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A Review of Hermite-Hadamard Inequality

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Abstract - *In this review we present the most important lines of development, around the well-known Hermite-Hadamard Inequality, as well as some open problems.*

Keywords: Hermite-Hadamard inequality; convex function

1. INTRODUCTION

One of the most attractive concepts in Mathematical Sciences is the convex function, present today in multiple mathematical areas ranging from Optimization to Function Theory and center of possibly the most fruitful nucleus in the study of inequalities. integral, as we will see later. Let's start by introducing the concept of a convex function as follows.

In what follows, [a,b] is a real, closed and bounded interval. A function f:[a,b] ⊆ R → R is said to be convex on the interval [a,b], if the inequality

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

holds. We say that f is concave if -f is convex.

We recommend that readers consult the work [21] where a fairly complete overview of the various notions and ramifications is presented. For example, the following graph that illustrates these extensions and derivations of the original concept is interesting.

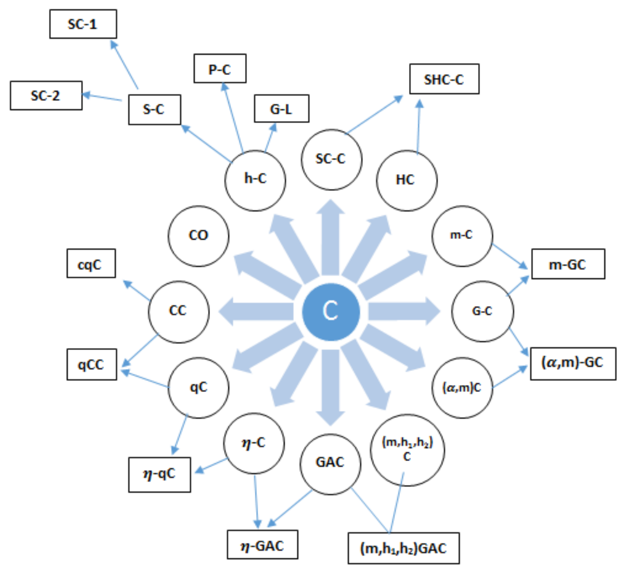

Fig -1: This figure describes different ramifications derived from the classical definition of convex function, obtained in recent years

With

{C} Classical convex function.
{CO} Convexity with respect to another function.
{SC-C} Strongly convex function with modulus C.
{HC} Harmonically convex function.
{SHC-C} Strongly harmonically convex function with modulus C.
{h-C} h-convex function.
{P-C} P- convex function.
{G-L} Godunova-Levin function.
{S-C} s-convex function.
{SC-1} s-convex function in the first sense.
{SC-2} s-convex function in the second sense.
{m-C} m-convex function.
{G-C} Geometrically convex function.

{m-GC} m-geometrically convex function.
{(α,m)-C} (α,m) convex function
{(α,m)-GC} (α,m) geometrically con|vex function.
{GAC} Geometric arithmetically convex function.
{(m,h1,h2)-C} (m, h1,h2) convex function.
{(m,h1,h2)-GAC} (m, h1,h2) geometric arithmetically convex function.
{η-C} η convex function.
{η-GAC} Generalized geometric arithmetically convex function with respect to η.
{qC} Quasi-convex function.
{CC} C- convex function.
{cqC} C- quasi-convex function.
{qCC} Quasi-convex function with respect to C.
{ η qC} η-quasi-convex function.

For a convex function f, the following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(u)du \leq \frac{f(a)+f(b)}{2} \quad (1)$$

is known in the literature as a Hermite-Hadamard integral inequality, so known in honor of the French mathematicians who published it, independently of each other ([11,12]).

This inequality has attracted the attention of researchers in recent decades and an increase in the number of publications referring to it has been appreciated. This development has occurred in four fundamental directions:

I) With new notions of convexity.

II) Using different integral operators.

III) Defining functionals, which allow obtaining new estimates of $f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx$ or $\frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}$.

IV) Using a more refined mesh, that is, instead of considering a and b, take other nodes in the interval.

In this paper we will take a tour of one of the most dynamic areas of current Mathematics (addresses I) and II) above) and we will show various work directions, perfectly defined and some open problems.

2. THE TOUR

I) New notions of convexity.

In [3] we present the following definitions:

Definiton 1. Let $h: [0,1] \to R$ be a nonnegative function, h≠0 and $f: I = [0, +\infty) \to [0, +\infty)$. If inequality

$$f(\sigma a + m(1-\sigma)b) \leq h^s(\sigma)f(a) + m\big(1 - h^s(\sigma)\big)f(b)$$

is fulfilled for all a,b∈I and σ ∈[0,1], where m ∈[0,1], s ∈(0,1]. Then is said function f is a (h,m)-convex modified of first type on I.

Definiton 2. Let $h: [0,1] \to R$ be a nonnegative function, h≠0 and $f: I = [0, +\infty) \to [0, +\infty)$. If inequality

$$f(\sigma a + m(1-\sigma)b) \leq h^s(\sigma)f(a) + m\big(1 - (\sigma)\big)^s f(b)$$

is fulfilled for all a,b∈I and σ ∈[0,1], where m ∈[0,1], s ∈(0,1]. Then is said function f is a (h,m)-convex modified of second type on I.

Considering the triple (h(z),m,s), we have the following particular cases of our definitions:

(z,1,1), then f is a convex function on [0,+∞).

(z,m,1), then f is a m-convex function on [0,+∞).

(z,1,s) and s∈ (0,1], then f is a s-convex function on [0,+∞).

(z,1,s) and s∈[-1,1], then f is a extended s-convex function on [0,+∞).

(z,m,s) and s∈(0,1], then f is a (s,m)-convex function on [0,+∞).

(zα,1,s) with α∈(0,1], then f is a (α,s)-convex function on [0,+∞).

(zα,m,1) with α∈(0,1], then f is a (α,m)-convex function on [0,+∞).

(zα,m,s) with α∈(0,1], then f is a s-(α,m)-convex function on [0,+∞).

(h(z),m,1), then we have a variant of the (h,m)-convex function on [0,+∞).

That is, our definition contains as particular cases, many of the notions of convexity reported in the literature. It is clear then, that studying the inequality (1) under the notion of modified (h,m)-convex functions, allows us to obtain more general results than those known.

II) Different integral operators.

In different papers, we have used various operators that are generalizations of the classical Riemann Integral, of the Riemann-Liouville Fractional Integral and others. To cite just one, consider the following weighted operator:

Definition 3. Let $f \in L1(a,b)$ and let $w: [0,+\infty) \to [0,+\infty)$ be a continuous function with first order derivatives integrables on $[0,+\infty)$. Then the weighted fractional integrals are defined by (right and left, respectively):

$$J_{a+}^{w}\psi(\xi) = \int_a^{\xi} w'\left(\frac{\xi-\tau}{b-a}\right)\psi(\tau)\,d\tau, \quad \xi > a,$$

$$J_{b-}^{w}\psi(\xi) = \int_{\xi}^{b} w'\left(\frac{\tau-\xi}{b-a}\right)\psi(\tau)\,d\tau, \quad \xi < b.$$

Obviously if $w'(t)=1$, we obtain the Riemann Integral, while if $w'(t) = t^{\alpha-1}$ we obtain the Riemann-Liouville Fractional Integral.
In this way, we can generalize results reported for different integral operators.

Interested readers may consult [1-10], [13-20] and [22-25], for varied results.

3. CONCLUSIONS

We have presented a group of results that illustrate one of the most dynamic directions in Mathematical Sciences today: the Hermite-Hadamard Inequality for convex functions. Obviously, this is not exhaustive, for example, new results for the Katugampola Fractional Integral can be obtained using these notions of convexity or some new ones.

On the other hand, obtaining new refinements for inequality (1) in the class of (h,m)-convex functions of the first type is an open problem.

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