(\mathcal{S}\boldsymbol{u}(t),\boldsymbol{u}(t),\boldsymbol{u}_n(t))-j(\mathcal{S}\boldsymbol{u}(t),\boldsymbol{u}(t),\frac{g_n}{g}\boldsymbol{u}(t))\end{aligned}$$

and, therefore,

$$
\begin{aligned}
(A\boldsymbol{u}_n(t) - Au(t), \boldsymbol{u}_n(t) - \boldsymbol{u}(t))_V &\leq \Big(\frac{g_n}{g} - 1\Big)(A\boldsymbol{u}_n(t) - A\boldsymbol{u}(t), \boldsymbol{u}(t))_V \\
&+\Big(\frac{g_n}{g} - 1\Big) j_n(\mathcal{S}\boldsymbol{u}_n(t), \boldsymbol{u}_n(t), \boldsymbol{u}(t)) + j_n(\mathcal{S}\boldsymbol{u}_n(t), \boldsymbol{u}_n(t), \boldsymbol{u}(t)) \\
&-j_n(\mathcal{S}\boldsymbol{u}_n(t), \boldsymbol{u}_n(t), \boldsymbol{u}_n(t)) + j(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{u}_n(t)) \\
&-j(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{u}(t)) + \Big(1 - \frac{g_n}{g}\Big) j(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{u}(t)).
\end{aligned} \tag{4.34}
$$

Note that Lemma 4.2 implies the bound

$$\|\boldsymbol{u}_n(t)\|_V \leq c. \tag{4.35}$$

Here and below in this proof c denotes a positive constant which could depend on m, on the solution $\boldsymbol{u}$ and the rest of the problem data, but is independent on n and t, and whose value may change from line to line. We now use the properties of A, $\mathcal{S}$ and j as well as inequality (4.35) to see that

$$\Big(\frac{g_n}{g} - 1\Big)(A\boldsymbol{u}_n(t) - Au(t), \boldsymbol{u}(t))_V \leq c|g_n - g|, \tag{4.36}$$

$$\Big(\frac{g_n}{g} - 1\Big) j_n(\mathcal{S}\boldsymbol{u}_n(t), \boldsymbol{u}_n(t), \boldsymbol{u}(t)) \leq c|g_n - g|, \tag{4.37}$$

$$\Big(1 - \frac{g_n}{g}\Big) j(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{u}(t)) \leq c|g_n - g|, \tag{4.38}$$

$$
\begin{aligned}
&j_n(\mathcal{S}\boldsymbol{u}_n(t), \boldsymbol{u}_n(t), \boldsymbol{u}(t)) - j_n(\mathcal{S}\boldsymbol{u}_n(t), \boldsymbol{u}_n(t), \boldsymbol{u}_n(t)) \\
&\quad + j(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{u}_n(t)) - j(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{u}(t)) \\
&\qquad \leq c\|\mathcal{S}\boldsymbol{u}_n(t) - \mathcal{S}\boldsymbol{u}(t)\|_Z \|\boldsymbol{u}_n(t) - \boldsymbol{u}(t)\|_V + \widetilde{c}\|\boldsymbol{u}_n(t) - \boldsymbol{u}(t)\|_V^2 + c|\mu_n - \mu|,
\end{aligned} \tag{4.39}
$$

where $\widetilde{c}$ is a positive constant such that $\widetilde{c} \leq m_{\mathcal{A}}$, if n is large enough. We now combine inequalities (4.34), (4.36)–(4.39), then use the strongly monotonicity of A to deduce that

$$\|\boldsymbol{u}_n(t) - \boldsymbol{u}(t)\|_V^2 \leq c\|\mathcal{S}\boldsymbol{u}_n(t) - \mathcal{S}\boldsymbol{u}(t)\|_Z \|\boldsymbol{u}_n(t) - \boldsymbol{u}(t)\|_V + c(|g_n - g| + |\mu_n - \mu|).$$

Next, the elementary inequality $x^2 \leq ax + b \implies x \leq a + \sqrt{b}$ implies that

$$\|\boldsymbol{u}_n(t) - \boldsymbol{u}(t)\|_V \leq c\,\|\mathcal{S}\boldsymbol{u}_n(t) - \mathcal{S}\boldsymbol{u}(t)\|_Z + c\sqrt{|g_n - g| + |\mu_n - \mu|},$$

and, therefore, inequality (3.26) and the Gronwall argument yield

$$\max_{t\in[0,m]} \|\boldsymbol{u}_n(t) - \boldsymbol{u}(t)\|_V \leq c\sqrt{|g_n - g| + |\mu_n - \mu|}),$$

for n large enough. Finally, assumption (4.11) and (3.1) show that the convergence (4.33) holds, which concludes the proof. □

5. Optimization problems

Theorems 4.1 and 4.5 can be used to study a large number of optimization problems associated to Problem $\mathcal{P}$. To provide a first general example, we assume in what follows that (3.9), (3.10), (3.12) and (3.13) hold and let $K_0 \subset L^2(\Omega)^d$, $K_2 \subset L^2(\Gamma_2)^d$. Moreover, let θ_0 and θ_2 be such that

$$\theta_0 \in C(\mathbb{R}_+, \mathbb{R}), \qquad \theta_2 \in C(\mathbb{R}_+, \mathbb{R}), \tag{5.1}$$

and, in addition, consider two reals intervals K_3 and K_4 such that

$$K_3 = [a_3, b_3] \text{ with } 0 < a_3 \leq b_3, \quad K_4 = [\xi_4, \zeta_4] \text{ with } 0 \leq \xi_4 \leq \zeta_4 \leq \frac{m_{\mathcal{A}}}{L_p\|\gamma\|^2}\,. \tag{5.2}$$

We denote in what follows by X the Hilbert space $X = L^2(\Omega)^d \times L^2(\Gamma_3)^d \times \mathbb{R} \times \mathbb{R}$ endowed with the canonical inner product and let $\mathcal{K} = K_0 \times K_2 \times K_3 \times K_4 \subset X$. For every element $k = (\boldsymbol{b}_0, \boldsymbol{b}_2, g, \mu) \in \mathcal{K}$ we consider the Problem $\mathcal{P}$ with the data $\boldsymbol{f}_0 = \theta_0 \boldsymbol{b}_0$, $\boldsymbol{f}_2 = \theta_2 \boldsymbol{b}_2$, g and μ. Using (5.1) and (5.2) we see that the assumptions of Theorem 3.2 are satisfied and, therefore, this problem has a unique solution, denoted $\boldsymbol{u}_k$, with regularity $\boldsymbol{u}_k \in C(\mathbb{R}_+, U)$.

Consider now a cost functional $\mathcal{L} : V \to \mathbb{R}$ and an arbitrary time moment $t \in \mathbb{R}_+$. We formulate the following optimization problem.

Problem $\mathcal{O}_1$. Find $k^* \in \mathcal{K}$ such that

$$\mathcal{L}(\boldsymbol{u}_{k^*}(t)) = \min_{k \in \mathcal{K}} \mathcal{L}(\boldsymbol{u}_k(t)).$$

In the study of this problem we assume that

$$K_0 \text{ is a bounded weakly closed set of } L^2(\Omega)^d. \tag{5.3}$$

$$K_2 \text{ is a bounded weakly closed set of } L^2(\Gamma_2)^d. \tag{5.4}$$

$$\mathcal{L} : V \to \mathbb{R} \text{ is a lower semicontinuous function.} \tag{5.5}$$

We have the following existence result.

Theorem 5.1. *Assume that (3.9), (3.10), (3.12), (3.13), (5.1)–(5.5). Then, Problem $J_t : \mathcal{K} \to \mathbb{R}$ has at least one solution.*

Proof. Consider the function $J_t : \mathcal{K} \to \mathbb{R}$ defined by

$$J_t(k) = \mathcal{L}(\boldsymbol{u}_k(t)) \qquad \text{for all } k \in \mathcal{K}.$$

Assume that $\{k_n\} = \{(\boldsymbol{b}_{0n}, \boldsymbol{b}_{2n}, g_n, \mu_n)\}$ is a sequence of elements of $\mathcal{K}$ which converges weakly in X to the element $k = \{(\boldsymbol{b}_0, \boldsymbol{b}_2, g, \mu)\} \in \mathcal{K}$. Then, $\boldsymbol{b}_{0n} \rightharpoonup \boldsymbol{b}_0$ in $L^2(\Omega)^d$, $\boldsymbol{b}_{2n} \rightharpoonup \boldsymbol{b}_2$ in $L^2(\Gamma_2)^d$, $g_n \to g$, $\mu_n \to \mu$ and, therefore, denoting $\boldsymbol{f}_{0n} = \theta_0 \boldsymbol{b}_{0n}$, $\boldsymbol{f}_{2n} = \theta_2 \boldsymbol{b}_{2n}$, $\boldsymbol{f}_0 = \theta_0 \boldsymbol{b}_0$, $\boldsymbol{f}_2 = \theta_2 \boldsymbol{b}_2$, we see that conditions (4.9)–(4.11) are satisfied. We now use Theorem 4.1 to deduce that $\boldsymbol{u}_{k_n}(t) \to \boldsymbol{u}_k(t)$ in V. Thus, using assumption (5.5) we obtain that

$$\liminf \mathcal{L}(\boldsymbol{u}_{k_n}(t)) \geq \mathcal{L}(\boldsymbol{u}_k(t)),$$

which shows that the function J_t is weakly lower semicontinuous on $\mathcal{K}$. Moreover, assumptions (5.2)–(5.4) guarantee that $\mathcal{K}$ is a bounded weakly closed subset on X. Theorem 5.1 follows now from a version of the well known Weierstrass theorem (Theorem 7.3.4 in [34], for instance). □

Example 5.2. A first example of Problem $\mathcal{O}_1$ can be obtained by taking

$$\mathcal{L}(\boldsymbol{u}) = \int_{\Gamma_3} \|u_\nu - \phi\|^2 \, da \qquad \text{for all } \boldsymbol{u} \in V, \tag{5.6}$$

where $\phi \in L^2(\Gamma_3)$ is a given function such that $\phi(\boldsymbol{x}) \geq 0$ for a.e. $\boldsymbol{x} \in \Gamma_3$. With this choice, the mechanical interpretation of Problem $\mathcal{O}_1$ is the following: given set $\mathcal{K}$ and a contact process of the form (2.1)–(2.6), we are looking for a body force $\boldsymbol{f}_0 = \theta_0 \boldsymbol{b}_0^*$, a traction $\boldsymbol{f}_2 = \theta_2 \boldsymbol{b}_2^*$, a bound g^* and a friction coefficient μ^* such that $k^* = (\boldsymbol{b}_0^*, \boldsymbol{b}_2^*, g^*, \mu^*) \in \mathcal{K}$ and the corresponding penetration of the body in the foundation at the given moment t is as close as possible to the given penetration ϕ. Note that the function $\mathcal{L} : V \to \mathbb{R}$ defined by (5.6) is continuous, hence it satisfies condition (5.5). Therefore, Theorem 5.1 guarantees the existence of the solutions to the corresponding optimization problem.

Example 5.3. A second example of Problem $\mathcal{O}_1$ can be obtained by taking

$$\mathcal{L}(\boldsymbol{u}) = \int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2\,dx \qquad \text{for all } \boldsymbol{u} \in V. \tag{5.7}$$

With this choice, the mechanical interpretation of Problem $\mathcal{O}_1$ is the following: given set $\mathcal{K}$ and a contact process of the form (2.1)–(2.6), we are looking for a body force $\boldsymbol{f}_0 = \theta_0 \boldsymbol{b}_0^*$, a traction $\boldsymbol{f}_2 = \theta_2 \boldsymbol{b}_2^*$, a bound g^* and a friction coefficient μ^* such that $k^* = (\boldsymbol{b}_0^*, \boldsymbol{b}_2^*, g^*, \mu^*) \in \mathcal{K}$ and the corresponding deformation in the body at the given moment t is as small as possible. Note that the function (5.7) satisfies condition (5.5). Therefore, Theorem 5.1 guarantees the existence of the solutions to the corresponding optimization problem.

We now move to a second kind of optimization problems, based on Theorem 4.5. We assume in what follows that (3.9)–(3.13) hold and consider two reals intervals K_3 and K_4 such that (5.2) hold. Let $\mathcal{K} = K_3 \times K_4 \subset \mathbb{R}^2$. For every element $(g, \mu) \in \mathcal{K}$ we consider the Problem $\mathcal{P}$ with the data g and μ. Using (5.2) we see that the assumptions of Theorem 3.2 are satisfied and, therefore, this problem has a unique solution, denoted $\boldsymbol{u}(g, \mu)$, with regularity $\boldsymbol{u}(g, \mu) \in C(\mathbb{R}_+, U)$.

Consider now a cost functional and $\mathcal{L} : C(\mathbb{R}_+; V) \to \mathbb{R}$ together with the following optimization problem.

Problem $\mathcal{O}_2$. Find $(g^*, u^*) \in \mathcal{K}$ such that

$$\mathcal{L}\big(\boldsymbol{u}(g^*, \mu^*)\big) = \min_{(g,\mu)\in\mathcal{K}} \mathcal{L}\big(\boldsymbol{u}(g, \mu)\big). \tag{5.8}$$

Here, for each $(g, \mu) \in \mathcal{K}$ we use the notation $\boldsymbol{u}(g, \mu)$ for the solution of inequality (3.17) which is also the unique solution of the inequality (3.23). In the study of Problem $\mathcal{O}_2$ we assume that

$$\begin{cases} \mathcal{L} : C(\mathbb{R}_+; V) \to \mathbb{R} \text{ is a lower semicontinuous function, i.e.,} \\ \boldsymbol{u}_n \to \boldsymbol{u} \quad \text{in} \quad C(\mathbb{R}_+; V) \implies \liminf\limits_{n\to\infty} \mathcal{L}(\boldsymbol{u}_n) \geq \mathcal{L}(\boldsymbol{u}). \end{cases} \tag{5.9}$$

We have the following existence result.

Theorem 5.4. *Assume that* (3.9)–(3.13), (5.2) and (5.9) *hold. Then, Problem $\mathcal{O}_2$ has at least one solution.*

Proof. Consider the function $J : \mathcal{K} \to \mathbb{R}$ defined by

$$J(g, \mu) = \mathcal{L}\big(\boldsymbol{u}(g, \mu)\big) \qquad \text{for all } (g, \mu) \in \mathcal{K}.$$

Theorem 4.5 guarantees that the map $(g, \mu) \mapsto \boldsymbol{u}(g, \mu) : \mathcal{K} \to C(\mathbb{R}_+; V)$ is continuous. Therefore, using assumption (5.9) we deduce that the function J is lower semicontinuous on $\mathcal{K}$. Recall also that the set $\mathcal{K}$ is a compact subset of $\mathbb{R}^2$. Theorem 5.4 follows now from the Weierstrass theorem. □

Example 5.5. An example of Problem $\mathcal{O}_2$ can be obtained by taking

$$\mathcal{L}(\boldsymbol{u}) = \int_0^T \Big(\int_\Omega \|\boldsymbol{\sigma}_u(t, \boldsymbol{x})\|^2\,dx\Big)\,ds \qquad \text{for all } \boldsymbol{u} \in C(\mathbb{R}_+; V), \tag{5.10}$$

where $T > 0$ is given and, for each $\boldsymbol{u} \in C(\mathbb{R}_+; V)$, $\boldsymbol{\sigma}_u \in C(\mathbb{R}_+; Q)$ represents the stress function defined by the viscoelastic constitutive equation (2.1). With this choice, the mechanical interpretation of Problem $\mathcal{O}_2$ is the following: given a contact process of the form (2.1)–(2.6) and a compact set $\mathcal{K} = K_3 \times K_4 \in \mathbb{R}^2$ described in (5.2), we are looking for a bound g^* and a friction coefficient μ^* such that $(g^*, \mu^*) \in \mathcal{K}$ and the corresponding stress in the body during the time interval of interest $[0, T]$ is as small as possible. It is easy to see that the function $\mathcal{L}$ defined by (5.10) satisfies condition (5.9). Indeed, consider a sequence $\{\boldsymbol{u}_n\} \subset C(\mathbb{R}_+;V)$ such that $\boldsymbol{u}_n \to \boldsymbol{u}$ in $C(\mathbb{R}_+;V)$. We use (2.1) to see that

$$\boldsymbol{\sigma}_n(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_n(s))\,ds,$$
$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds,$$

for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$. Then, assumptions (3.9) and (3.10) imply that $\boldsymbol{\sigma}_n \to \boldsymbol{\sigma}$ in $C(\mathbb{R}_+;Q)$. Therefore, $\mathcal{L}(\boldsymbol{u}_n) \to \mathcal{L}(\boldsymbol{u})$, which shows that condition (5.9) is satisfied. Theorem 5.1 guarantees now the existence of the solutions to the corresponding optimization problem.

Acknowledgments

This project has received funding from the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie Grant Agreement No. 823731 CONMECH. This research was also supported by the National Natural Science Foundation of China (11771067) and the Applied Basic Project of Sichuan Province (2019YJ0204).

References

[1] G. Duvaut, J.L. Lions, Inequalities in Mechanics and Physics, Springer-Verlag, Berlin, 1976.
[2] C. Eck, J. Jarušek, M. Krbeč, Unilateral Contact Problems: Variational Methods and Existence Theorems, in: Pure and Applied Mathematics, vol. 270, Chapman/CRC Press, New York, 2005.
[3] S. Migórski, A. Ochal, M. Sofonea, Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, in: Advances in Mechanics and Mathematics, vol. l26, Springer, New York, 2013.
[4] P.D. Panagiotopoulos, Inequality Problems in Mechanics and Applications, Birkhäuser, Boston, 1985.
[5] M. Sofonea, A. Matei, Mathematical Models in Contact Mechanics, in: London Mathematical Society Lecture Note Series, vol. 398, Cambridge University Press, 2012.
[6] A. Capatina, Variational Inequalities Frictional Contact Problems, in: Advances in Mechanics and Mathematics, vol. 31, Springer, New York, 2014.
[7] J. Lu, Y.B. Xiao, N.J. Huang, A stackelberg quasi-equilibrium problem via quasi-variational inequalities, Carpathian J. Math. 34 (2018) 355–362.
[8] W. Li, et al., A class of differential inverse quasi-variational inequalities in finite dimensional spaces, J. Nonlinear Sci. Appl. 10 (2017) 4532–4543.
[9] M. Sofonea, S. Migórski, Variational-Hemivariational Inequalities with Applications, in: Pure and Applied Mathematics, Chapman & Hall/CRC Press, Boca Raton-London, 2018.
[10] Y.M. Wang, et al., Equivalence of well-posedness between systems of hemivariational inequalities and inclusion problems, J. Nonlinear Sci. Appl. 9 (2016) 1178–1192.
[11] W.X. Zhang, D.R. Han, S.L. Jiang, A modified alternating projection based prediction–correction method for structured variational inequalities, Appl. Numer. Math. 83 (2014) 12–21.
[12] A. Amassad, D. Chenais, C. Fabre, Optimal control of an elastic contact problem involving Tresca friction law, Nonlinear Anal. TMA 48 (2002) 1107–1135.
[13] A. Matei, S. Micu, Boundary optimal control for nonlinear antiplane problems, Nonlinear Anal. TMA 74 (2011) 1641–1652.
[14] A. Matei, S. Micu, Boundary optimal control for a frictional contact problem with normal compliance, Appl. Math. Optim. 78 (2018) 379–401.
[15] A. Matei, S. Micu, C. Niţă, Optimal control for antiplane frictional contact problems involving nonlinearly elastic materials of Hencky type, Math. Mech. Solids 23 (2018) 308–328.
[16] M. Sofonea, Y.B. Xiao, Boundary optimal control of a nonsmooth frictionless contact problem, Comput. Math. Appl. (2019) http://dx.doi.org/10.1016/j.camwa.2019.02.027.
[17] M. Sofonea, Y.B. Xiao, Optimization problems for elastic contact models with unilateral constraints, Z. Angew. Math. Phys. 70 (1) (2019) http://dx.doi.org/10.1007/s00033-018-1046-2.
[18] M. Sofonea, Optimal control of variational-hemivariational inequalities, Appl. Math. Optim., in press, http://dx.doi.org./10.1007/s00245-017-9450-0.
[19] M. Sofonea, A. Matei, Y.B. Xiao, Optimal control for a class of mixed variational problems, submitted for publication.
[20] A. Touzaline, Optimal control of a frictional contact problem, Acta Math. Appl. Sin. Engl. Ser. 31 (2015) 991–1000.
[21] Y.B. Xiao, M. Sofonea, On the optimal control of variational-hemivariational inequalities, J. Math. Anal. Appl. 475 (2019) 364–384.

[22] W. Han, M. Sofonea, Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity, in: Studies in Advanced Mathematics, vol. 30, Americal Mathematical Society, Providence, RI-International Press, Somerville, MA, 2002.

[23] J. Haslinger, I. Hlaváček, J. Nečas, Numerical methods for unilateral problems in solid mechanics, in: P.G. Ciarlet, J.L. Lions (Eds.), Handbook of Numerical Analysis, Vol. IV, North-Holland, Amsterdam, 1996, pp. 313–485.

[24] I. Hlaváček, et al., Solution of Variational Inequalities in Mechanics, Springer-Verlag, New York, 1988.

[25] N. Kikuchi, J.T. Oden, Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods, SIAM, Philadelphia, 1988.

[26] Y.B. Xiao, M. Sofonea, Generalized penalty method for elliptic variational–hemivariational inequalities, Appl. Math. Optim. (2019) http://dx.doi.org/10.1007/s00245-019-09563-4.

[27] M. Sofonea, Y.B. Xiao, Fully history-dependent quasivariational inequalities in contact mechanics, Appl. Anal. 95 (2016) 2464–2484.

[28] M. Shillor, M. Sofonea, J.J. Telega, Models and Analysis of Quasistatic Contact, in: Lect. Notes Phys., vol. 655, Springer, Berlin, 2004.

[29] R. Hu, et al., Equivalence results of well-posedness for split variational-hemivariational inequalities, J. Nonlinear Convex Anal. 20 (2019) 447–459.

[30] Q.Y. Shu, R. Hu, Y.B. Xiao, Metric characterizations for well-posedness of split hemivariational inequalities, J. Inequal. Appl. 2018 (2018) 190, http://dx.doi.org/10.1186/s13660-018-1761-4.

[31] Y.B. Xiao, N.J. Huang, M.M. Wong, Well-posedness of hemivariational inequalities and inclusion problems, Taiwanese J. Math. 15 (2011) 1261–1276.

[32] M. Sofonea, A. Matei, History-dependent quasivariational inequalities arising in contact mechanics, Eur. J. Appl. Math. 22 (2011) 471–491.

[33] M. Sofonea, Y.B. Xiao, Tykhonov well-posedness of elliptic variational-hemivariational inequalities, submitted for publication.

[34] A.J. Kurdila, M. Zabarankin, Convex Functional Analysis, Birkhäuser, Basel, 2005.