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Solvability of parabolic variational-hemivariational inequalities involving space-fractional Laplacian

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ABSTRACT

The paper is devoted to investigate a new kind of variational-hemivariational inequality of parabolic type driven by a generalized space-fractional Laplace operator, and two multi-valued terms which are expressed by the Clarke generalized gradient and convex subgradient, respectively. An existence theorem of weak solutions to the problem is established by applying a surjectivity theorem for the sum of operators combined with results from nonsmooth analysis.

1. Introduction

In many complicated physical processes and engineering applications, mathematical models of problems are formulated as inequalities instead of the more commonly seen equations. Many problems are focused on the study of variational inequalities and hemivariational inequalities. Generally speaking, variational inequalities are referred to those inequality problems with a convex framework, while hemivariational inequalities are involved in those systems with nonconvex and nonsmooth structure. In the recent years, the study of variational and hemivariational inequalities has been considered extensively in variety of mathematical theory analysis and engineering applications, see [9,10,12–15,18–21,23,26,27,29,30]. On

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the other hand, the problems involving fractional and nonlocal operators constitute a source of extensive interest in recent years, since they provide precise description of some physical phenomena for viscoelastic materials such as fractional Kelvin-Voigt constitutive laws and fractional Maxwell model, electrodynamics biotechnology, aerodynamics and control of dynamical systems, see [2,8,16,17,22,28,31,32].

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $s\in(0,1)$ be such that $N>2s$, $\Omega^{\complement}:=\mathbb{R}^N\backslash\Omega$, and $0<T<\infty$. The purpose of the paper is to explore the existence of solutions for the following space-fractional variational-hemivariational inequality of parabolic type:

$$\begin{cases} u'+\mathcal{L}_K u+\partial J(u)+\partial_C\varphi(u)\ni f & \text{in } \Omega\times(0,T)\\ u(x,t)=0 & \text{in } \Omega^{\complement}\times(0,T)\\ u(x,0)=u_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $u'=\frac{\partial}{\partial t}$ and the operator $\mathcal{L}_K$ stands for the generalized nonlocal space-fractional Laplace operator defined as follows

$$\mathcal{L}_K u(x):=-\int_{\mathbb{R}^N}(u(x+y)+u(x-y)-2u(x))K(y)\,dy \ \text{ for a.e. } x\in\mathbb{R}^N,$$

for all $u\in X_0$, the space X_0 is given in Section 2. Moreover, for problem (1), the kernel function K is assumed to satisfy the following condition:

H(K): $K:\mathbb{R}^N\backslash\{0\}\to(0,+\infty)$ is such that

(i) the function $x\mapsto\min\{|x|^2,1\}K(x)$ belongs to $L^1(\mathbb{R}^N)$.
(ii) for all $x\in\mathbb{R}^N\backslash\{0\}$, there exists a constant $m_K>0$ such that

$$K(x)\geq m_K|x|^{-(N+2s)}.$$

(iii) for each $x\in\mathbb{R}^N\backslash\{0\}$, we have $K(x)=K(-x)$.

The terms ∂J and $\partial_C\varphi$ denote the generalized subdifferential operator in the sense of Clarke, see [5], for a locally Lipschitz functional J, and convex subdifferential operator for a proper, convex functional φ, respectively.

To highlight the level of generalization of problem (1), we present below its several particular cases.

(i) If the kernel function K is specialized to

$$K(x):=|x|^{-(N+2s)} \ \text{ for all } x\in\mathbb{R}^N\backslash\{0\},$$

and for some $s\in(0,1)$ such that $2s<N$, i.e., the generalized fractional nonlocal Laplace operator $\mathcal{L}_K$ becomes the classical fractional Laplace operator $(-\Delta)^s$,

$$(-\Delta)^s u(x):=-\int_{\mathbb{R}^N}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\,dy \ \text{ for a.e. } x\in\mathbb{R}^N,$$

then problem (1) reduces to the following parabolic variational-hemivariational inequality involving fractional Laplace operator

$$\begin{cases} u'+(-\Delta)^s u+\partial J(u)+\partial_C\varphi(u)\ni f & \text{in } \Omega\times(0,T)\\ u(x,t)=0 & \text{in } \Omega^{\complement}\times(0,T)\\ u(x,0)=u_0 & \text{in } \Omega. \end{cases}$$

(ii) If the convex functional $\varphi\equiv0$, then the fractional variational-hemivariational inequality of parabolic type (1) reduces to the following "purely" fractional parabolic hemivariational inequality

$$\begin{cases} u'+\mathcal{L}_K u+\partial J(u)\ni f & \text{in } \Omega\times(0,T)\\ u(x,t)=0 & \text{in } \Omega^{\complement}\times(0,T)\\ u(x,0)=u_0 & \text{in } \Omega. \end{cases}$$

(iii) If the locally Lipschitz functional $J\equiv0$, then the fractional variational-hemivariational inequality of parabolic type (1) converts to the following fractional parabolic variational inequality

$$\begin{cases} u'+\mathcal{L}_K u+\partial_C\varphi(u)\ni f & \text{in } \Omega\times(0,T)\\ u(x,t)=0 & \text{in } \Omega^{\complement}\times(0,T)\\ u(x,0)=u_0 & \text{in } \Omega. \end{cases}$$

In fact, as far as we know, until now, there is no reference which deals with all of the special cases listed above. Based on this motivation, the aim of the paper is to examine the existence of weak solution to problem (1).

The rest of the paper is organized as follows. In Section 2, we will recall some preliminary material needed in the investigation of the inequality problem. Section 3 is devoted to study existence of weak solutions to problem (1) by using a surjectivity result for the sum of operators combined with the theory of convex and nonsmooth analysis.

2. Preliminaries and hypotheses

In this section we recall the main preliminary material and notation needed in the study of problem (1).

First, we review the following concepts from nonlinear analysis.

Definition 1 [21, Section 3.4]. Let E be a reflexive Banach space with its dual E^* and $A : D(A) \subset E \to 2^{E^*}$ be a multivalued function, where $D(A) = \{u \in E \mid Au \neq \emptyset\}$ stands for the domain of A. We say that

(i) A is monotone, if

$$\langle u^* - v^*, u - v \rangle_{E^* \times E} \geq 0 \text{ for all } u^* \in Au, v^* \in Av \text{ and } u, v \in D(A).$$

(ii) A is maximal monotone, if it is monotone and it has a maximal graph in the sense of inclusion among all monotone operators, namely, the inequality

$$\langle u^* - w^*, u - v \rangle_{E^* \times E} \geq 0 \text{ for all } u^* \in Au \text{ and } u \in D(A),$$

implies $v \in D(A)$ and $w^* \in Av$.

(iii) A is pseudomonotone with respect to $D(L)$ (or L-pseudomonotone), for a linear, maximal monotone operator L: $D(L) \subset E \to E^*$, if
 (a) for each $u \in E$, the set Au is nonempty, closed, and convex in E^*;
 (b) A is upper semicontinuous from each finite dimensional subspace of E into E^* endowed with its weak topology;
 (c) for each sequences $\{u_n\} \subset D(L)$ and $\{u_n^*\} \subset E^*$ with

$$\begin{cases} u_n \to u \text{ weakly in } E, \\ Lu_n \to Lu \text{ weakly in } E^*, \\ u_n^* \in Au_n \text{ for all } n \in \mathbb{N}, \\ u_n^* \to u^* \text{ weakly in } E^*, \\ \limsup\limits_{n \to \infty} \langle u_n^*, u_n - u \rangle_{E^* \times E} \leq 0, \end{cases}$$

 we have $u^* \in Au$ and $\lim_{n \to \infty} \langle u_n^*, u_n \rangle_{E^* \times E} = \langle u^*, u \rangle_{E^* \times E}$.

Next, we recall some basic tools from convex analysis and nonsmooth analysis.

Definition 2 [21, Definition 3.31]. Let E be a Banach space with its dual E^*, and $\varphi : E \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. The mapping $\partial_C \varphi : E \to 2^{E^*}$ defined by

$$\partial_C \varphi(u) = \{ u^* \in E^* \mid \langle u^*, v - u \rangle_{E^* \times E} \leq \varphi(v) - \varphi(u) \text{ for all } v \in E \}$$

is called the subdifferential of φ. Any element $u^* \in \partial_C \varphi(u)$ is called a subgradient of φ at u.

Definition 3 [21, Definitions 3.21 and 3.22]. Let $J : E \to \mathbb{R}$ be a locally Lipschitz continuous functional and $u, v \in E$. We denote by $J^0(u; v)$ the generalized (Clarke) directional derivative of J at the point u in the direction v defined by

$$J^0(u; v) = \limsup_{w \to u, \, t \downarrow 0} \frac{J(w + tv) - J(w)}{t}.$$

The generalized Clarke gradient $\partial J : E \to 2^{E^*}$ of $J : E \to \mathbb{R}$ at $u \in E$ is defined by

$$\partial J(u) = \{ \xi \in E^* \mid J^0(u; v) \geq \langle \xi, v \rangle_{E^* \times E} \text{ for all } v \in E \}.$$

Theorem 4 *[6, Theorem 6.3.19, p. 48]. Let E be a real Banach space and $\varphi : E \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous functional. Then $\partial_C \varphi : E \to 2^{E^*}$ is a maximal monotone operator.*

Proposition 5 *[21, Proposition 3.23]. Let $J : E \to \mathbb{R}$ be locally Lipschitz of rank $L_u > 0$ near $u \in E$. Then, we have*

(a) *the function $v \mapsto J^0(u; v)$ is positively homogeneous, subadditive, and satisfies*

$$|J^0(u; v)| \leq L_u \|v\|_E \text{ for all } v \in E;$$

(b) *$(u, v) \mapsto J^0(u; v)$ is upper semicontinuous;*
(c) *$\partial J(u)$ is a nonempty, convex, and weakly * compact subset of E^* with $\|\xi\|_{E^*} \leq L_u$ for all $\xi \in \partial J(u)$;*
(d) *for all $v \in E$, we have $J^0(u; v) = \max\{\langle \xi, v \rangle_{E^* \times E} \mid \xi \in \partial J(u)\}$.*

Additionally, we consider the important concept of strongly-quasi boundedness for multivalued operators.

Definition 6 [7, Definition 2.14]. Let E be a reflexive Banach space with its dual E^* and $A : D(A) \subset E \to 2^{E^*}$ be a multivalued mapping. A is called to be strongly-quasi bounded, if for each $M > 0$, there exists $K_M > 0$ satisfying if $u \in D(A)$ and $u^* \in Au$ are such that

$$\langle u^*, u \rangle_{E^* \times E} \leq M \text{ and } \|u\|_E \leq M,$$

then $\|u^*\|_{E^*} \leq K_M$.

In fact, it is not easy to verify that a multivalued operator is strongly-quasi bounded by using the definition. However, Browder-Hess in [3, Proposition 14] provided the following criterion to validate the strongly-quasi boundedness.

Proposition 7 *[3, Proposition 14]. Let E be a reflexive Banach space with its dual E^*. If $A : D(A) \subset E \to 2^{E^*}$ is a monotone operator such that $0 \in \mathrm{int}D(A)$, then A is strongly-quasi bounded.*

Finally, we recall the following surjectivity result for the sum of operators in Banach spaces, which will play a significant role in the proof of our main result.

Theorem 8 *[7, Theorem 3.1]. Let E be a reflexive, strictly convex Banach space, L: $D(L) \subset E \to E^*$ be a linear, densely defined and maximal monotone operator, $A : E \to 2^{E^*}$ be a bounded and L-pseudomonotone operator such that*

$$\langle Au, u\rangle_{E^*\times E} \ge r(\|u\|_E)\|u\|_E \quad \text{for all } u \in E,$$

where $r : \mathbb{R}_+ \to \mathbb{R}$ is a function satisfying $r(s) \to +\infty$ as $s \to +\infty$. If the multivalued mapping $B : D(B) \subset E \to 2^{E^*}$ is a maximal monotone operator which is strongly-quasi bounded and $0 \in B(0)$, then $L + A + B$ is surjective, namely, $R(L + A + B) = E^*$.

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, and $s \in (0, 1)$ be such that $N > 2s$. In what follows, we adopt the symbols $\mathcal{S} := (\mathbb{R}^N\backslash\Omega) \times (\mathbb{R}^N\backslash\Omega)$, $\mathcal{P} := \mathbb{R}^{2N}\backslash\mathcal{S}$, and $2_s^* := \frac{2N}{N-2s}$ to denote the fractional critical exponent. Also, we denote by $u|_\Omega$ the function u restricted to the domain Ω. Consider the function space

$$X := \{u : \mathbb{R}^N \to \mathbb{R} \mid u|_\Omega \in L^2(\Omega) \text{ and } (u(x) - u(y))^2K(x - y) \in L^2(\mathcal{P})\}.$$

It is obvious, see [25], that X is a normed linear space endowed with the norm

$$\|u\|_X := \|u\|_{L^2(\Omega)} + \left(\int_{\mathcal{P}} |u(x) - u(y)|^2K(x - y)\, dy\, dx\right)^{\frac{1}{2}}$$

for all $u \in X$. Since the boundary condition for problem (1) is the generalized Dirichlet boundary, so, we also introduce a subspace of X, given by

$$X_0 := \{ u \in X \mid u = 0 \text{ for a.e. } x \in \Omega^{\complement} \}.$$

Lemma 9 [25]**.** *Let $s \in (0, 1)$ and Ω be a bounded, open subset of $\mathbb{R}^N$ with Lipschitz boundary and $N > 2s$. Then, we have*

(i) *X_0 is a Hilbert space with the inner product*

$$\langle u, v\rangle_{X_0} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} [u(x) - u(y)][v(x) - v(y)]K(x - y)\, dx\, dy$$

for all $u, v \in X_0$.

(ii) *If $p \in [1, 2_s^*]$, then there exists a positive constant $c(p)$ such that*

$$\|u\|_{L^p(\mathbb{R}^N)} \le c(p)\|u\|_{X_0} \quad \text{for all } u \in X_0.$$

(iii) *The embedding from X_0 to $L^p(\mathbb{R}^N)$ is compact if $p \in [1, 2_s^*)$.*

Let X_0^* be the dual space of X_0. Note that $X_0 \subset L^2(\Omega) \subset X_0^*$ and $2 < 2_s^*$, so from Lemma 9, we can see that the embedding from X_0 to $L^2(\Omega)$ is compact. Besides, we introduce the function spaces

$$\mathcal{X}_0 = L^2(0, T; X_0), \quad \mathcal{X}_0^* = L^2(0, T; X_0^*) \text{ and } \mathcal{W} = \left\{u \in \mathcal{X}_0 \mid \frac{\partial u}{\partial t} \in \mathcal{X}_0^*\right\},$$

where the time derivative $\frac{\partial u}{\partial t}$ is understood in the sense of vector-valued distributions. Moreover, [21, Proposition 2.54] reveals that the function space $\mathcal{W}$ endowed the norm

$$\|u\|_{\mathcal{W}} := \|u\|_{\mathcal{X}_0} + \left\|\frac{\partial u}{\partial t}\right\|_{\mathcal{X}_0^*} \quad \text{for all } u \in \mathcal{W},$$

is a Banach space, and the embeddings $\mathcal{W} \subset L^2(0, T; L^2(\Omega))$ and $\mathcal{W} \subset C(0, T; L^2(\Omega))$ are compact and continuous, respectively.

3. Existence of solutions

In this section, we shall focus our attention to examine existence of weak solutions to problem (1).

For $0 < T < \infty$, we denote $\Lambda := \Omega \times (0, T)$. We impose the following assumptions for the data of problem (1).

$\underline{H(j)}$: $j : \Omega \times (0, T) \times \mathbb{R} \to \mathbb{R}$ is such that $j(\cdot, 0) \in L^1(\Lambda)$ and

(i) for each $r \in \mathbb{R}$, the function $(x, t) \mapsto j(x, t, r)$ is measurable on Λ;
(ii) for a.e. $(x, t) \in \Lambda$, the functional $r \mapsto j(x, t, r)$ is locally Lipschitz;
(iii) there exist $c_j > 0$, $p \ge 1$, and $a \in L^{\frac{p}{p-1}}(\Lambda)$ with $a(x, t) \ge 0$ satisfying

$$|\xi| \le a(x, t) + c_j|r|^{p-1} \quad \text{for all } \xi \in \partial j(x, t, r) \text{ and for a.e. } (x, t) \in \Lambda.$$

<u>$H(0)$</u>: $f \in \mathcal{X}_0^*$ and $u_0 \in \mathrm{int}D(\varphi)$.
<u>$H(\varphi)$</u>: $\varphi : X_0 \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous functional such that

$$0 \in \partial_C \varphi(u_0).$$

Define the functional $J : \mathcal{X}_0 \to \mathbb{R}$ by

$$J(u) := \int_\Lambda j(x,t,u(x,t))\,dx\,dt \ \text{ for all } u \in \mathcal{X}_0. \tag{2}$$

Under the hypothesis $H(j)$, from [21, Theorem 3.47], we readily obtain the following result.

Proposition 10. *Assume that hypothesis $H(j)$ is fulfilled. Then the functional J defined in (2) is locally Lipschitz and there exists a constant $c_J > 0$ such that for all $u, v \in L^p(\Lambda)$*

$$\begin{cases} J^0(u;v) \le c_J\left(1+\|u\|_{L^p(\Lambda)}^{p-1}\right)\|v\|_{L^p(\Lambda)}, \\ \|\xi\|_{L^{p'}(\Lambda)} \le c_J\left(1+\|u\|_{L^p(\Lambda)}^{p-1}\right) \ \text{ for all } \xi \in \partial\left(J|_{L^p(\Lambda)}\right)(u), \end{cases} \tag{3}$$

where p' is the conjugate exponent of p, i.e., $\frac{1}{p}+\frac{1}{p'}=1$.

The definition of weak solution for problem (1) reads as follows.

Definition 11. We say that $u \in \mathcal{W}$ is a weak solution to problem (1), if $u(x,0) = u_0(x)$ in Ω, and the following inequality holds

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} \frac{\partial u(x,t)}{\partial t}\left(v(x,t)-u(x,t)\right)dx\,dt + J^0(u; v-u) + \int_0^T \varphi(v(t))\,dt \\ &\quad - \int_0^T \varphi(u(t))\,dt + \int_0^T \int_{\mathbb{R}^N} \left(v(x,t)-u(x,t)\right)\mathcal{L}_K u(x,t)\,dx\,dt \\ &\ge \int_0^T \int_{\mathbb{R}^N} f(x,t)\left(v(x,t)-u(x,t)\right)dx\,dt \end{aligned}$$

for all $v \in \mathcal{X}_0$.

The main result of the paper is delivered below.

Theorem 12. *Assume that $H(K)$, $H(j)$, $H(0)$, and $H(\varphi)$ hold. If $1 \le p < 2$ or $p = 2$ with $c_J c(2)^2 < 1$, where $c(2) > 0$ is given in Lemma 9(ii), then problem (1) admits a weak solution in the sense of Definition 11.*

Proof. First, we define the operator $\mathcal{A} : \mathcal{X}_0 \to \mathcal{X}_0^*$ by

$$\begin{aligned} (\mathcal{A}u)(v) :&= \int_0^T \int_{\mathbb{R}^N} v(x,t)\mathcal{L}_K\left(u(x,t)+u_0(x)\right)dx\,dt \\ &= -\int_0^T \int_{\mathbb{R}^{2N}} v(x,t)[u(x+y,t)+u(x-y,t)-2u(x,t)]K(y)\,dy\,dx\,dt \\ &\quad - \int_0^T \int_{\mathbb{R}^{2N}} v(x,t)[u_0(x+y)+u_0(x-y)-2u_0(x)]K(y)\,dy\,dx\,dt \end{aligned} \tag{4}$$

for all $u, v \in \mathcal{X}_0$. We verify that $\mathcal{A}$ is a linear and continuous operator. Let $u, v \in \mathcal{X}_0$ and $z, w \in X_0$. Note that

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} w(x)\left[z(x+y)+z(x-y)-2z(x)\right]K(y)\,dy\,dx \\ &= \int_{\mathbb{R}^{2N}} w(x)\left[z(x+y)-z(x)\right]K(y)\,dy\,dx + \int_{\mathbb{R}^{2N}} w(x)\left[z(x-y)-z(x)\right]K(y)\,dy\,dx \\ &= \int_{\mathbb{R}^{2N}} w(x)\left[z(y)-z(x)\right]K(x-y)\,dy\,dx + \int_{\mathbb{R}^{2N}} w(x)\left[z(y)-z(x)\right]K(y-x)\,dy\,dx \\ &= \int_{\mathbb{R}^{2N}} w(x)\left[z(y)-z(x)\right]K(x-y)\,dy\,dx + \int_{\mathbb{R}^{2N}} w(y)\left[z(x)-z(y)\right]K(x-y)\,dy\,dx \\ &= -\int_{\mathbb{R}^{2N}} \left[w(x)-w(y)\right]\left[z(x)-z(y)\right]K(x-y)\,dy\,dx \end{aligned}$$

so, we have

$$(\mathcal{A}u)(v) = \int_0^T \int_{\mathbb{R}^{2N}} [u(x,t) - u(y,t)][v(x,t) - v(y,t)]K(x-y)\,dy\,dx\,dt$$
$$+ \int_0^T \int_{\mathbb{R}^{2N}} [u_0(x) - u_0(y)][v(x,t) - v(y,t)]K(x-y)\,dy\,dx\,dt$$
$$= \langle u + \widetilde{u}_0, v\rangle_{\mathcal{X}_0},$$

where $\widetilde{u}_0(t,x) = u_0(x)$ for all $(t,x) \in \Lambda$. Hence, we infer that $\mathcal{A}$ is a linear and continuous operator and

$$\|\mathcal{A}u\|_{\mathcal{X}_0^*} \le \|u\|_{\mathcal{X}_0} + \|\widetilde{u}_0\|_{\mathcal{X}_0} \text{ for all } u \in \mathcal{X}_0. \tag{5}$$

Additionally, we have

$$\langle \mathcal{A}(u) - \mathcal{A}(v), u - v\rangle_{\mathcal{X}_0} = \langle u + \widetilde{u}_0, u - v\rangle_{\mathcal{X}_0} - \langle v + \widetilde{u}_0, u - v\rangle_{\mathcal{X}_0} = \|u - v\|_{\mathcal{X}_0}^2$$

for all $u, v \in \mathcal{X}_0$, which indicates that $\mathcal{A}$ is strongly monotone as well. □

Next, we introduce the operator $L : D(L) \subset \mathcal{X}_0 \to \mathcal{X}_0^*$ defined by

$$Lu = \frac{\partial u}{\partial t},$$

which is closed, linear, densely defined, and maximal monotone, see [6, Section 8.5]. Here, the domain $D(L)$ of L is given by

$$D(L) := \{u \in \mathcal{W} \mid u(0) = 0\}.$$

Moreover, we consider the functional $\Psi : \mathcal{X}_0 \to \overline{\mathbb{R}}$ given by

$$\Psi(u) := \int_0^T \varphi(u(t) + u_0)\,dt \text{ for all } u \in \mathcal{X}_0. \tag{6}$$

We shall prove the following claims.

Claim 1. *The multivalued operator $\mathcal{A}(\cdot) + \partial J(\cdot + \widetilde{u}_0) : \mathcal{X}_0 \to 2^{\mathcal{X}_0^*}$ is bounded and pseudomonotone with respect to $D(L)$ (i.e., L-pseudomonotone).*

In fact, from Propositions 5 and 10, we deduce that the set $\mathcal{A}u + \partial J(u + \widetilde{u}_0)$ is nonempty, closed, and convex in $\mathcal{X}_0^*$ for all $u \in \mathcal{X}_0$. Next, from (5), for all $u \in \mathcal{X}_0$ and $\xi \in \partial J(u + \widetilde{u}_0)$, we obtain the following estimate

$$\|\mathcal{A}u + \xi\|_{\mathcal{X}_0^*} \le \|\mathcal{A}u\|_{\mathcal{X}_0^*} + \|\xi\|_{\mathcal{X}_0^*}$$
$$\le \|u\|_{\mathcal{X}_0} + \|\widetilde{u}_0\|_{\mathcal{X}_0} + c_J\big(1 + c(p)^{p-1}(\|u\|_{\mathcal{X}_0} + \|\widetilde{u}_0\|_{\mathcal{X}_0})^{p-1}\big),$$

which clearly implies that the mapping $\mathcal{A}(\cdot) + \partial J(\cdot + \widetilde{u}_0) : \mathcal{X}_0 \to 2^{\mathcal{X}_0^*}$ is bounded. Moreover, since $\mathcal{A}$ is linear and continuous (hence demicontinuous as well) and ∂J is upper upper semicontinuous from $\mathcal{X}_0$ to w–$\mathcal{X}_0^*$, see Propositions 5 and 10, it is easy to demonstrate that $\mathcal{A}(\cdot) + \partial J(\cdot + \widetilde{u}_0) : \mathcal{X}_0 \to 2^{\mathcal{X}_0^*}$ is upper semicontinuous from $\mathcal{X}_0$ to w–$\mathcal{X}_0^*$.

It remains to verify the condition (c) in Definition 1(iii). Let $\{u_n\} \subset D(L)$ and $\{u_n^*\} \subset \mathcal{X}_0^*$ be such that $u_n \to u$ weakly in $\mathcal{X}_0$, $Lu_n \to Lu$ weakly in $\mathcal{X}_0^*$, $u_n^* \in \mathcal{A}u_n + \partial J(u_n + \widetilde{u}_0)$ with $u_n^* \to u^*$ weakly in $\mathcal{X}_0^*$, and

$$\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle_{\mathcal{X}_0} \le 0. \tag{7}$$

Then, we are able to find a sequence $\{\xi_n\} \subset \mathcal{X}_0^*$ such that $\xi_n \in \partial J(u_n + \widetilde{u}_0)$ and

$$u_n^* = \mathcal{A}(u_n) + \xi_n \text{ for each } n \in \mathbb{N}.$$

From (7) and the above equality, it yields

$$\limsup_{n\to\infty} \langle \mathcal{A}u_n, u_n - u\rangle_{\mathcal{X}_0} + \liminf_{n\to\infty} \langle \xi_n, u_n - u\rangle_{\mathcal{X}_0} \le 0. \tag{8}$$

Since $X_0 \subset L^2(\Omega) \subset X_0^*$ and the embedding of X_0 to $L^2(\Omega)$ is compact, see Lemma 9, we have

$$u_n \to u \text{ strongly in } L^2(\Lambda).$$

Furthermore, invoking [4, Theorem 2.2], one has

$$\partial(J|_{\mathcal{X}_0})(u) \subset \partial(J|_{L^2(\Lambda)})(u) \text{ for all } u \in \mathcal{X}_0.$$

This results in

$$\langle \xi_n, u_n - u\rangle_{\mathcal{X}_0} = \langle \xi_n, u_n - u\rangle_{L^2(\Lambda)}. \tag{9}$$

Further, Proposition 10 and the boundedness of $\{u_n\}$ in $\mathcal{X}_0$ entail that the sequence $\{\xi_n\}$ is bounded both in $L^2(\Lambda)$ and $\mathcal{X}_0^*$. Then, by (9), we pass to the limit as $n \to \infty$ to get

$$\lim_{n\to\infty} \langle \xi_n, u_n - u\rangle_{\mathcal{X}_0} = \lim_{n\to\infty} \langle \xi_n, u_n - u\rangle_{L^2(\Lambda)} = 0.$$

This convergence combined with (8) and the monotonicity of $\mathcal{A}$ implies

$$\limsup_{n\to\infty} \|u_n - u\|^2_{\mathcal{X}_0} = \limsup_{n\to\infty} \langle \mathcal{A}u_n - \mathcal{A}u, u_n - u\rangle_{\mathcal{X}_0} + \lim_{n\to\infty} \langle \mathcal{A}u, u_n - u\rangle_{\mathcal{X}_0} \le 0.$$

Hence $u_n \to u$ strongly in $\mathcal{X}_0$. On the other side, the reflexivity of $\mathcal{X}_0^*$ and boundedness of $\{\xi_n\} \subset \mathcal{X}_0^*$ allow to assume, at least for a subsequence, that

$$\xi_n \to \xi \text{ weakly in } \mathcal{X}_0^* \text{ for some } \xi \in \mathcal{X}_0^*.$$

Since ∂J is upper semicontinuous from $\mathcal{X}_0$ to w–$\mathcal{X}_0^*$ and it has convex and closed values, it is closed from $\mathcal{X}_0$ to w–$\mathcal{X}_0^*$, see [11, Theorem 1.1.4]. Therefore, we obtain $\xi \in \partial J(u + \widetilde{u}_0)$.

To conclude, we have $u^* = \xi + \mathcal{A}u \in \mathcal{A}u + \partial J(u + \widetilde{u}_0)$ and

$$\langle u_n^*, u_n\rangle_{\mathcal{X}_0} = \langle \xi_n + \mathcal{A}(u_n), u_n\rangle_{\mathcal{X}_0} \to \langle \xi + \mathcal{A}(u), u\rangle_{\mathcal{X}_0} = \langle u^*, u\rangle_{\mathcal{X}_0},$$

i.e., the operator $\mathcal{A}(\cdot) + \partial J(\cdot + \widetilde{u}_0) : \mathcal{X}_0 \to 2^{\mathcal{X}_0^*}$ is L-pseudomonotone.

Claim 2. *There exists a function $r : \mathbb{R}_+ \to \mathbb{R}$ with $r(s) \to +\infty$ as $s \to +\infty$ such that*

$$\langle \mathcal{A}u + \partial J(u + \widetilde{u}_0), u\rangle_{\mathcal{X}_0} \ge r(\|u\|_{\mathcal{X}_0})\|u\|_{\mathcal{X}_0}$$

for all $u \in \mathcal{X}_0$.

From [4, Theorem 2.2] and Proposition 10, for all $u \in \mathcal{X}_0$, one has

$$\begin{aligned}
&\langle \mathcal{A}u + \partial J(u + \widetilde{u}_0), u\rangle_{\mathcal{X}_0} = \langle \mathcal{A}u, u\rangle_{\mathcal{X}_0} + \langle \partial J(u + \widetilde{u}_0), u\rangle_{\mathcal{X}_0}\\
&\ge \|u\|^2_{\mathcal{X}_0} - \|\widetilde{u}_0\|_{\mathcal{X}_0}\|u\|_{\mathcal{X}_0} + \langle \partial J(u + \widetilde{u}_0), u\rangle_{L^p(\Lambda)}\\
&\ge \|u\|^2_{\mathcal{X}_0} - \|\widetilde{u}_0\|_{\mathcal{X}_0}\|u\|_{\mathcal{X}_0} - \|\partial J(u + \widetilde{u}_0)\|_{L^p(\Lambda)}\|u\|_{L^p(\Lambda)}\\
&\ge \|u\|^2_{\mathcal{X}_0} - \|\widetilde{u}_0\|_{\mathcal{X}_0}\|u\|_{\mathcal{X}_0} - c_J(1 + \|u + \widetilde{u}_0\|^{p-1}_{L^p(\Lambda)})\|u\|_{L^p(\Lambda)}\\
&\ge \|u\|^2_{\mathcal{X}_0} - \|\widetilde{u}_0\|_{\mathcal{X}_0}\|u\|_{\mathcal{X}_0} - c_Jc(p)\|u\|_{\mathcal{X}_0} - c_Jc(p)^p\|u\|^p_{\mathcal{X}_0} - c_Jc(p)^p\|\widetilde{u}_0\|^{p-1}_{\mathcal{X}_0}\|u\|_{\mathcal{X}_0}\\
&= \|u\|_{\mathcal{X}_0}\big(\|u\|_{\mathcal{X}_0} - \|\widetilde{u}_0\|_{\mathcal{X}_0} - c_Jc(p) - c_Jc(p)^p\|\widetilde{u}_0\|^{p-1}_{\mathcal{X}_0} - c_Jc(p)^p\|u\|^{p-1}_{\mathcal{X}_0}\big).
\end{aligned}$$

Let $r(s) = s - \|\widetilde{u}_0\|_{\mathcal{X}_0} - c_Jc(p) - c_Jc(p)^p\|\widetilde{u}_0\|^{p-1}_{\mathcal{X}_0} - c_Jc(p)^p s^{p-1}$ for $s \in \mathbb{R}$. It follows from the condition $1 \le p < 2$ or $p = 2$ with $c_Jc(2)^2 < 1$ that the function r is coercive, namely, $r(s) \to +\infty$ as $s \to +\infty$. This shows Claim 2.

Claim 3. *$\Psi : \mathcal{X}_0 \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous functional.*

It obvious that $\Psi \not\equiv +\infty$, because $0 \in D(\Psi)$, see hypothesis $H(\varphi)$. Let $u_1, u_2 \in \mathcal{X}_0$ and $\lambda \in (0, 1)$ be arbitrary. The convexity of φ implies

$$\begin{aligned}
\Psi(\lambda u_1 + (1-\lambda)u_2) &= \int_0^T \varphi(\lambda u_1(t) + (1-\lambda)u_2(t) + u_0)\,dt\\
&= \int_0^T \varphi(\lambda(u_1(t) + u_0) + (1-\lambda)(u_2(t) + u_0))\,dt\\
&\le \lambda \int_0^T \varphi(u_1(t) + u_0)\,dt + (1-\lambda)\int_0^T \varphi(u_2(t) + u_0)\,dt\\
&= \lambda\Psi(u_1) + (1-\lambda)\Psi(u_2),
\end{aligned}$$

hence Ψ is convex. Let $\{u_n\} \subset \mathcal{X}_0$ be such that $u_n \to$ u *in* $\mathcal{X}_0$. Since φ is convex and lower semicontinuous, it follows from [24, Lemma 2.5] that there are two constants $m_1, m_2 \in \mathbb{R}$ satisfying

$$\varphi(z) \ge m_1\|z\|_{X_0} + m_2 \text{ for all } z \in X_0.$$

Invoking the Hölder inequality, we obtain

$$\begin{aligned}
\Psi(u_n) &= \int_0^T \varphi(u_n(t) + u_0)\,dt \ge m_2T + m_1\int_0^T \|u_n(t) + u_0\|_{X_0}\,dt\\
&\ge m_2T - |m_1|T^{\frac{1}{2}}\|u_n + \widetilde{u}_0\|_{\mathcal{X}_0} \ge m_0,
\end{aligned}$$

for some $m_0>0$, which is independent of $\{u_n\}$. From the convergence $u_n\to u$ in $\mathcal{X}_0$, we may assume, by passing to a subsequence if necessary, that $u_n(t)\to u(t)$ in X_0 for a.e. $t\in(0,T)$. This combined with the Fatou lemma, see e.g. [21, Theorem 1.64], gives

$$\liminf_{n\to\infty}\Psi(u_n)=\liminf_{n\to\infty}\int_0^T\varphi(u_n(t)+u_0)\,dt\ge\int_0^T\liminf_{n\to\infty}\varphi(u_n(t)+u_0)\,dt$$
$$\ge\int_0^T\varphi(u(t)+u_0)\,dt=\Psi(u).$$

Then, Ψ is a proper convex and lower semicontinuous functional.

Claim 4. *$\partial_C\Psi:\mathcal{X}_0\to 2^{\mathcal{X}_0^*}$ is a maximal monotone operator, which is strongly-quasi bounded and $0\in\partial_C\Psi(0)$.*

From Claim 3, we know that Ψ is a proper, convex and lower semicontinuous functional. Invoking Theorem 4, it implies that $\partial_C\Psi:\mathcal{X}_0\to 2^{\mathcal{X}_0^*}$ is a maximal monotone operator. The hypothesis $H(\varphi)$ yields directly that $0\in\partial_C\Psi(0)$. Next, we shall show that the mapping $\partial_C\Psi$ is strongly-quasi bounded. In fact, from e.g. [1, Proposition 2.7], we have

$$\operatorname{int}D(\Psi)\subset D(\partial_C\Psi),$$

which gives $\operatorname{int}D(\Psi)\subset\operatorname{int}D(\partial_C\Psi)$. Now, assumptions $H(\varphi)$, $H(0)$, and Proposition 7 guarantee that the mapping $\partial_C\Psi$ is strongly-quasi bounded.

Having verified Claims 1–4, we are now in a position to apply the surjectivity result, Theorem 8. We deduce that there exists a function $w\in\mathcal{W}$ with $w(0)=0$ solving the following inclusion problem

$$\begin{cases}Lw+\mathcal{A}w+\partial J(w+\widetilde{u}_0)+\partial_C\Psi(w)\ni f \quad\text{in}\quad \mathcal{X}_0^*\\ w(0)=0.\end{cases}\tag{10}$$

Claim 5. *If $w\in\mathcal{W}$ is a solution to problem* (10)*, then the function $u=w+u_0$ is a weak solution to problem* (1)*.*

Let $w\in\mathcal{W}$ be a solution to problem (10). Hence, $u=w+u_0$ solves the following problem

$$\begin{cases}\dfrac{\partial u}{\partial t}+\mathcal{A}(u-\widetilde{u}_0)+\partial J(u)+\partial_C\Psi(u-\widetilde{u}_0)\ni f\\ u(0)=u_0.\end{cases}\tag{11}$$

Recalling the definition of generalized Clarke subdifferential and convex subdifferential, it yields

$$\langle\partial J(u),v-u\rangle_{\mathcal{X}_0}\le J^0(u;v-u)\quad\text{for all } v\in\mathcal{X}_0,\tag{12}$$

and

$$\begin{aligned}\langle\partial_C\Psi(u-\widetilde{u}_0),v-u\rangle_{\mathcal{X}_0}&=\langle\partial_C\Psi(u-\widetilde{u}_0),v-\widetilde{u}_0-(u-\widetilde{u}_0)\rangle_{\mathcal{X}_0}\\&\le\Psi(v-\widetilde{u}_0)-\Psi(u-\widetilde{u}_0)\quad\text{for all } v\in\mathcal{X}_0.\end{aligned}\tag{13}$$

Combining (11)–(13), definitions (4), and (6), we have

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^N}\frac{\partial u(x,t)}{\partial t}\big(v(x,t)-u(x,t)\big)\,dx\,dt+J^0(u;v-u)+\int_0^T\varphi(v(t))\,dt\\&-\int_0^T\varphi(u(t))\,dt+\int_0^T\int_{\mathbb{R}^N}\big(v(x,t)-u(x,t)\big)\mathcal{L}_K u(x,t)\,dx\,dt\\&\ge\int_0^T\int_{\mathbb{R}^N}f(x,t)\big(v(x,t)-u(x,t)\big)\,dx\,dt\quad\text{for all}\quad v\in\mathcal{X}_0.\end{aligned}$$

This means that $u=w+u_0$ is a weak solution to problem (1), which completes the proof of the theorem.

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