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Optimization problems for elastic contact models with unilateral constraints

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Abstract. The aim of this paper is to provide some results in the study of an abstract optimization problem in reflexive Banach spaces and to illustrate their use in the analysis and control of static contact problems with elastic materials. We start with a simple model problem which describes the equilibrium of an elastic body in unilateral contact with a foundation. We derive a variational formulation of the model which is in the form of minimization problem for the stress field. Then we introduce the abstract optimization problem for which we prove existence, uniqueness and convergence results. The proofs are based on arguments of lower semicontinuity, monotonicity, convexity, compactness and Mosco convergence. Finally, we use these abstract results to deduce both the unique solvability of the contact model and the existence and the convergence of the optimal pairs for an associated optimal control problem.

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1. Introduction

Optimization methods represent a mathematical tool intensively used in applied mathematics, in both the analysis and numerical approximation of various nonlinear boundary value problems. They are used in solid and fluid mechanics and in engineering sciences as well. Comprehensive references in the field include [4,8,9,29].

Process of contact between deformable bodies arises in industry and everyday life. Their mathematical modeling leads to strongly elliptic or evolutionary nonlinear boundary value problems. References in the field include [6,7,10,13,20,22,27,34] and, more recently, [3,18,25]. There, various existence and uniqueness results have been proved, by using arguments of variational and hemivariational inequalities. In part of these references, the numerical analysis of the models was also provided, together with error estimates and convergence results. Moreover, numerical simulations which represents an evidence of the theoretical results have been presented, together with their mechanical interpretations. Results on optimal control for various contact problems with elastic materials could be found in [1,3,16,23,28,31,32,36] and the references therein. Abstract results in the study of variational and hemivariational inequalities, together with various applications, can be found in the recent papers [11,14,15,18,21,26,33,35,37], for instance.

Our aim in this paper is twofold. The first one is to study a general class of optimization problems in abstract reflexive Banach spaces for which we provide existence, uniqueness and convergence results. The second one is to illustrate how these abstract results can be applied in the analysis and optimal control of contact problems with unilateral constraints. In this way, we construct and develop mathematical tools useful in contact mechanics and illustrate the cross-fertilization between models and applications, on the one hand, and the nonlinear functional analysis, on the other hand.

The results we present here represent a continuation of [2,5], where the analysis and control of new models which describe the equilibrium of an elastic body in contact with a foundation has been carried out. The model considered in [2] was frictionless and its variational formulation was in the form of elliptic variational inequality with unilateral constraints. The model considered in [5] was frictional and, therefore, the variational formulation of the problem was in the form of elliptic quasivariational inequality. In both models, the unknown was the displacement field. We provided the unique solvability of the models, the continuous dependence of the solution with respect to the data and discussed related optimal control problems. To this end, we used arguments of analysis and control for elliptic variational and quasivariational inequalities.

In the current paper, we use a different approach, based on abstract optimization results that we state and prove here for the first time. Due to their generality, these results have an interest on their own and, therefore, they could be useful in the study of various elliptic problems. Nevertheless, our aim is to illustrate their use in the analysis and control of elastic contact problems. To this end, we consider an elastic frictionless contact problem with unilateral constraints for which the abstract results work. In contrast to the models in [2,5], the model we consider in this paper leads to a variational formulation in which the unknown is the stress field, which consists one of the traits of novelty of the current work.

The paper is structured as follows. In Sect. 2, we introduce the contact model, list the assumptions on the data and derive its variational formulation, in terms of the stress. In Sect. 3, we state an abstract optimization problem in reflexive Banach spaces, for which we prove existence, uniqueness and convergence results, gathered in Theorems 3.2 and 3.3. These abstract results are useful in both the analysis and the control of the contact model as we illustrate in the last two sections of the manuscript. Indeed, in Sect. 4 we use Theorem 3.3 to prove the unique weak solvability of the contact model as well as the continuous dependence of the solution with respect to the data. In Sect. 5, we use Theorem 3.2 to prove the existence of optimal pairs for an associated optimal control problem and, under additional assumptions, a convergence result. We end this paper with Sect. 6 in which we present some concluding remarks.

2. The contact model

We consider an elastic body which occupies the domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$, with smooth boundary Γ. The boundary Γ is divided into three measurable disjoint parts Γ_1, Γ_2 and Γ_3 such that $meas\,(\Gamma_1) > 0$. The body is fixed on Γ_1, is acted upon by given surface tractions on Γ_2, and could arrive in frictionless contact with an obstacle on Γ_3. Denote by $\mathbb{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. Then, the mathematical model we consider to describe the equilibrium of the body in this physical setting is the following.

Problem. *$\mathcal{P}$. Find a displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ such that*

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathcal{A}\boldsymbol{\sigma} + \beta\,(\boldsymbol{\sigma} - P_B\boldsymbol{\sigma}) \qquad \text{in} \quad \Omega, \tag{2.1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in} \quad \Omega, \tag{2.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on} \quad \Gamma_1, \tag{2.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \qquad \text{on} \quad \Gamma_2, \tag{2.4}$$

$$u_\nu \le g, \quad \sigma_\nu + p \le 0, \quad (u_\nu - g)(\sigma_\nu + p) = 0 \qquad \text{on} \quad \Gamma_3, \tag{2.5}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \qquad \text{on} \quad \Gamma_3. \tag{2.6}$$

We now provide a description of the equations and boundary conditions in Problem $\mathcal{P}$ where, for simplicity, we do not mention the dependence of various functions with respect to the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$.

First, Eq. (2.1) represents the elastic constitutive law of the material in which $\mathcal{A}$ is the elasticity tensor, β is a given coefficient, B is a nonempty closed convex subset of $\mathbb{S}^d$, $P_B : \mathbb{S}^d \to B$ denotes the projection operator and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ represents the linear strain tensor. Such kind of constitutive laws represents a regularization of the well-known Hencky law, in which the stress tensor is constrained to remain in the convex set B. Details on this matter can be found in [30]. Equation (2.2) is the equation of equilibrium in which $\boldsymbol{f}_0$ denotes the density of body forces and Div is the divergence operator for tensor valued functions. We use it here since the contact process is assumed to be static and, therefore, the inertial term in the equation of motion is neglected. Conditions (2.3), (2.4) represent the displacement and traction boundary conditions, respectively, where $\boldsymbol{\nu}$ denotes the unit outward normal to Γ and $\boldsymbol{f}_2$ represents the density of surface tractions.

Next, condition (2.5) represents a version of the Signorini condition. Here g denotes the gap between the body's surface and the obstacle, measured on the outward normal, and p is a given function, say a pressure. Moreover, the index ν and τ denote the normal and tangential components of vectors and tensors, respectively. This condition describes the contact with a rigid body covered by a fluid of pressure p and was considered in [17] and the references therein. Finally, condition (2.6) represents the frictionless condition in which $\boldsymbol{\sigma}_\tau$ denotes the tangential stress.

To provide the analysis of Problem $\mathcal{P}$, we need to introduce further notation. Thus, we recall that inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\begin{aligned} &\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i\ , \qquad \|\boldsymbol{v}\| = (\boldsymbol{v}\cdot\boldsymbol{v})^{\frac12} \quad \forall \quad \boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^d,\\ &\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}\ , \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{\frac12} \quad \forall \quad \boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^d, \end{aligned}$$

where the indices i, j run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. The zero element of the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$ will be denoted by $\mathbf{0}$. We use the standard notation for Sobolev and Lebesgue spaces associated with Ω and Γ. In particular, we use the spaces $L^2(\Omega)^d$, $L^2(\Gamma_2)^d$, $L^2(\Gamma_3)$ and $H^1(\Omega)^d$, endowed with their canonical inner products and associated norms. Moreover, we recall that for an element $\boldsymbol{v}\in H^1(\Omega)^d$ we still write $\boldsymbol{v}$ for the trace $\gamma\boldsymbol{v}\in L^2(\Gamma)^d$ of $\boldsymbol{v}$ to Γ. In addition, we consider the following spaces:

$$\begin{aligned} &V = \{\,\boldsymbol{v}\in H^1(\Omega)^d:\ \boldsymbol{v}=\mathbf{0}\ \text{ on }\ \Gamma_1\,\},\\ &Q = \{\,\boldsymbol{\sigma}=(\sigma_{ij}):\ \sigma_{ij}=\sigma_{ji}\in L^2(\Omega)\,\}. \end{aligned}$$

The spaces V and Q are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u},\boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x, \qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,\mathrm{d}x. \tag{2.7}$$

Recall that here are below $\boldsymbol{\varepsilon}$ and Div will represent the deformation and the divergence operators, respectively, i.e.,

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac12\,(u_{i,j}+u_{j,i}), \quad \mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j}),$$

where an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable $\boldsymbol{x}=(x_i)$, e.g., $u_{i,j} = \partial u_i/\partial x_j$. The associated norms on these spaces are denoted by $\|\cdot\|_V$ and $\|\cdot\|_Q$. Also, recall that the completeness of the space V follows from the assumption $meas\,(\Gamma_1)>0$ which allows the use of Korn's inequality.

We denote by $\mathbf{0}_V$ the zero element of V and we recall that, for an element $\boldsymbol{v}\in V$, the normal and tangential components on Γ are given by $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$, respectively. For a regular function $\boldsymbol{\sigma}:\Omega\to\mathbb{S}^d$, we have $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$ and, moreover, the following Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x + \int_\Omega \mathrm{Div}\,\boldsymbol{\sigma}\cdot\boldsymbol{v}\,\mathrm{d}x = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{v}\,da \quad \text{for all} \quad \boldsymbol{v}\in H^1(\Omega)^d. \tag{2.8}$$

In the study of contact problem (2.1)–(2.6), we assume that the elasticity tensor $\mathcal{A}$ is symmetric, bounded and positively defined, i.e.,

$$\left\{\begin{array}{l}\text{(a) } \mathcal{A} = (\mathcal{A}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d.\\ \text{(b) } \mathcal{A}_{ijkl} = \mathcal{A}_{klij} = \mathcal{A}_{jikl} \in L^\infty(\Omega),\ 1 \le i,j,k,l \le d.\\ \text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that}\\ \qquad \mathcal{A}\boldsymbol{\tau}\cdot\boldsymbol{\tau} \ge m_{\mathcal{A}}\|\boldsymbol{\tau}\|^2 \quad \forall \quad \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega.\end{array}\right. \tag{2.9}$$

We also assume that the set B and the rest of the data satisfy the following conditions.

$$B \text{ is a closed convex subset of } \mathbb{S}^d \text{ such that } \mathbf{0} \in B. \tag{2.10}$$

$$\beta \in L^\infty(\Omega), \qquad \beta(\boldsymbol{x}) \ge 0 \quad \text{a.e. } \boldsymbol{x} \in \Omega. \tag{2.11}$$

$$\boldsymbol{f}_0 \in L^2(\Omega)^d. \tag{2.12}$$

$$\boldsymbol{f}_2 \in L^2(\Gamma_2)^d. \tag{2.13}$$

$$p \in L^2(\Gamma_3). \tag{2.14}$$

$$g \ge 0. \tag{2.15}$$

Note that, if p is interpreted as a pressure, from physical point of view we have to assume that $p(\boldsymbol{x}) \ge 0$ a.e. $\boldsymbol{x} \in \Gamma_3$. Nevertheless, this additional assumption is not needed from mathematical point of view and, therefore, we skip it here and in Sect. 4. We shall consider it only in Sect. 5 where we deal with an optimal control problem associated with $\mathcal{P}$, in which p represents the control.

Finally, we assume that there exists an element $\boldsymbol{\theta} \in V$ such that

$$\boldsymbol{\theta} = \boldsymbol{\nu} \qquad \text{on} \qquad \Gamma_3, \tag{2.16}$$

and we refer the reader to [12,24] for examples and details on this condition.

We now introduce the form $a : Q \times Q \to \mathbb{R}$ and the function $j : Q \to \mathbb{R}$ defined by

$$a(\boldsymbol{\sigma},\boldsymbol{\tau}) = \int_\Omega \mathcal{A}\boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,\mathrm{d}x \quad \forall \quad \boldsymbol{\sigma},\, \boldsymbol{\tau} \in Q, \tag{2.17}$$

$$j(\boldsymbol{\sigma}) = \frac{1}{2}\int_\Omega \beta\,\|\boldsymbol{\sigma} - P_B\boldsymbol{\sigma}\|^2\,\mathrm{d}x \quad \forall \quad \boldsymbol{\sigma} \in Q. \tag{2.18}$$

Next, we consider the product Hilbert space

$$Y = L^2(\Omega)^d \times L^2(\Gamma_2)^d \times L^2(\Gamma_3) \times \mathbb{R}$$

endowed with the canonical inner product denoted by $(\cdot,\cdot)_Y$. and the associated norm $\|\cdot\|_Y$. Moreover, assumptions (2.12)–(2.15) lead us to consider the subset of Y defined by

$$\Lambda = \{\, \boldsymbol{\eta} = (\boldsymbol{f}_0, \boldsymbol{f}_2, p, g) \in Y \ :\ g \ge 0 \,\}. \tag{2.19}$$

Then, for $\boldsymbol{\eta} = (\boldsymbol{f}_0, \boldsymbol{f}_2, p, g) \in \Lambda$ we introduce the element $\boldsymbol{f}(\boldsymbol{\eta}) \in V$, the sets $U(\boldsymbol{\eta}) \subset V$, $\Sigma(\boldsymbol{\eta}) \subset Q$ and the function $J(\cdot,\boldsymbol{\eta}) : Q \to \mathbb{R}$ defined by

$$(\boldsymbol{f}(\boldsymbol{\eta}), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2\cdot\boldsymbol{v}\,da - \int_{\Gamma_3} p v_\nu\,da \quad \forall \quad \boldsymbol{v} \in V, \tag{2.20}$$

$$U(\boldsymbol{\eta}) = \{\, \boldsymbol{v} \in V \ :\ v_\nu \le g \ \text{ a.e. on } \Gamma_3 \,\}, \tag{2.21}$$

$$\Sigma(\boldsymbol{\eta}) = \{\, \boldsymbol{\tau} \in Q : (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(g\boldsymbol{\theta}))_Q \ge (\boldsymbol{f}(\boldsymbol{\eta}), \boldsymbol{v} - g\boldsymbol{\theta})_V \quad \forall \quad \boldsymbol{v} \in U(\boldsymbol{\eta}) \,\}, \tag{2.22}$$

$$J(\boldsymbol{\sigma},\boldsymbol{\eta}) = \frac{1}{2}\,a(\boldsymbol{\sigma},\boldsymbol{\sigma}) + j(\boldsymbol{\sigma}) - (\boldsymbol{\varepsilon}(g\boldsymbol{\theta}), \boldsymbol{\sigma})_Q \qquad \forall \quad \boldsymbol{\sigma} \in Q. \tag{2.23}$$

We now derive the variational formulation of Problem Q and, to this end, we assume that $(\boldsymbol{u},\boldsymbol{\sigma})$ are sufficiently regular functions which satisfy (2.1)–(2.6), $\boldsymbol{\eta}\in\Lambda$ is fixed and $\boldsymbol{v}\in U(\boldsymbol{\eta})$. Then, using (2.8), equation (2.2) and the boundary conditions (2.3), (2.4) we find that

$$\int_\Omega \boldsymbol{\sigma}\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}))\,\mathrm{d}x \tag{2.24}$$

$$=\int_\Omega \boldsymbol{f}_0\cdot(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}x+\int_{\Gamma_2}\boldsymbol{f}_2\cdot(\boldsymbol{v}-\boldsymbol{u})\,da+\int_{\Gamma_3}\boldsymbol{\sigma\nu}\cdot(\boldsymbol{v}-\boldsymbol{u})\,da.$$

On the other hand,

$$\int_{\Gamma_3}\boldsymbol{\sigma\nu}\cdot(\boldsymbol{v}-\boldsymbol{u})\,da=\int_{\Gamma_3}\boldsymbol{\sigma}_\nu(v_\nu-u_\nu)\,da+\int_{\Gamma_3}\boldsymbol{\sigma}_\tau\cdot(\boldsymbol{v}_\tau-\boldsymbol{u}_\tau)\,da$$

and, therefore, conditions (2.5) and (2.6) yield

$$\int_{\Gamma_3}\boldsymbol{\sigma\nu}\cdot(\boldsymbol{v}-\boldsymbol{u})\,da\geq\int_{\Gamma_3}p(u_\nu-v_\nu)\,da. \tag{2.25}$$

We now combine (2.24), (2.25) and then use notation (2.20) to deduce that

$$(\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}))_Q\geq(\boldsymbol{f}(\boldsymbol{\eta}),\boldsymbol{v}-\boldsymbol{u})_V\quad\forall\quad\boldsymbol{v}\in U(\boldsymbol{\eta}). \tag{2.26}$$

Moreover, assumption (2.16) allows us to write inequality (2.26) with $\boldsymbol{v}=g\boldsymbol{\theta}$, $\boldsymbol{v}=2\boldsymbol{u}-g\boldsymbol{\theta}$, both in $U(\boldsymbol{\eta})$, to see that

$$(\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}(g\boldsymbol{\theta}))_Q=(\boldsymbol{f}(\boldsymbol{\eta}),\boldsymbol{u}-g\boldsymbol{\theta})_V. \tag{2.27}$$

Next, adding (2.26) and (2.27) we find that

$$(\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(g\boldsymbol{\theta})_Q\geq(\boldsymbol{f}(\boldsymbol{\eta}),\boldsymbol{v}-g\boldsymbol{\theta})_V\quad\forall\quad\boldsymbol{v}\in U(\boldsymbol{\eta})$$

which implies that

$$\boldsymbol{\sigma}\in\Sigma(\boldsymbol{\eta}). \tag{2.28}$$

Let $\boldsymbol{\tau}\in\Sigma(\boldsymbol{\eta})$. We use (2.23) to see that

$$J(\boldsymbol{\tau},\boldsymbol{\eta})-J(\boldsymbol{\sigma},\boldsymbol{\eta})$$
$$=\frac{1}{2}\,a(\boldsymbol{\tau}-\boldsymbol{\sigma},\boldsymbol{\tau}-\boldsymbol{\sigma})+a(\boldsymbol{\sigma},\boldsymbol{\tau}-\boldsymbol{\sigma})+j(\boldsymbol{\tau})-j(\boldsymbol{\sigma})-(\boldsymbol{\varepsilon}(g\boldsymbol{\theta}),\boldsymbol{\tau}-\boldsymbol{\sigma})_Q$$

and, therefore, (2.9)(c) implies that

$$J(\boldsymbol{\tau},\boldsymbol{\eta})-J(\boldsymbol{\sigma},\boldsymbol{\eta})$$
$$\geq a(\boldsymbol{\sigma},\boldsymbol{\tau}-\boldsymbol{\sigma})+j(\boldsymbol{\tau})-j(\boldsymbol{\sigma})-(\boldsymbol{\varepsilon}(g\boldsymbol{\theta}),\boldsymbol{\tau}-\boldsymbol{\sigma})_Q. \tag{2.29}$$

On the other hand, it is well known that the function $j:Q\to\mathbb{R}$ is convex and Gâteaux differentiable and, moreover, $\nabla j(\boldsymbol{\sigma})=\beta(\boldsymbol{\sigma}-P_B\boldsymbol{\sigma})$ for all $\boldsymbol{\sigma}\in Q$. Therefore, the subgradient inequality yields

$$j(\boldsymbol{\tau})-j(\boldsymbol{\sigma})\geq(\beta(\boldsymbol{\sigma}-P_B\boldsymbol{\sigma}),\boldsymbol{\tau}-\boldsymbol{\sigma})_Q. \tag{2.30}$$

We now combine inequalities (2.29) and (2.30) then use the constitutive law (2.1) to see that

$$J(\boldsymbol{\tau},\boldsymbol{\eta})-J(\boldsymbol{\sigma},\boldsymbol{\eta})\geq(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}(g\boldsymbol{\theta}),\boldsymbol{\tau}-\boldsymbol{\sigma})_Q. \tag{2.31}$$

Note that $\boldsymbol{u}\in U(\boldsymbol{\eta})$. Then, inequality (2.31), equality (2.27) and definition (2.22) show that $J(\boldsymbol{\tau},\boldsymbol{\eta})\geq J(\boldsymbol{\sigma},\boldsymbol{\eta})$. Therefore, we deduce the following variational formulation of Problem $\mathcal{P}$.

Problem. $\mathcal{P}^V$. *Given $\boldsymbol{\eta}\in\Lambda$, find a stress field $\boldsymbol{\sigma}$ such that*

$$\boldsymbol{\sigma}\in\Sigma(\boldsymbol{\eta}),\qquad J(\boldsymbol{\sigma},\boldsymbol{\eta})\leq J(\boldsymbol{\tau},\boldsymbol{\eta})\qquad\forall\,\boldsymbol{\tau}\in\Sigma(\boldsymbol{\eta}). \tag{2.32}$$

The analysis of Problem $\mathcal{P}^V$, including existence, uniqueness and various convergence results, will be provided in the next section. Here we restrict ourselves to mention that a couple of functions $(\boldsymbol{u},\boldsymbol{\sigma})$ which satisfies (2.32) and (2.1) is called a weak solution of the elastic contact problem (2.1)–(2.6).

3. An abstract optimization problem

In this section, we consider an abstract optimization problem which includes as particular case Problem $\mathcal{P}^V$. We prove existence and convergence results which have interest in their own and are useful in the analysis and control of the contact problem $\mathcal{P}$. The functional framework is as follows. Assume that $(X,\|\cdot\|_X)$ is a reflexive Banach space, $(Y,\|\cdot\|_Y)$ a normed space, $\Lambda\subset Y$, $J: X\times\Lambda\to\mathbb{R}$ and, for each $\eta\in\Lambda$, $K(\eta)$ is a given subset of X. Then, the optimization problem under consideration is as follows.

Problem. $\mathcal{O}$. *Given $\eta\in\Lambda$, find u such that*

$$u\in K(\eta),\qquad J(u,\eta)=\min_{v\in K(\eta)} J(v,\eta). \tag{3.1}$$

Next, for each $n\in\mathbb{N}$, we consider a perturbation $\eta_n\in\Lambda$ of η, together with the following optimization problem.

Problem. $\mathcal{O}_n$. *Given $\eta_n\in\Lambda$, find u_n such that*

$$u_n\in K(\eta_n),\qquad J(u_n,\eta_n)=\min_{v\in K(\eta_n)} J(v,\eta_n). \tag{3.2}$$

Below in this paper, we denote by $\to$ and $\rightharpoonup$ the strong and weak convergence in various normed spaces, which will be specified. Moreover, in the study of problems $\mathcal{O}$ and $\mathcal{O}_n$ we consider the following assumptions.

(Λ) Λ is a nonempty weakly closed subset of Y.

(K) For each $\eta\in\Lambda$, $K(\eta)$ is a nonempty weakly closed subset of X.

(K^*) For each $\eta\in\Lambda$, $K(\eta)=K$ with $K\subset X$ given.

$$(J_1)\quad \left\{\begin{array}{l}\text{For all sequences } \{u_k\}\subset X \text{ and } \{\eta_k\}\subset\Lambda \text{ such that}\\ u_k\rightharpoonup u \text{ in } X,\ \eta_k\rightharpoonup\eta \text{ in } Y \text{ and for all } v\in X,\\ \text{the inequality below holds:}\\ \limsup\limits_{k\to\infty}\,[J(v,\eta_k)-J(u_k,\eta_k)]\le J(v,\eta)-J(u,\eta).\end{array}\right.$$

$$(J_2)\quad \left\{\begin{array}{l}\text{For all sequences } \{u_k\}\subset X \text{ and } \{\eta_k\}\subset\Lambda \text{ such that}\\ \|u_k\|_X\to\infty \text{ and } \eta_k\rightharpoonup\eta \text{ in } Y,\text{ one has } J(u_k,\eta_k)\to\infty.\end{array}\right.$$

$$(J_3)\quad \left\{\begin{array}{l}\text{For all sequence } \{\eta_k\}\subset\Lambda \text{ such that } \eta_k\rightharpoonup\eta \text{ in } Y\\ \text{and all } v\in X,\text{ one has } J(v,\eta_k)\to J(v,\eta).\end{array}\right.$$

$$(J_4)\quad \left\{\begin{array}{l}\text{For all sequences } \{v_k\}\subset X \text{ and } \{\eta_k\}\subset\Lambda \text{ such that}\\ v_k\to v \text{ in } X,\ \eta_k\rightharpoonup\eta \text{ in } Y \text{ one has}\\ J(v_k,\eta_k)-J(v,\eta_k)\to 0.\end{array}\right.$$

$$\eta_n\rightharpoonup\eta\quad\text{in}\quad Y. \tag{3.3}$$

$$K(\eta_n)\xrightarrow{M}K(\eta)\quad\text{in}\quad X, \tag{3.4}$$

where notation "$\xrightarrow{M}$" denotes the convergence in the sense of Mosco that we recall below, for the convenience of the reader.

Definition 3.1. Let X be a normed space, $\{K_n\}$ a sequence of nonempty subsets of X and K a nonempty subset of X. We say that the sequence $\{K_n\}$ converges to K in the Mosco sense if the following conditions hold.

$$(M_1)\qquad \begin{cases}\text{For each } v \in K, \text{ there exists a sequence } \{v_n\} \text{ such that}\\ v_n \in K_n \text{ for each } n \in \mathbb{N} \text{ and } v_n \to v \text{ in } X.\end{cases}$$

$$(M_2)\qquad \begin{cases}\text{For each sequence } \{v_n\} \text{ such that}\\ v_n \in K_n \text{ for each } n \in \mathbb{N} \text{ and } v_n \rightharpoonup v \text{ in } X, \text{ we have } v \in K.\end{cases}$$

Note that the convergence in the sense of Mosco depends on the topology of the normed space X and, for this reason, in (3.4) we write explicitely $K(\eta_n) \xrightarrow{M} K(\eta)$ in X. More details on this topic can be found in [19].

Our first result in this section is as follows.

Theorem 3.2. *Assume the hypotheses* (Λ), (K), (J_1) *and* (J_2). *Then, we have the following statements.*

(i) *Problem $\mathcal{O}$ has at least one solution and Problem $\mathcal{O}_n$ has at least one solution, for each $n \in \mathbb{N}$.*

(ii) *Under assumptions (3.3), (3.4), (J_3), (J_4) or, alternatively, under assumptions (3.3), (K^*), (J_3), if u_n is a solution of Problem $\mathcal{O}_n$, for each $n \in \mathbb{N}$, then there exists a subsequence of the sequence $\{u_n\}$, again denoted $\{u_n\}$, and an element $u \in X$, such that*

$$u_n \rightharpoonup u \quad \text{in} \quad X. \tag{3.5}$$

Moreover, u is a solution to Problem $\mathcal{O}$.

Proof. (i) Let $\lambda \in \Lambda$ be given. We take $\eta_k = \eta$ in (J_1) to see that, for all sequences $\{u_k\} \subset X$ such that $u_k \rightharpoonup u$ and for all $v \in X$,

$$\limsup_{k\to\infty}\,[J(v,\eta) - J(u_k,\eta)] \le J(v,\eta) - J(u,\eta),$$

which implies that

$$\liminf_{k\to\infty}\, J(u_k,\eta) \ge J(u,\eta).$$

It follows that the function $J(\cdot,\eta) : X \to \mathbb{R}$ is lower semicontinuous. Moreover, taking $\eta_k = \eta$ in (J_2) we deduce that $J(\cdot,\eta)$ is coercive. Recall also the assumption (K) on $K(\eta)$. The existence of at least one solution to Problem $\mathcal{O}$ is now a direct consequence of the well-known Weierstrass theorem. The existence of at least one solution to Problem $\mathcal{O}_n$ follows by the same argument, applied to the function $J(\cdot,\eta_n) : X \to \mathbb{R}$ with $\eta_n \in \Lambda$ given.

(ii) Assume now that, in addition, (3.3), (3.4), (J_3) and (J_4) hold. We claim that the sequence $\{u_n\}$ is bounded in X. Indeed, if $\{u_n\}$ is not bounded, then we can find a subsequence of the sequence $\{u_n\}$, again denoted $\{u_n\}$, such that $\|u_n\|_X \to \infty$. Therefore, using assumptions (3.3) and (J_2) we deduce that

$$J(u_n,\eta_n) \to \infty. \tag{3.6}$$

Let v be a given element in $K(\eta)$ and recall that assumption (3.4) implies that condition (M_1) holds. Thus, there exists a sequence $\{v_n\}$ such that $v_n \in K(\eta_n)$ for each $n \in \mathbb{N}$ and

$$v_n \to v \quad \text{in} \quad X. \tag{3.7}$$

Moreover, since u_n is a solution of Problem $\mathcal{O}_n$ we obtain that $J(u_n,\eta_n) \le J(v_n,\eta_n)$ and, therefore,

$$J(u_n,\eta_n) \le J(v_n,\eta_n) - J(v,\eta_n) + J(v,\eta_n) - J(v,\eta) + J(v,\eta) \qquad \forall\, n \in \mathbb{N}. \tag{3.8}$$

On the other hand, convergences (3.7) and (3.3) allow us to use assumption (J_4) to find that $J(v_n,\eta_n) - J(v,\eta_n) \to 0$ and, in addition, assumption (J_3) shows that $J(v,\eta_n) - J(v,\eta) \to 0$. Thus, inequality (3.8) implies that the sequence $\{J(u_n,\eta_n)\}$ is bounded, which contradicts (3.6). We conclude from above that the sequence $\{u_n\}$ is bounded in X and, therefore, there exists a subsequence of the sequence $\{u_n\}$, again denoted $\{u_n\}$, and an element $u \in X$, such that (3.5) holds.

We now prove that u is a solution of Problem $\mathcal{O}$. To this end, we use (3.5) and condition (M_2), guaranteed by assumption (3.4), to deduce that $u \in K(\eta)$. Next, we consider an arbitrary element $v \in K(\eta)$ and, using condition (M_1), we know that there exists a sequence $\{v_n\}$ such that $v_n \in K(\eta_n)$ for each $n \in \mathbb{N}$ and (3.7) holds. Since u_n is a solution to Problem $\mathcal{O}_n$ we have $J(u_n, \eta_n) \le J(v_n, \eta_n)$, which implies that

$$0 \le [J(v, \eta_n) - J(u_n, \eta_n)] + [J(v_n, \eta_n) - J(v, \eta_n)] \qquad \forall\, n \in \mathbb{N}. \tag{3.9}$$

We now use convergences (3.7), (3.3) and assumptions (J_1), (J_4) to see that

$$\limsup_{n\to\infty} [J(v, \eta_n) - J(u_n, \eta_n)] \le J(v, \eta) - J(u, \eta), \tag{3.10}$$

$$J(v_n, \eta_n) - J(v, \eta_n) \to 0 \tag{3.11}$$

We now combine (3.9)–(3.11) to deduce that u is a solution of Problem $\mathcal{O}$.

Alternatively, assume now that (3.3), (K^*), (J_3) hold. Then, $K(\eta_n) = K(\eta) = K$ and, since u_n is a solution of Problem $\mathcal{O}_n$, we have

$$J(u_n, \eta_n) \le J(v, \eta_n) \quad \forall \quad v \in K,\ n \in \mathbb{N}. \tag{3.12}$$

Using now assumption (J_3) we deduce that the sequence $\{J(u_n, \eta_n)\}$ is bounded, which contradicts (3.6). We conclude from above that the sequence $\{u_n\}$ is bounded in X and, therefore, there exists a subsequence of the sequence $\{u_n\}$, again denoted $\{u_n\}$, and an element $u \in X$, such that (3.5) holds.

We now prove that u is a solution of Problem $\mathcal{O}$. To this end, we use (3.5) and assumptions (K^*), (K) to see that $u \in K(\eta) = K$. Next, we consider an arbitrary element $v \in K(\eta)$. Since u_n is the solution to Problem $\mathcal{O}_n$ we have $J(u_n, \eta_n) \le J(v, \eta_n)$ which implies that

$$0 \le J(v, \eta_n) - J(u_n, \eta_n) \qquad \forall\, n \in \mathbb{N}. \tag{3.13}$$

We now use convergence (3.3) and assumption (J_1) to see that

$$\limsup_{n\to\infty} [J(v, \eta_n) - J(u_n, \eta_n)] \le J(v, \eta) - J(u, \eta). \tag{3.14}$$

We now combine (3.13) and (3.14) to deduce that u is a solution of Problem $\mathcal{O}$, which concludes the proof. $\square$

We now reinforce the conditions on the data by considering the following assumptions.

$(\widetilde{K})$ For each $\eta \in \Lambda$, $K(\eta) \subset X$ is a nonempty closed convex subset.

$(\widetilde{J})$ For each $\eta \in \Lambda$, $J(\cdot, \eta) : X \to \mathbb{R}$ is a strictly convex function.

$$(J^*) \qquad \begin{cases} \text{There exists } m > 0 \text{ such that} \\ (1-t)J(u, \eta) + tJ(v, \eta) - J((1-t)u + tv, \eta) \ge m\, t(1-t)\|u - v\|_X^2 \\ \text{for all } u,\ v \in X,\ \eta \in \Lambda,\ t \in [0, 1]. \end{cases}$$

Note that assumption $(\widetilde{K})$ implies assumption (K) and, moreover, assumption (J^*) implies assumption $(\widetilde{J})$.

Theorem 3.3. *Assume the hypotheses* (Λ), $(\widetilde{K})$, (J_1), (J_2) *and* $(\widetilde{J})$. *Then we have the following statements.*

(i) *Problem $\mathcal{O}$ has a unique solution u and Problem $\mathcal{O}_n$ has a unique solution u_n, for each $n \in \mathbb{N}$.*

(ii) *Under assumptions (3.3), (3.4), (J_3), (J_4) or, alternatively, under assumptions (3.3), (K^*), (J_3), the sequence $\{u_n\}$ converges weakly to u, i.e., $u_n \rightharpoonup u$ in X.*

(iii) *Under assumptions (3.3), (3.4), (J_3), (J_4), (J^*) or, alternatively, under assumptions (3.3), (K^*), (J_3), (J^*), the sequence $\{u_n\}$ converges strongly to u, i.e., $u_n \to u$ in X.*

Proof. (i) For the existence part, we use arguments similar to those used in the proof of Theorem 3.2 (i), (ii). Since the modifications are straightforward, we skip the details. The uniqueness part follows from the strictly convexity of the functions $J(\cdot, \eta)$ and $J(\cdot, \eta_n)$, guaranteed by assumption $(\widetilde{J})$.

(ii) Assume now that, in addition, (3.3), (3.4), (J_3), (J_4) or, alternatively, (3.3), (K^*), (J_3) hold. Then, a careful analysis of the proof of Theorem 3.2 ii) reveals that in both cases the sequence $\{u_n\}$ is bounded and any weakly convergent subsequence of $\{u_n\}$ converges to a solution of Problem $\mathcal{P}$. On the other hand, Problem $\mathcal{P}$ has a unique solution, denoted u, as proved in the first part of the theorem. The weak convergence of the whole sequence $\{u_n\}$ to u is now a consequence of a standard argument.

(iii) Assume now that (3.3), (3.4), (J_3), (J_4), (J^*) hold and let $\{\widetilde{u}_n\}$ be a sequence such that $\widetilde{u}_n \in K(\eta_n)$ for each $n \in \mathbb{N}$ and

$$\widetilde{u}_n \to u \quad \text{in} \quad X. \tag{3.15}$$

Recall that the existence of such sequence follows from assumption (M_1), guaranteed by condition (3.4). Then, using (J^*) with $t = \frac{1}{2}$ we find that

$$m\|\widetilde{u}_n - u_n\|_X^2 \le 2\left[J(\widetilde{u}_n, \eta_n) - J\Big(\frac{\widetilde{u}_n + u_n}{2}, \eta_n\Big)\right] + 2\left[J(u_n, \eta_n) - J\Big(\frac{\widetilde{u}_n + u_n}{2}, \eta_n\Big)\right]$$

and, since u_n is a minimizer for the function $J(\cdot, \eta_n)$ on $K(\eta_n)$, it follows that

$$m\|\widetilde{u}_n - u_n\|_X^2 \le 2\Big[J(\widetilde{u}_n, \eta_n) - J\Big(\frac{\widetilde{u}_n + u_n}{2}, \eta_n\Big)\Big], \tag{3.16}$$

which implies that

$$m\|\widetilde{u}_n - u_n\|_X^2 \le 2\Big[J(u, \eta_n) - J\Big(\frac{\widetilde{u}_n + u_n}{2}, \eta_n\Big)\Big] + 2\Big[J(\widetilde{u}_n, \eta_n) - J(u, \eta_n)\Big] \tag{3.17}$$

for all $n \in \mathbb{N}$.

Next, we use convergences (3.3), (3.5), (3.15) and assumptions (J_1), (J_4) to deduce that

$$\limsup_{n\to\infty} \left[J(u, \eta_n) - J\Big(\frac{\widetilde{u}_n + u_n}{2}, \eta_n\Big)\right] \le 0, \tag{3.18}$$

$$J(\widetilde{u}_n, \eta_n) - J(u, \eta_n) \to 0. \tag{3.19}$$

We now combine inequalities (3.17)–(3.19) to find that $\widetilde{u}_n - u_n \to 0$ in X, which implies that $u_n \to u$ in X due to (3.15).

Alternatively, assume now that assumptions (3.3), (K^*) and (J_3) hold. Then, $K(\eta_n) = K(\eta) = K$ and, therefore, we are allowed to take $\widetilde{u}_n = u$ in (3.16) to obtain that

$$m\|u - u_n\|_X^2 \le 2\Big[J(u, \eta_n) - J\Big(\frac{u + u_n}{2}, \eta_n\Big)\Big] \qquad \forall \quad n \in \mathbb{N}. \tag{3.20}$$

On the other hand, the part ii) shows that the sequence $\{u_n\}$ converges weakly to u, i.e., $u_n \rightharpoonup u$ in X, which implies that

$$\frac{u_n + u}{2} \rightharpoonup u \quad \text{in} \quad X. \tag{3.21}$$

We use convergences (3.3), (3.21) and assumption (J_1) with $v = u$ to deduce that

$$\limsup_{n\to\infty} \left[J(u, \eta_n) - J\Big(\frac{u_n + u}{2}, \eta_n\Big)\right] \le 0. \tag{3.22}$$

Thus, it follows from inequalities (3.20) and (3.22) that $u_n \to u$ in X which concludes the proof. □

4. Existence, uniqueness and convergence results

In this section, we use Theorem 3.3 to study Problem $\mathcal{P}^V$. To this end, for each $n \in \mathbb{N}$ we consider a perturbation $\boldsymbol{f}_{0n}$, $\boldsymbol{f}_{2n}$, p_n, g_n of the data $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, p, g, respectively, such that

$$\boldsymbol{f}_{0n} \in L^2(\Omega)^d, \tag{4.1}$$

$$\boldsymbol{f}_{2n} \in L^2(\Gamma_2)^d, \tag{4.2}$$

$$p_n \in L^2(\Gamma_3), \tag{4.3}$$

$$g_n \ge 0. \tag{4.4}$$

Moreover, we assume that the following convergences hold.

$$\boldsymbol{f}_{0n} \rightharpoonup \boldsymbol{f}_0 \quad \text{in} \quad L^2(\Omega)^d, \tag{4.5}$$
$$\boldsymbol{f}_{2n} \rightharpoonup \boldsymbol{f}_2 \quad \text{in} \quad L^2(\Gamma_2)^d, \tag{4.6}$$
$$p_n \rightharpoonup p \quad \text{in} \quad L^2(\Gamma_3), \tag{4.7}$$
$$g_n \to g. \tag{4.8}$$

Denote $\boldsymbol{\eta}_n = (\boldsymbol{f}_{0n}, \boldsymbol{f}_{2n}, p_n, g_n)$, which clearly belongs to Λ. With these data, we consider the following perturbation of Problem $\mathcal{P}^V$.

Problem. $\mathcal{P}_n^V$. *Given $\boldsymbol{\eta}_n \in \Lambda$, find a stress field $\boldsymbol{\sigma}_n$ such that*

$$\boldsymbol{\sigma}_n \in \Sigma(\boldsymbol{\eta}_n), \qquad J(\boldsymbol{\sigma}_n, \boldsymbol{\eta}_n) \le J(\boldsymbol{\tau}, \boldsymbol{\eta}_n) \qquad \forall\, \boldsymbol{\tau} \in \Sigma(\boldsymbol{\eta}_n). \tag{4.9}$$

We have the following existence, uniqueness and convergence result.

Theorem 4.1. *Assume (2.9)–(2.16), (4.1)–(4.4). Then, the following statement hold.*

(i) *Problem $\mathcal{P}^V$ has a unique solution $\boldsymbol{\sigma}$ and, for each $n \in \mathbb{N}$, Problem $\mathcal{P}_n^V$ has a unique solution $\boldsymbol{\sigma}_n$.*
(ii) *If (4.5)–(4.8) hold, then the sequence $\{\boldsymbol{\sigma}_n\}$ converges strongly to $\boldsymbol{\sigma}$, i.e., $\boldsymbol{\sigma}_n \to \boldsymbol{\sigma}$ in X.*

To provide the proof of Theorem 4.1, we need some preliminary results that we present in what follows. First, we note that assumption (2.9) on the elasticity tensor implies that the bilinear form a is symmetric, continuous and coercive with constant $m_{\mathcal{A}}$. Therefore,

$$\boldsymbol{\sigma}_k \to \boldsymbol{\sigma} \ \text{ in } \ Q, \quad \boldsymbol{\tau}_k \to \boldsymbol{\tau} \ \text{ in } \ Q \quad \Longrightarrow \quad a(\boldsymbol{\sigma}_k, \boldsymbol{\tau}_k) \to a(\boldsymbol{\sigma}, \boldsymbol{\tau}). \tag{4.10}$$
$$a(\boldsymbol{\tau}, \boldsymbol{\tau}) \ge m_{\mathcal{A}} \|\boldsymbol{\tau}\|_Q^2 \qquad \forall\, \boldsymbol{\tau} \in Q. \tag{4.11}$$

This implies that the function $\boldsymbol{\tau} \mapsto a(\boldsymbol{\tau}, \boldsymbol{\tau})$ is weakly lower semicontinuous on X, i.e.,

$$\boldsymbol{\tau}_k \rightharpoonup \boldsymbol{\tau} \ \text{ in } \ Q \quad \Longrightarrow \quad \liminf_{k\to\infty} a(\boldsymbol{\tau}_k, \boldsymbol{\tau}_k) \ge a(\boldsymbol{\tau}, \boldsymbol{\tau}). \tag{4.12}$$

On the other hand, it is easy to see that the function j is convex and continuous and, therefore, it is weakly lower semicontinuous. Hence,

$$\boldsymbol{\tau}_k \rightharpoonup \boldsymbol{\tau} \ \text{ in } \ Q \quad \Longrightarrow \quad \liminf_{k\to\infty} j(\boldsymbol{\tau}_k) \ge j(\boldsymbol{\tau}), \tag{4.13}$$
$$\boldsymbol{\tau}_k \to \boldsymbol{\tau} \ \text{ in } \ Q \quad \Longrightarrow \quad \lim_{k\to\infty} j(\boldsymbol{\tau}_k) = j(\boldsymbol{\tau}). \tag{4.14}$$

We are now in position to provide the proof of Theorem 4.1.

Proof. We shall use Theorem 3.3 with $X = Q$ and $K(\cdot) = \Sigma(\cdot)$, Λ and J being defined by (2.19) and (2.23), respectively. To this end, in what follows we check the validity of assumptions of this theorem.

First, it is easy to see that the set Λ is a nonempty closed convex subset of Y which implies that condition (Λ) holds. On the other hand, for each $\boldsymbol{\eta} \in \Lambda$ the set $\Sigma(\boldsymbol{\eta})$ is a closed convex subset of Q and, moreover, since

$$(\boldsymbol{\varepsilon}(\boldsymbol{f}(\boldsymbol{\eta})), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q = (\boldsymbol{f}(\boldsymbol{\eta}), \boldsymbol{v})_V \qquad \forall\, \boldsymbol{v} \in V,$$

we deduce that $\boldsymbol{\varepsilon}(\boldsymbol{f}(\boldsymbol{\eta})) \in \Sigma(\boldsymbol{\eta})$. Therefore, condition $(\widetilde{K})$ is satisfied. In addition, a simple calculation based on the definitions (2.17)–(2.18), the properties of the form a and the convexity of the function j shows that

$$(1-t)J(\boldsymbol{\sigma}, \boldsymbol{\eta}) + tJ(\boldsymbol{\tau}, \boldsymbol{\eta}) - J((1-t)\boldsymbol{\sigma} + t\boldsymbol{\tau}, \boldsymbol{\eta}) \ge \frac{1}{2}\, t(1-t)\, a(\boldsymbol{\sigma} - \boldsymbol{\tau}, \boldsymbol{\sigma} - \boldsymbol{\tau})$$

for all $\boldsymbol{\sigma}$, $\boldsymbol{\tau} \in X$, $\boldsymbol{\eta} \in \Lambda$, $t \in [0, 1]$. We combine this inequality with inequality (4.11) to see that condition (J^*) holds.

Assume now that $\{\boldsymbol{\sigma}_k\} \subset Q$ and $\{\boldsymbol{\eta}_k\} \subset \Lambda$ are two sequences such that $\boldsymbol{\sigma}_k \rightharpoonup \boldsymbol{\sigma}$ in Q, $\boldsymbol{\eta}_k = (\boldsymbol{f}_{0k}, \boldsymbol{f}_{2k}, p_k, g_k) \rightharpoonup \boldsymbol{\eta} = (\boldsymbol{f}_0, \boldsymbol{f}_2, p, g)$ in Y and let $\boldsymbol{\tau} \in Q$. Since

$$J(\boldsymbol{\tau}, \boldsymbol{\eta}_k) - J(\boldsymbol{\sigma}_k, \boldsymbol{\eta}_k) = \frac{1}{2}\, a(\boldsymbol{\tau}, \boldsymbol{\tau}) - \frac{1}{2}\, a(\boldsymbol{\sigma}_k, \boldsymbol{\sigma}_k) + j(\boldsymbol{\tau}) - j(\boldsymbol{\sigma}_k) - (\boldsymbol{\varepsilon}(g_k \boldsymbol{\theta}), \boldsymbol{\tau} - \boldsymbol{\sigma}_k)_Q,$$

using (4.12), (4.13) and the convergence $g_k \to g$, we deduce that

$$\begin{aligned}&\limsup_{k\to\infty}\, [J(\boldsymbol{\tau}, \boldsymbol{\eta}_k) - J(\boldsymbol{\sigma}_k, \boldsymbol{\eta}_k)]\\ &\quad le \frac{1}{2}\, a(\boldsymbol{\tau}, \boldsymbol{\tau}) - \frac{1}{2}\, \liminf_{k\to\infty}\, a(\boldsymbol{\sigma}_k, \boldsymbol{\sigma}_k) + j(\boldsymbol{\tau}) - \liminf_{k\to\infty} j(\boldsymbol{\sigma}_k) - (\boldsymbol{\varepsilon}(g\boldsymbol{\theta}), \boldsymbol{\tau} - \boldsymbol{\sigma})_Q\\ &\quad le \frac{1}{2}\, a(\boldsymbol{\tau}, \boldsymbol{\tau}) - \frac{1}{2}\, a(\boldsymbol{\sigma}, \boldsymbol{\sigma}) + j(\boldsymbol{\tau}) - j(\boldsymbol{\sigma}) - (\boldsymbol{\varepsilon}(g\boldsymbol{\theta}), \boldsymbol{\tau} - \boldsymbol{\sigma})_Q = J(\boldsymbol{\tau}, \boldsymbol{\eta}) - J(\boldsymbol{\sigma}, \boldsymbol{\eta}).\end{aligned}$$

It follows from here that condition (J_1) is satisfied.

On the other hand, for any sequences $\{\boldsymbol{\sigma}_k\} \subset Q$ and $\{\boldsymbol{\eta}_k\} \subset \Lambda$, using inequality (4.11) and the positivity of j we have

$$\begin{aligned}J(\boldsymbol{\sigma}_k, \boldsymbol{\eta}_k) &= \frac{1}{2}\, a(\boldsymbol{\sigma}_k, \boldsymbol{\sigma}_k) + j(\boldsymbol{\sigma}_k) - (\boldsymbol{\varepsilon}(g_k \boldsymbol{\theta}), \boldsymbol{\sigma}_k)_Q\\ &\geq \frac{m_{\mathcal{A}}}{2}\, \|\boldsymbol{\sigma}_k\|_Q^2 - g_k \|\boldsymbol{\varepsilon}(\boldsymbol{\theta})\|_Q \|\boldsymbol{\sigma}_k\|_Q.\end{aligned} \tag{4.15}$$

Assume now that and $\boldsymbol{\eta}_k \rightharpoonup \boldsymbol{\eta}$ in Y. Then $\{\boldsymbol{\eta}_k\}$ is bounded in Y which implies that $\{g_k\}$ is bounded in $\mathbb{R}$. Therefore, if $\|\boldsymbol{\sigma}_k\|_Q \to \infty$, inequality (4.15) shows that $J(\boldsymbol{\sigma}_k, \boldsymbol{\eta}_k) \to \infty$. We conclude from above that condition (J_2) is satisfied, too.

Let $\{\boldsymbol{\eta}_k\} \subset \Lambda$ be a sequence such that $\boldsymbol{\eta}_k \rightharpoonup \boldsymbol{\eta}$ in Y and let $\boldsymbol{\tau} \in Q$. We have

$$J(\boldsymbol{\tau}, \boldsymbol{\eta}_k) - J(\boldsymbol{\tau}, \boldsymbol{\eta}) = (g - g_k)(\boldsymbol{\varepsilon}(\boldsymbol{\theta}), \boldsymbol{\tau})$$

and, using the convergence $g_k \to g$ we obtain that $J(\boldsymbol{\tau}, \boldsymbol{\eta}_k) - J(\boldsymbol{\tau}, \boldsymbol{\eta}) \to 0$ which shows that condition (J_3) holds.

Assume now that $\{\boldsymbol{\tau}_k\} \subset Q$ and $\{\boldsymbol{\eta}_k\} \subset \Lambda$ are two sequences such that $\boldsymbol{\tau}_k \to \boldsymbol{\tau}$ in Q and $\boldsymbol{\eta}_k \rightharpoonup \boldsymbol{\eta}$ in Y. We have

$$J(\boldsymbol{\tau}_k, \boldsymbol{\eta}_k) - J(\boldsymbol{\tau}, \boldsymbol{\eta}_k) = \frac{1}{2}\, a(\boldsymbol{\tau}_k, \boldsymbol{\tau}_k) - \frac{1}{2}\, a(\boldsymbol{\tau}, \boldsymbol{\tau}) + j(\boldsymbol{\tau}_k) - j(\boldsymbol{\tau}) - g_k(\boldsymbol{\varepsilon}(\boldsymbol{\theta}), \boldsymbol{\tau}_k - \boldsymbol{\tau})_Q$$

and, using convergences (4.10) and (4.14), we deduce that

$$J(\boldsymbol{\tau}_k, \boldsymbol{\eta}_k) - J(\boldsymbol{\tau}, \boldsymbol{\eta}_k) \to 0,$$

which shows that condition (J_4) holds.

Assume in what follows that (4.5)–(4.8) hold. Then

$$\boldsymbol{\eta}_n = (\boldsymbol{f}_{0n}, \boldsymbol{f}_{2n}, p_n, g_n) \rightharpoonup \boldsymbol{\eta} = (\boldsymbol{f}_0, \boldsymbol{f}_2, p, g) \quad \text{in} \quad Y, \tag{4.16}$$

which shows that condition (3.3) holds, too.

We now prove that

$$\Sigma(\boldsymbol{\eta}_n) \xrightarrow{M} \Sigma(\boldsymbol{\eta}) \quad \text{in} \quad Q. \tag{4.17}$$

To this end, we introduce the sets

$$U_0 = \{\ \boldsymbol{v} \in V \ :\ v_\nu \leq 0, \quad \text{a.e. on} \ \ \Gamma_3\}, \tag{4.18}$$

$$\Sigma_0 = \{\, \boldsymbol{\tau} \in Q : (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \geq 0 \quad \forall\, \boldsymbol{v} \in U_0\,\}. \tag{4.19}$$

Then, it is easy to see that $\boldsymbol{v} \in U(\boldsymbol{\eta})$ if and only if $\boldsymbol{v} - g\boldsymbol{\theta} \in U_0$ and, therefore, definition (2.22) implies that $\boldsymbol{\tau} \in \Sigma(\boldsymbol{\eta})$ if and only if $\boldsymbol{\tau} - \boldsymbol{\varepsilon}(\boldsymbol{f}(\boldsymbol{\eta})) \in \Sigma_0$. We conclude from here that

$$\Sigma(\boldsymbol{\eta}) = \boldsymbol{\varepsilon}(\boldsymbol{f}(\boldsymbol{\eta})) + \Sigma_0, \tag{4.20}$$

$$\Sigma(\boldsymbol{\eta}_n) = \boldsymbol{\varepsilon}(\boldsymbol{f}(\boldsymbol{\eta}_n)) + \Sigma_0 \qquad \forall \quad n \in \mathbb{N}. \tag{4.21}$$

Let $\boldsymbol{\tau} \in \Sigma(\boldsymbol{\eta})$. Then, using (4.20) it follows that there exists an element $\boldsymbol{\tau}_0 \in \Sigma_0$ such that $\boldsymbol{\tau} = \boldsymbol{\varepsilon}(\boldsymbol{f}(\boldsymbol{\eta})) + \boldsymbol{\tau}_0$. Define $\boldsymbol{\tau}_n = \boldsymbol{\varepsilon}(\boldsymbol{f}(\boldsymbol{\eta}_n)) + \boldsymbol{\tau}_0$ and note that (4.21) shows that $\boldsymbol{\tau}_n \in \Sigma(\boldsymbol{\eta}_n)$. Moreover, an elementary calculus combined with definition (2.20) show that

$$\begin{aligned}
\|\boldsymbol{\tau}_n - \boldsymbol{\tau}\|_Q^2 &= \|\boldsymbol{\varepsilon}(\boldsymbol{f}(\boldsymbol{\eta}_n)) - \boldsymbol{\varepsilon}(\boldsymbol{f}(\boldsymbol{\eta}))\|_Q^2 = \|\boldsymbol{f}(\boldsymbol{\eta}_n) - \boldsymbol{f}(\boldsymbol{\eta})\|_V^2 \\
&= (\boldsymbol{f}(\boldsymbol{\eta}_n), \boldsymbol{f}(\boldsymbol{\eta}_n) - \boldsymbol{f}(\boldsymbol{\eta}))_V - (\boldsymbol{f}(\boldsymbol{\eta}), \boldsymbol{f}(\boldsymbol{\eta}_n) - \boldsymbol{f}(\boldsymbol{\eta}))_V \\
&= \int_\Omega (\boldsymbol{f}_{0n} - \boldsymbol{f}_0) \cdot (\boldsymbol{f}(\boldsymbol{\eta}_n) - \boldsymbol{f}(\boldsymbol{\eta}))\, \mathrm{d}x + \int_{\Gamma_2} (\boldsymbol{f}_{2n} - \boldsymbol{f}_2) \cdot (\boldsymbol{f}(\boldsymbol{\eta}_n) - \boldsymbol{f}(\boldsymbol{\eta}))\, da \\
&\quad - \int_{\Gamma_3} (p_n - p)(\boldsymbol{f}(\boldsymbol{\eta}_n) - \boldsymbol{f}(\boldsymbol{\eta}))_\nu \, da.
\end{aligned}$$

Note that (4.16) implies that $\boldsymbol{f}(\boldsymbol{\eta}_n) \rightharpoonup \boldsymbol{f}(\boldsymbol{\eta})$ in V and, using a compactness argument, it follows that $\boldsymbol{f}(\boldsymbol{\eta}_n) \to \boldsymbol{f}(\boldsymbol{\eta})$ in $L^2(\Omega)^d$, $\boldsymbol{f}(\boldsymbol{\eta}_n) \to \boldsymbol{f}(\boldsymbol{\eta})$ in $L^2(\Gamma_2)^d$ and $(\boldsymbol{f}(\boldsymbol{\eta}_n) - \boldsymbol{f}(\boldsymbol{\eta}))_\nu \to 0$ in $L^2(\Gamma_3)^d$. Therefore, using the previous equality and convergence (4.16) it is easy to see that $\|\boldsymbol{\tau}_n - \boldsymbol{\tau}\|_Q^2 \to 0$ which shows that condition (M_1) in Definition 3.1 is satisfied.

Assume now that the sequence $\{\boldsymbol{\tau}_n\} \subset \Sigma(\boldsymbol{\eta}_n)$ is such that $\boldsymbol{\tau}_n \rightharpoonup \boldsymbol{\tau}$ in Q. Then $\boldsymbol{\tau}_n = \boldsymbol{\varepsilon}(\boldsymbol{f}(\boldsymbol{\eta}_n)) + \boldsymbol{\tau}_{0n}$ with $\boldsymbol{\tau}_{0n} \in \Sigma_0$ and, since $\boldsymbol{f}(\boldsymbol{\eta}_n) \rightharpoonup \boldsymbol{f}(\boldsymbol{\eta})$ in V, we deduce that $\boldsymbol{\varepsilon}(\boldsymbol{f}(\boldsymbol{\eta}_n)) \rightharpoonup \boldsymbol{\varepsilon}(\boldsymbol{f}(\boldsymbol{\eta}))$ in Q. It follows from here that the sequence $\{\boldsymbol{\tau}_{0n}\} \subset \Sigma_0$ converges weakly in Q and, since Σ_0 is a weakly closed subset of Q, we deduce that it converges weakly to an element $\boldsymbol{\tau}_0 \in \Sigma_0$. This implies that $\boldsymbol{\tau} = \boldsymbol{\varepsilon}(\boldsymbol{f}(\boldsymbol{\eta})) + \boldsymbol{\tau}_0$ and, using (4.21) we deduce that $\boldsymbol{\tau} \in \Sigma(\boldsymbol{\eta})$. This shows that condition (M_2) in Definition 3.1 is satisfied and, therefore, convergence (4.17) holds.

To conclude, it follows from above that conditions (Λ), $(\widetilde{K})$, (J_1)–(J_4), (J^*), (3.3) and (3.4) hold. Theorem 4.1 is now a direct consequence of Theorem 3.3. □

Note that Theorem 4.1 provides the unique weak solvability of Problem $\mathcal{P}$. In addition, it shows that the weak solution of this contact problem depends continuously on the densities of the applied forces, the given pressure and the gap g, which is important from mechanical point of view.

5. An optimal control problem

Theorem 3.2 can be used in the study of optimal control problems associated with the contact Problem $\mathcal{P}$. Several examples can be considered. Nevertheless, in this section we restrict ourselves to the study of a representative example, i.e., a boundary control problem in which the control is the function p, interpreted as the pressure of the fluid which covers the obstacle.

Below in this section, in contrast to the previous sections of the manuscript, we consider the product Hilbert space $Y = L^2(\Omega)^d \times L^2(\Gamma_2)^d \times \mathbb{R}$, endowed with the canonical inner product $(\cdot,\cdot)_Y$ and the associated norm $\|\cdot\|_Y$. A generic element of Y will be denoted by $\boldsymbol{\xi} = (\boldsymbol{f}_0, \boldsymbol{f}_2, g)$. Moreover, we use the notation Λ for the set

$$\Lambda = \{\, \boldsymbol{\xi} = (\boldsymbol{f}_0, \boldsymbol{f}_2, g) \in Y \,:\, g \geq 0 \,\} \tag{5.1}$$

and let W be a given set such that

$$W \subset \{\, p \in L^2(\Gamma_3) \,:\, p(\boldsymbol{x}) \geq 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3 \,\}.$$

Let $\boldsymbol{\xi} = (\boldsymbol{f}_0, \boldsymbol{f}_2, g)$ and $p \in W$ be given and denote by $\boldsymbol{\sigma}(\boldsymbol{f}_0, \boldsymbol{f}_2, p, g) = \boldsymbol{\sigma}(p, \boldsymbol{\xi})$ the solution of Problem $\mathcal{P}^V$, under the assumption of Theorem 4.1. With these notations, we define the set of admissible pairs for Problem $\mathcal{P}^V$ by equality

$$\mathcal{V}_{ad}(\boldsymbol{\xi}) = \{\, (\boldsymbol{\sigma}, p) \,:\, p \in W \quad \text{and} \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}(p, \boldsymbol{\xi}) \,\}. \tag{5.2}$$

In other words, a pair $(\boldsymbol{\sigma}, p)$ belongs to $\mathcal{V}_{ad}(\boldsymbol{\xi})$ if and only if $p \in W$ and, moreover, $\boldsymbol{\sigma}$ is the solution of Problem $\mathcal{P}^V$ with $\boldsymbol{\eta} = (\boldsymbol{f}_0, \boldsymbol{f}_2, p, g)$. Consider also a cost functional $\mathcal{L} : Q \times L^2(\Gamma_3) \to \mathbb{R}$. Then, the optimal control problem we are interested in is the following.

Problem. $\mathcal{Q}$. *Given $\boldsymbol{\xi} \in \Lambda$, find $(\boldsymbol{\sigma}^*, p^*) \in \mathcal{V}_{ad}(\boldsymbol{\xi})$ such that*

$$\mathcal{L}(\boldsymbol{\sigma}^*, p^*) = \min_{(\boldsymbol{\sigma}, p) \in \mathcal{V}_{ad}(\boldsymbol{\xi})} \mathcal{L}(\boldsymbol{\sigma}, p). \tag{5.3}$$

To solve Problem $\mathcal{Q}$, we consider the following assumptions.

$$W \quad \textit{is a nonempty weakly closed subset of} \quad L^2(\Gamma_3). \tag{5.4}$$

$$\begin{cases} \textit{For all sequences } \{\boldsymbol{\sigma}_k\} \subset Q \textit{ and } \{p_k\} \subset L^2(\Gamma_3) \textit{ such that} \\ \boldsymbol{\sigma}_k \to \boldsymbol{\sigma} \quad \textit{in} \quad Q, \quad p_k \rightharpoonup p \quad \textit{in} \quad L^2(\Gamma_3) \textit{ we have} \\ \text{(a)} \quad \liminf\limits_{k\to\infty} \mathcal{L}(\boldsymbol{\sigma}_k, p_k) \geq \mathcal{L}(\boldsymbol{\sigma}, p). \\ \text{(b)} \quad \lim\limits_{k\to\infty} \big[\mathcal{L}(\boldsymbol{\sigma}_k, p_k) - \mathcal{L}(\boldsymbol{\sigma}, p_k)\big] = 0. \end{cases} \tag{5.5}$$

$$\begin{cases} \textit{There exists } h : L^2(\Gamma_3) \to \mathbb{R} \textit{ such that} \\ \text{(a)} \quad \mathcal{L}(\boldsymbol{\sigma}, p) \geq h(p) \quad \forall\, \boldsymbol{\sigma} \in Q, \ p \in L^2(\Gamma_3). \\ \text{(b)} \quad \|p_k\|_{L^2(\Gamma_3)} \to +\infty \implies h(p_k) \to \infty. \end{cases} \tag{5.6}$$

Example 5.1. A typical example of function $\mathcal{L}$ which satisfies conditions (5.5)–(5.6) is obtained by taking

$$\mathcal{L}(\boldsymbol{\sigma}, p) = f(\boldsymbol{\sigma}) + h(p) \qquad \forall\, \boldsymbol{\sigma} \in Q, \ p \in L^2(\Gamma_3)$$

where $f : Q \to \mathbb{R}$ is a continuous positive function and $h : L^2(\Gamma_3) \to \mathbb{R}$ is a lower semicontinuous coercive function, i.e., it satisfies condition (5.6)(b). We conclude that our results below are valid for such type of cost functionals.

Next, for each $n \in \mathbb{N}$ we consider a perturbation $\boldsymbol{\xi}_n = (\boldsymbol{f}_{0n}, \boldsymbol{f}_{2n}, g_n) \in \Lambda$ of $\boldsymbol{\xi}$. With these data, we consider the following perturbation of Problem $\mathcal{Q}$.

Problem. $\mathcal{Q}_n$. *Given $\boldsymbol{\xi}_n \in \Lambda$, Find $(\boldsymbol{\sigma}_n^*, p_n^*) \in \mathcal{V}_{ad}(\boldsymbol{\xi}_n)$ such that*

$$\mathcal{L}(\boldsymbol{\sigma}_n^*, p_n^*) = \min_{(\boldsymbol{\sigma}, p) \in \mathcal{V}_{ad}(\boldsymbol{\xi}_n)} \mathcal{L}(\boldsymbol{\sigma}, p). \tag{5.7}$$

Note that in the statement of this problem, the set of admissible pairs is defined by

$$\mathcal{V}_{ad}(\boldsymbol{\xi}_n) = \{\, (\boldsymbol{\sigma}, p) \ : \ p \in W \quad \text{and} \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}(p, \boldsymbol{\xi}_n) \,\}. \tag{5.8}$$

We have the following existence, uniqueness and convergence result.

Theorem 5.2. *Assume (2.9)–(2.16), (4.1)–(4.4) and (5.4)–(5.6). Then, the following statement hold.*

(i) *Problem $\mathcal{Q}$ has at least one solution $(\boldsymbol{\sigma}^*, p^*)$ and, for each $n \in \mathbb{N}$, Problem $\mathcal{Q}_n$ has at least one solution $(\boldsymbol{\sigma}_n^*, p_n^*)$.*

(ii) *If (4.5), (4.6), (4.8) hold and $(\boldsymbol{\sigma}_n^*, p_n^*)$ is a solution of Problem $\mathcal{Q}_n$, for each $n \in \mathbb{N}$, then there exists a subsequence of the sequence $\{(\boldsymbol{\sigma}_n^*, p_n^*)\}$, again denoted $\{(\boldsymbol{\sigma}_n^*, p_n^*)\}$, and an element $(\boldsymbol{\sigma}^*, p^*)$, such that*

$$\boldsymbol{\sigma}_n^* \to \boldsymbol{\sigma}^* \quad \text{in} \quad Q. \tag{5.9}$$

$$p_n^* \rightharpoonup p^* \quad \text{in} \quad L^2(\Gamma_3). \tag{5.10}$$

Moreover, $(\boldsymbol{\sigma}^*, p^*)$ is a solution to Problem $\mathcal{Q}$.

To provide the proof of Theorem 5.2, we need some preliminary results that we present in what follows. First, for each $\boldsymbol{\xi} \in \Lambda$ we consider the function $J(\cdot, \boldsymbol{\xi}) : L^2(\Gamma_3) \to \mathbb{R}$ defined by

$$J(p, \boldsymbol{\xi}) = \mathcal{L}(\boldsymbol{\sigma}(p, \boldsymbol{\xi}), p) \qquad \forall\, p \in L^2(\Gamma_3), \tag{5.11}$$

together with the following auxiliary problem.

Problem. $\mathcal{R}$. *Given $\boldsymbol{\xi} \in \Lambda$, find $p^* \in W$ such that*

$$J(p^*, \boldsymbol{\xi}) = \min_{p \in W} J(p, \boldsymbol{\xi}). \tag{5.12}$$

Then, using definitions (5.11) and (5.2) it is easy to see that

$$\begin{cases} (\boldsymbol{\sigma}^*, p^*) \textit{ is a solution of Problem } \mathcal{Q} \textit{ if and only if} \\ p^* \textit{ is a solution of Problem } \mathcal{R} \textit{ and } \boldsymbol{\sigma}^* = \boldsymbol{\sigma}(p^*, \boldsymbol{\xi}). \end{cases} \tag{5.13}$$

Next, for each $n \in \mathbb{N}$ we consider the following problem.

Problem. $\mathcal{R}_n$. *Given $\boldsymbol{\xi}_n \in \Lambda$, find $p_n^* \in W$ such that*

$$J(p_n^*, \boldsymbol{\xi}_n) = \min_{p \in W} J(p, \boldsymbol{\xi}_n). \tag{5.14}$$

Then, using again (5.11) and definition (5.8) it is easy to see that

$$\begin{cases} (\boldsymbol{\sigma}_n^*, p_n^*) \textit{ is a solution of Problem } \mathcal{Q}_n \textit{ if and only if} \\ p_n^* \textit{ is a solution of Problem } \mathcal{R}_n \textit{ and } \boldsymbol{\sigma}_n^* = \boldsymbol{\sigma}(p_n^*, \boldsymbol{\xi}_n). \end{cases} \tag{5.15}$$

We are now in position to provide the proof of Theorem 5.2.

Proof. We shall use Theorem 3.2 with $X = L^2(\Gamma_3)$ and $K(\cdot) = W$, Λ and J being defined by (5.1) and (5.11), respectively. To this end, in what follows we check the validity of assumptions in Theorem 3.2.

First, it is easy to see that the set Λ is a nonempty closed convex subset of Y which implies that condition (Λ) holds. On the other hand, assumption (5.4) guarantees that condition (K) is satisfied and, since W does not depend on $\boldsymbol{\xi}$, we deduce that condition (K^*) holds, too.

Assume now that $\{p_k\} \subset L^2(\Gamma_3)$ and $\{\boldsymbol{\xi}_k\} \subset \Lambda$ are two sequences such that $p_k \rightharpoonup p$ in $L^2(\Gamma_3)$, $\boldsymbol{\xi}_k = (\boldsymbol{f}_{0k}, \boldsymbol{f}_{2k}, g_k) \rightharpoonup \boldsymbol{\xi} = (\boldsymbol{f}_0, \boldsymbol{f}_2, g)$ in Y and let $q \in L^2(\Gamma_3)$. Then, using (5.11) we have

$$J(q, \boldsymbol{\xi}_k) - J(p_k, \boldsymbol{\xi}_k) = \mathcal{L}(\boldsymbol{\sigma}(q, \boldsymbol{\xi}_k), q) - \mathcal{L}(\boldsymbol{\sigma}(p_k, \boldsymbol{\xi}_k), p_k) \qquad \forall\, k \in \mathbb{N}. \tag{5.16}$$

Note also that Theorem 4.1 guarantees the convergences $\boldsymbol{\sigma}(q, \boldsymbol{\xi}_k) \to \boldsymbol{\sigma}(q, \boldsymbol{\xi})$ in Q, $\boldsymbol{\sigma}(p_k, \boldsymbol{\xi}_k) \to \boldsymbol{\sigma}(p, \boldsymbol{\xi})$ in Q. Therefore, assumption (5.5) and definition (5.11) imply that

$$\lim_{k \to \infty} \mathcal{L}(\boldsymbol{\sigma}(q, \boldsymbol{\xi}_k), q) = \mathcal{L}(\boldsymbol{\sigma}(q, \boldsymbol{\xi}), q) = J(q, \boldsymbol{\xi}), \tag{5.17}$$

$$\limsup_{k \to \infty} [-\mathcal{L}(\boldsymbol{\sigma}(p_k, \boldsymbol{\xi}_k), p_k)] \le -\mathcal{L}(\boldsymbol{\sigma}(p, \boldsymbol{\xi}), p) = -J(p, \boldsymbol{\xi}). \tag{5.18}$$

We now pass to the upper limit in (5.16) and use relations (5.17), (5.18) to find that

$$\limsup_{k \to \infty} \left[J(q, \boldsymbol{\xi}_k) - J(p_k, \boldsymbol{\xi}_k)\right] \le J(q, \boldsymbol{\xi}) - J(p, \boldsymbol{\xi}),$$

which shows that condition (J_1) is satisfied.

On the other hand, for any sequences $\{p_k\} \subset X$ and $\{\boldsymbol{\xi}_k\} \subset \Lambda$, using inequality (5.6)(a) we have

$$J(p_k, \boldsymbol{\xi}_k) = \mathcal{L}(\boldsymbol{\sigma}(p_k, \boldsymbol{\xi}_k), p_k) \ge h(p_k).$$

Therefore, if $\|p_k\|_{L^2(\Gamma_3)} \to \infty$, from (5.6)(b) we deduce that $J(p_k, \boldsymbol{\xi}_k) \to \infty$ which shows that condition (J_2) is satisfied, too.

Let $\{\boldsymbol{\xi}_k\} \subset \Lambda$ be a sequence such that $\boldsymbol{\xi}_k \rightharpoonup \boldsymbol{\xi}$ in Y and let $q \in L^2(\Gamma_3)$. We have

$$J(q, \boldsymbol{\xi}_k) - J(q, \boldsymbol{\xi}) = \mathcal{L}(\boldsymbol{\sigma}(q, \boldsymbol{\xi}_k), q) - \mathcal{L}(\boldsymbol{\sigma}(q, \boldsymbol{\xi}), q)$$

and, using the convergence $\boldsymbol{\sigma}(q, \boldsymbol{\xi}_k) \to \boldsymbol{\sigma}(q, \boldsymbol{\xi})$, guaranteed by Theorem 4.1, we deduce by assumption (5.5)(b) that

$$J(q, \boldsymbol{\xi}_k) - J(q, \boldsymbol{\xi}) \to 0.$$

This shows that condition (J_3) holds.

Note also that, if (4.5), (4.6), (4.8) hold, then $\boldsymbol{\xi}_n = (\boldsymbol{f}_{0n}, \boldsymbol{f}_{2n}, g_n) \rightharpoonup \boldsymbol{\xi} = (\boldsymbol{f}_0, \boldsymbol{f}_2, g)$ in Y, which shows that condition (3.3) holds.

We conclude from above that all the assumptions (Λ), (K), (J_1), (J_2), (3.3), (K^*), (J_3) of Theorem 3.2 are valid for the study of the optimization problems $\mathcal{Q}$ and $\mathcal{Q}_n$. Theorem 5.2 is now a direct consequence of Theorem 3.2, combined with the equivalences (5.13) and (5.15) and the convergence result in Theorem 4.1. □

We end this section with two examples of optimal control problems for which the result provided by Theorem 5.2 hold. In both examples, for a given $\boldsymbol{\sigma} \in Q$ we denote by $\boldsymbol{u}(\boldsymbol{\sigma})$ the unique displacement field $\boldsymbol{u} \in V$ associated with $\boldsymbol{\sigma}$ via the constitutive law (2.1). Note that the existence and uniqueness of $\boldsymbol{u}$ follows by using arguments similar to those used in Theorem 5.13 in [25], and therefore, we skip its proof. Nevertheless, we recall that, under assumptions (2.9)–(2.15), the operator $\boldsymbol{\sigma} \mapsto \boldsymbol{u}(\boldsymbol{\sigma}) : Q \to V$ is Lipschitz continuous.

Example 5.3. Let α and δ be strictly positive constants and let $\phi \in L^2(\Gamma_3)$ be given. Define

$$W = \{\, p \in L^2(\Gamma_3) \ : \ p(\boldsymbol{x}) \geq 0 \ \text{ a.e. } \boldsymbol{x} \in \Gamma_3\} \tag{5.19}$$

$$\mathcal{L}(\boldsymbol{\sigma}, p) = \alpha \int_{\Gamma_3} (u_\nu(\boldsymbol{\sigma}) - \phi)^2 \, da + \delta \int_{\Gamma_3} p^2 \, da \tag{5.20}$$

for all $(\boldsymbol{\sigma}, p) \in Q \times L^2(\Gamma_3)$. With this choice, the mechanical interpretation of Problem $\mathcal{Q}$ is the following: we are looking for a surface pressure $p \in W$ acting on Γ_3 such that the corresponding normal displacement u_ν is as close as possible to the "desired displacement" ϕ. Furthermore, this choice has to fulfill a minimum expenditure condition which is taken into account by the last term in (5.20). Note that in this case conditions (5.4)–(5.6) are satisfied and, therefore, Theorem 5.2 can be applied to obtain that the corresponding Problem $\mathcal{Q}$ has at least one solution.

Example 5.4. Let $W(\cdot)$ be defined by (5.19), $\alpha > 0$, $\delta > 0$ and let

$$\mathcal{L}(\boldsymbol{\sigma}, p) = \alpha \int_{\Omega} \|\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{\sigma}))\|^2 \, \mathrm{d}x + \delta \int_{\Gamma_3} p^2 \, da \tag{5.21}$$

for all $(\boldsymbol{\sigma}, p) \in Q \times L^2(\Gamma_3)$. With this choice, the mechanical interpretation of Problem $\mathcal{Q}$ is the following : we are looking for a surface pressure p acting on Γ_3 such that the corresponding deformation in the body is as small as possible. And, again, this choice has to fulfill a minimum expenditure condition which is taken into account by the last term in (5.21). Note that in this case conditions (5.4)–(5.6) are satisfied and, therefore, Theorem 5.2 guarantees the existence of at least one solution of the corresponding optimal control problem.

6. Conclusion

We studied an abstract optimization problem for which we provided existence, uniqueness and convergence results. The proofs were based on arguments of monotonicity, lower semicontinuity and Mosco convergence. To illustrate the usefulness of these abstract results, we applied them in the study of the analysis and control of a mathematical model which describes the equilibrium of an elastic body in frictionless contact with an obstacle, the so-called foundation.

The study presented in this paper give rise to several open problems that we describe in what follows. Any progress in these directions will complete our work and will open the way for new advances and ideas. First, it would be interesting to derive necessary optimality conditions in the study of Problem $\mathcal{Q}$. Due to the nonsmooth and nonconvex feature of the functional $\mathcal{L}$, the treatment of this problem requires the use of its approximation by smooth optimization problems. And, in this matter, the abstract convergence results for the optimal pairs in this paper could be a crucial tool. Another interesting continuation of the results presented in this paper would be their extension to frictional models of contact. Such models lead, in general, to variational and hemivariational inequalities in which the unknown is the displacement field. Considering optimal control for quasistatic or dynamic models of contact would be another problem which deserves future research. Such models lead to evolutionary variational, hemivariational or variational–hemivariational inequalities, as shown in the recent references [18,26].

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