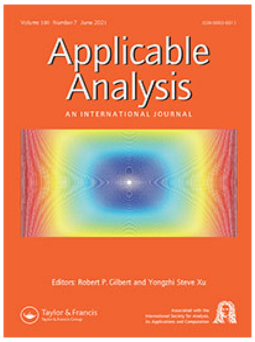

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Penalty method for a class of differential variational inequalities

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ABSTRACT

The purpose of this paper is to investigate a class of differential variational inequalities involving a constraint set in Banach spaces. A well-posedness result for the inequality is obtained, including the existence, uniqueness, and stability of the solution in mild sense. Further, we introduce a penalized problem without constraints and prove that the solution to the original inequality can be approached, as a parameter converges to zero, by the solution of the penalized problem. Finally, an application to a comprehensive obstacle parabolic-elliptic system delineates the abstract results.

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1. Introduction

The notion of differential variational inequalities (DVIs, for short) was initially introduced by Aubin-Cellina [1] in 1984. However, DVIs were firstly systematically studied in Euclidean spaces by Pang-Stewart [2] in 2008. In the paper [2], the authors pointed out that DVIs can be a powerful mathematical tool to represent models involving both dynamics and constraints in the form of inequalities, which arise in many applied problems in our real life, for instance, mechanical impact problems, electrical circuits with ideal diodes, the Coulomb friction problems for contacting bodies, economical dynamics, dynamic traffic networks, and so on. Since then, more and more scholars have been attracted to devote to the treatment of both theoretical and numerical aspects of the differential variational inequalities as well as its applications in economical dynamics system and contact mechanics problems. Amongst the results, we mention: Liu-Zeng-Motreanu [3–5] and Liu-Migórski-Zeng [6] proved the existence of solutions for a class of differential mixed variational inequalities in Banach spaces through applying the theory of semigroups, Filippov implicit function lemma and fixed point theorems for condensing multivalued operators; Migórski-Zeng [7] applied a temporally semi-discrete method based on the backward Euler difference scheme and a feedback iterative technique to address a new kind of problems, which consists of a hemivariational inequality of parabolic type combined with a nonlinear evolution equation in the framework of an evolution triple of spaces; Chen-Wang [8] in 2014 used the idea of DVIs to investigate a dynamic Nash equilibrium problem

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Dedicated to Professor Dumitru Motreanu on the occasion of his 70th birthday.

of multiple players with shared constraints and dynamic decision processes; Zeng-Liu-Migórski [9] utilized Rothe method combined with surjectivity of multivalued pseudomonotone operators and properties of the Clarke generalized gradient to establish the existence of solutions to a class of fractional differential hemivariational inequalities in Banach spaces and then applied the abstract results to study a frictional quasistatic contact problem for viscoelastic materials with adhesion; via using theory of measure of noncompactness and fixed point theorems, Ke-Loi-Obukhovskii [10] validated the existence of decay solutions for a new kind of fractional differential variational inequalities; Gwinner [11] in 2013 established a stability result of a class of differential variational inequalities by using the monotonicity method and technique of the Mosco convergence; and Liu-Loi-Obukhovskii [12] in 2013 studied the existence and global bifurcation problems for periodic solutions to a class of differential variational inequalities in finite-dimensional spaces by invoking the topological methods from the theory of multivalued maps and some versions of the method of guiding functions. For more details on these topics, the reader is welcome to consult [13–24] and the references therein.

The gap function methods as a powerful mathematical tool has been employed to various variational inequality problems. It is well-known that gap function methods play a crucial role in transforming a variational inequality problem into an optimization problem, by constructing a nonnegative function g, which is called to be gap function, such that $g(y) = 0$ if and only if y is a solution to the variational inequality, see cf. [25–30]. Then, methods solving an optimization problem can be exploited for finding a solution of a variational inequality problem. On the other side, from the view of numerical analysis in Mechanics, it is quite difficult to obtain the numerical solutions for the inequality problems in Mechanics that if inequality problems have constraints. Despite gap function methods can be a useful technique to deliver a variational inequality to an optimization problem, we have to admit that it is not a efficient approach to transform a variational inequality with constraints to a constraint free problem. To break the barrier, a critical technique, penalty method, was introduced. Generally speaking, the main feature of the penalty method is that constraints in a problem are enforced by penalty through a limiting procedure and the penalized problems are constraint free. The penalized problems have unique solutions which converge to the solution of the original problem, as the penalty parameter tends to zero. Recently, penalty methods have been a critical mathematical tool to handle a variety of problems, such as the study of derivation of optimality conditions in inverse problems and optimal control problems, see e.g. [31–33]. To the best of the author's knowledge, there are many references which use penalty methods to study only a single variational inequality. However, there are no results concerning penalty methods for differential variational inequalities. Therefore, for the first time, we apply penalty techniques to study differential variational inequalities.

Actually, the present paper presents a continuation of Liu-Zeng-Motreanu [3–5]. In those papers, the authors just proved the existence of solutions in mild sense for differential variational inequality, problem (1), however, they did not provide the uniqueness and stability of the solution. To fill this gap, in this paper, we first establish a well-posedness result for problem (1), including the existence, uniqueness, and stability of the solution. On the other hand, it results in a big challenge to obtain the numerical solutions for the inequality problems in Mechanics that if inequality problems have constraints. Face this challenge, the well-known penalty method has been an essential approach to overcome this difficulty. Based on this motivation, then, by using penalty technique, we introduce a penalized problem corresponding to problem (1). Further, we prove that the unique solution to the original problem (1) can be approached, as a parameter converges to zero, by the unique solution of the approximated problem (6) (see Section 4). Finally, we apply the abstract results obtained in Sections 3 and 4 to a comprehensive obstacle parabolic-elliptic system.

The rest of the paper is organized as follows. In Section 2, we deliver the functional framework and formulation of the differential variational inequality and recall the main preliminary material needed in what follows. Section 3 is devoted to establish a well-posedness result of problem (1). The main results on existence, uniqueness, and convergence for a penalized differential variational inequality (see problem (6) in Section 4) are provided in Theorem 4.1 of Section 4. Finally, in Section 5, an application of our results to an obstacle parabolic-elliptic system is discussed.

2. Problem statement and preliminary material

Let $(E, \|\cdot\|_E)$ and $(V, \|\cdot\|_V)$ be two reflexive and separable Banach spaces, $T > 0$, and K be a nonempty, closed, and convex subset of V . Let V^* be the dual space of V and denote the dual pair between V^* and V by $\langle\cdot,\cdot\rangle$. In what follows, we assume that $A\colon D(A) \subset E \to E$ is the infinitesimal generator of a C_0-semigroup $\{e^{At}\}_{t\geq 0}$ in E. Given functions $f\colon [0, T] \times E \times V \to E$, $g\colon [0, T] \times E \times V \to V^*$ and $\varphi\colon V \to \mathbb{R}$, in this paper, we are interested to investigate the following abstract differential variational inequality: Find functions $x\colon [0, T] \to E$ and $u\colon [0, T] \to V$ such that

$$\begin{aligned}
&x'(t) = Ax(t) + f(t, x(t), u(t)) && \text{for a.e. } t \in [0, T],\\
&u(t) \in S(K, g(t, x(t), \cdot), \varphi) && \text{for a.e. } t \in [0, T],\\
&x(0) = x_0,
\end{aligned} \tag{1}$$

where $S(K, g(t, x(t), \cdot), \varphi)$ represents the solution set of the following mixed variational inequality: Find $u(t) \in K$ such that

$$\langle g(t, x(t), u(t)), v - u(t)\rangle + \varphi(v) - \varphi(u(t)) \geq 0 \ \text{ for all } v \in K.$$

From our previous work [3–6], we give the definition of solutions of problem (1) in the sense of mild.

Definition 2.1: A pair of functions (x, u), with $x \in C(0, T; E)$ and $u\colon [0, T] \to K$ measurable, is said to be a mild solution of problem (1) if

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} f(s, x(s), u(s))\, \mathrm{d}s$$

for all $t \in [0, T]$, where $u(s) \in S(K, g(s, x(s), \cdot), \varphi)$ for a.e. $s \in [0, T]$. If (x, u) is a mild solution of the problem (1), then x is called to be the mild trajectory and u is the variational control trajectory.

Next, we recall some important notation, definitions, and preliminary materials, which will be needed in the sequel. For more details, we refer to [34–37].

Definition 2.2: Let $(X, \|\cdot\|_X)$ be a reflexive Banach space with its dual X^* and $A\colon X \to X^*$. We say that

(i) A is monotone, if for all $u, v \in X$, we have $\langle Au - Av, u - v\rangle \geq 0$.
(ii) A is strongly monotone with constant $m_A > 0$, if $\langle Au - Av, u - v\rangle \geq m_A\|u - v\|_X^2$ for all u, $v \in X$.
(iii) A is pseudomonotone, if A is a bounded operator and for every sequence $\{x_n\} \subseteq X$ converging weakly to $x \in X$ such that $\limsup\langle Ax_n, x_n - x\rangle \leq 0$, we have $\langle Ax, x - y\rangle \leq \liminf\langle Ax_n, x_n - y\rangle$ for all $y \in X$.
(iv) A is hemicontinuous, if for all u, v, $w \in X$, the function $\lambda \mapsto \langle A(u + \lambda v), w\rangle$ is continuous on $[0, 1]$.

Obviously, $A\colon X \to X^*$ is pseudomonotone if and only if A is bounded and $x_n \to x$ weakly in X with $\limsup\langle Ax_n, x_n - x\rangle \leq 0$ imply $\lim\langle Ax_n, x_n - x\rangle = 0$ and $Ax_n \to Ax$ weakly in X^*. Furthermore, if $A \in \mathcal{L}(X, X^*)$ (the class of linear bounded maps) is nonnegative, then it is pseudomonotone.

Let X be a Banach space with its dual space X^*. A function $f\colon X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is called to be proper, convex, and lower semicontinuous, if it fulfills, respectively, the following conditions

$$\mathrm{D}(f) := \{u \in X : f(u) < +\infty\} \neq \emptyset,$$
$$f(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda)f(v) \quad \text{for all } \lambda \in [0, 1] \quad \text{and } u, v \in X,$$

$$f(u) \le \liminf_{n\to\infty} f(u_n) \quad \text{for all sequences } \{u_n\} \subset X \quad \text{with } u_n \to u.$$

The convex subdifferential of a proper and convex function $f\colon X \to \overline{\mathbb{R}}$ is defined by

$$\partial f(u) = \begin{cases} \left\{\xi \in X^* : f(v) - f(u) \ge \langle \xi, v-u\rangle \quad \text{for all } v \in X\right\}, & \text{if } u \in \mathrm{D}(f) \\ \emptyset, & \text{elsewhere.} \end{cases}$$

Obviously, from [36, Proposition 3.33], we can see that if f is Gâteaux differentiable at a point $u \in X$, then we have $\partial f(u) = \{Df(u)\}$, where $Df(u)$ is the Gâteaux derivative of f at u.

Now, we recall the notion of the penalty operator, see [38].

Definition 2.3: Let X be a Banach space and K be a nonempty subset of X. An operator $P\colon X \to X^*$ is said to be a penalty operator of set K if P is bounded, demicontinuous, monotone and $K = \{u \in X : Pu = 0\}$.

Note that, if K is a nonempty, closed and convex subset of reflexive Banach space X, then the operator $P = J(I - P_K)$ is a penalty operator of K, where $J\colon X \to X^*$ is the duality map on X defined by

$$J(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|_X^2 = \|x^*\|_{X^*}^2\} \quad \text{for } x \in X,$$

I is the identity map on X, and $P_K\colon X \to X$ is the projection operator of K (see [35, Proposition 1.3.27]).

3. Well-posedness for differential variational inequalities

In this section, we shall pay our attention to establish a well-posedness result for problem (1), including the existence, uniqueness, and stability of the solution.

Under the functional framework mentioned in Section 1, we also need the following assumptions for the data of problem (1).

$\underline{H(\varphi)}$: The functional $\varphi\colon V \to \mathbb{R}$ is a convex and lower semicontinuous functional.

$\underline{H(g)}$: $g\colon [0,T] \times E \times V \to V^*$ is such that

(i) for all $(t,x) \in [0,T] \times E$, the mapping $u \mapsto g(t,x,u)$ is hemicontinuous and strongly monotone with constant $m_g > 0$;

(ii) if K is unbounded in V , there exist an element $v^* \in K$ such that

$$\liminf_{u\in K,\, \|u\|_V\to\infty} \frac{\langle g(t,x,u), u - v^*\rangle + \varphi(u) - \varphi(v^*)}{\|u\|_V} \to +\infty$$

for all $(t,x) \in [0,T] \times E$;

(iii) there exists a constant $L_g > 0$ such that

$$\|g(t_1,x_1,u) - g(t_2,x_2,u)\|_{V^*} \le L_g\left(|t_1 - t_2| + \|x_1 - x_2\|_E\right)$$

for all $t_1, t_2 \in [0,T]$, $u \in V$ and $x_1, x_2 \in E$.

$\underline{H(f)}$: The nonlinear function $f\colon [0,T] \times E \times V \to E$ satisfies the following properties:

(i) for all $(x,u) \in E \times V$, the function $t \mapsto f(t,x,u)$ is measurable on $[0,T]$;

(ii) the function $t \mapsto f(t,0,0)$ belongs to $L^1(0,T;E)$;

(iii) there exists a function $\psi \in L^1_+(0,T)$ such that

$$\|f(t,x_1,u_1) - f(t,x_2,u_2)\|_E \le \psi(t)\left(\|x_1 - x_2\|_E + \|u_1 - u_2\|_V\right)$$

for a.e. $t \in [0,T]$ and all $(x_1,u_1), (x_2,u_2) \in E \times V$.

Let $M_A > 0$ be such that $\sup_{t\in[0,T]} \|e^{At}\| \leq M_A$. We now present the main result in the section as follows.

Theorem 3.1: *Assume that $H(g), H(\varphi)$, and $H(f)$ are fulfilled. Then we have*

(i) *for each initial point $x_0 \in E$, differential variational inequality* (1) *possesses a unique solution $(x, u) \in C(0, T; E) \times C(0, T; V)$;*
(ii) *the map $x_0 \mapsto (x, u)(x_0) \colon E \to C(0, T; E) \times C(0, T; V)$ is Lipschitz continuous.*

Proof: (i) It follows from [3, Theorem 3.7] that differential variational inequality (1) admits a solution (x, u), where $x \in C(0, T; E)$ and $u\colon [0, T] \to V$ is measurable.

In fact, we shall show that the variational control trajectory u belongs to $C(0, T; V)$. To this end, let (x, u) be a solution of problem (1) and $t_1, t_2 \in [0, T]$. Hence, one has

$$\langle g(t, x(t), u(t)), v - u(t)\rangle + \varphi(v) - \varphi(u(t)) \geq 0 \quad \text{for all } v \in K \tag{2}$$

and for all $t \in [0, T]$. Inserting $t = t_1$ with $v = u(t_2)$, and $t = t_2$ with $v = u(t_1)$ into the above inequality, respectively, we sum the resulting inequalities to yield

$$\langle g(t_1, x(t_1), u(t_1)) - g(t_2, x(t_2), u(t_2)), u(t_1) - u(t_2)\rangle \leq 0.$$

From the strong monotonicity and Lipschitz continuity of g, one has

$$\begin{aligned} m_g\|u(t_1) - u(t_2)\|_V^2 &\leq \langle g(t_1, x(t_1), u(t_1)) - g(t_1, x(t_1), u(t_2)), u(t_1) - u(t_2)\rangle \\ &\leq \langle g(t_2, x(t_2), u(t_2)) - g(t_1, x(t_1), u(t_2)), u(t_1) - u(t_2)\rangle \\ &\leq L_g(|t_1 - t_2| + \|x(t_1) - x(t_2)\|_E)\|u(t_1) - u(t_2)\|_V. \end{aligned}$$

Hence, we have

$$\|u(t_1) - u(t_2)\|_V \leq \frac{L_g}{m_g}(|t_1 - t_2| + \|x(t_1) - x(t_2)\|_E).$$

Recall that $x \in C(0, T; E)$, so, the above inequality reveals that the variational control trajectory u is a continuous function too.

Next, we will show that the uniqueness of solution for problem (1). We assume that (x_1, u_1) and (x_2, u_2) are two solutions of problem (1). So, it has

$$x_i(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} f(s, x_i(s), u_i(s))\, \mathrm{d}s \tag{3}$$

$$\langle g(t, x_i(t), u_i(t)), v - u_i(t)\rangle + \varphi(v) - \varphi(u_i(t)) \geq 0 \quad \text{for all } v \in K \tag{4}$$

for all $t \in [0, T]$. Putting $v = u_2(t)$ into (4) for $i = 1$, and $v = u_1(t)$ into (4) for $i = 2$, we apply the Lipschitz continuity and strong monotonicity of g again to deduce

$$\begin{aligned} m_g\|u_1(t) - u_2(t)\|_V^2 &\leq \langle g(t, x_1(t), u_1(t)) - g(t, x_1(t), u_2(t)), u_1(t) - u_2(t)\rangle \\ &\leq \langle g(t, x_2(t), u_2(t)) - g(t, x_1(t), u_2(t)), u_1(t) - u_2(t)\rangle \\ &\leq \|g(t, x_2(t), u_2(t)) - g(t, x_1(t), u_2(t))\|_{V^*}\|u_1(t) - u_2(t)\|_V \\ &\leq L_g\|x_1(t) - x_2(t)\|_E\|u_1(t) - u_2(t)\|_V \ \text{ for all } t \in [0, T]. \end{aligned}$$

Hence, it has

$$\|u_1(t) - u_2(t)\|_V \leq \frac{L_g}{m_g}\|x_1(t) - x_2(t)\|_E \quad \text{for all } t \in [0, T]. \tag{5}$$

On the other hand, from equalities (3) for $i = 1{,}2$, we utilize hypothesis $H(f)$ (iii) to deliver

$$\begin{aligned}\|x_1(t) - x_2(t)\|_E &\leq \int_0^t \left\| e^{A(t-s)}(f(s, x_1(s), u_1(s)) - f(s, x_2(s), u_2(s)))\right\|_E \, \mathrm{d}s \\ &\leq M_A \int_0^t \psi(s)\,(\|x_1(s) - x_2(s)\|_E + \|u_1(s) - u_2(s)\|_V)\, \mathrm{d}s \quad \text{for all } t \in [0, T].\end{aligned}$$

Taking account into (5), the above inequality can be restated to

$$\|x_1(t) - x_2(t)\|_E \leq M_A \int_0^t \psi(s)\left(1 + \frac{L_g}{m_g}\right)\|x_1(s) - x_2(s)\|_E \, \mathrm{d}s$$

for all $t \in [0, T]$. Invoking Gronwall's inequality, we have $x_1 = x_2$. This gets together with inequality (5) to guarantee $(x_1, u_1) = (x_2, u_2)$, namely, differential variational inequality (1) has a unique solution $(x, u) \in C(0, T; E) \times C(0, T; V)$.

(ii) Let x_0^1 and x_0^2 be two initial points in E. From (i), we know that there exist unique solutions (x_1, u_1) and (x_2, u_2) corresponding to initial points x_0^1 and x_0^2, respectively. Therefore, we have

$$\begin{aligned}x_1(t) &= e^{At}x_0^1 + \int_0^t e^{A(t-s)} f(s, x_1(s), u_1(s))\, \mathrm{d}s \\ x_2(t) &= e^{At}x_0^2 + \int_0^t e^{A(t-s)} f(s, x_2(s), u_2(s))\, \mathrm{d}s\end{aligned}$$

for all $t \in [0, T]$. Subtract the above inequalities, we are able to find

$$\begin{aligned}\|x_1(t) - x_2(t)\|_E &\leq M_A \int_0^t \left\|f(s, x_1(s), u_1(s)) - f(s, x_2(s), u_2(s))\right\|_E \, \mathrm{d}s + M_A\|x_0^1 - x_0^2\|_E \\ &\leq M_A\|x_0^1 - x_0^2\|_E + M_A \int_0^t \psi(s)\,(\|x_1(s) - x_2(s)\|_E + \|u_1(s) - u_2(s)\|_V)\, \mathrm{d}s\end{aligned}$$

for all $t \in [0, T]$. Combining this with inequality (5) we get

$$\|x_1(t) - x_2(t)\|_E \leq M_A\|x_0^1 - x_0^2\|_E + M_A \int_0^t \psi(s)\left(1 + \frac{L_g}{m_g}\right)\|x_1(s) - x_2(s)\|_E \, \mathrm{d}s$$

for all $t \in [0, T]$. We are now in a position to apply Gronwall's inequality again to reveal

$$\begin{aligned}\max_{t\in[0,T]} \|x_1(t) - x_2(t)\|_E &\leq M_A\|x_0^1 - x_0^2\|_E \exp\left(M_A \int_0^T \psi(s)\left(1 + \frac{L_g}{m_g}\right) \mathrm{d}s\right) \\ &= M_A\|x_0^1 - x_0^2\|_E \exp\left(M_A\left(1 + \frac{L_g}{m_g}\right)\|\psi\|_{L^1(0,T)}\right).\end{aligned}$$

Moreover, we put the above estimate to (5) to obtain

$$\max_{t\in[0,T]} \|u_1(t) - u_2(t)\|_V \le \frac{L_g}{m_g} M_A \|x_0^1 - x_0^2\|_E \exp\left(M_A \left(1 + \frac{L_g}{m_g}\right) \|\psi\|_{L^1(0,T)} \right).$$

To conclude, we can see that the map $x_0 \mapsto (x,u)(x_0)\colon E \to C(0,T;E) \times C(0,T;V)$ is Lipschitz continuous with the Lipschitz constant $L > 0$,

$$L = \left(1 + \frac{L_g}{m_g}\right) M_A \exp\left(M_A \left(1 + \frac{L_g}{m_g}\right) \|\psi\|_{L^1(0,T)} \right),$$

which completes the proof of the theorem. ■

4. A penalization result for differential variational inequalities

In this section, we are interested in establishing of a penalization result for the differential variational inequality (1). Now, we introduce the penalized problem corresponding to problem (1), which is to find functions $x_\rho\colon [0,T] \to E$ and $u_\rho\colon [0,T] \to V$ such that

$$\begin{aligned} & x_\rho'(t) = Ax_\rho(t) + f(t, x_\rho(t), u_\rho(t)) \quad \text{for a.e. } t \in [0,T], \\ & u_\rho(t) \in V \quad \text{satisfying} \\ & \langle g(t, x_\rho(t), u_\rho(t)), v - u_\rho(t)\rangle + \frac{1}{\rho}\langle Pu_\rho(t), v - u_\rho(t)\rangle + \varphi(v) \\ & \quad - \varphi(u_\rho(t)) \ge 0 \text{ for all } v \in V \quad \text{and for all } t \in [0,T], \\ & x_\rho(0) = x_0, \end{aligned} \tag{6}$$

where $\rho > 0$ and $P\colon V \to V^*$ is a penalty operator of constraint set K, see Definition 2.3.

We have the following main results on existence, uniqueness, and convergence for problem (6).

Theorem 4.1: *Assume that $H(g)$, $H(\varphi)$, and $H(f)$ are fulfilled, and $P\colon V \to V^*$ is a penalty operator of K. Then we have*

(i) *for each $\rho > 0$, there exists a unique solution $(x_\rho, u_\rho) \in C(0,T;E) \times C(0,T;V)$ to problem (6);*
(ii) *(x_ρ, u_ρ) converges to the unique solution (x,u) of problem (1) in the following sense*

$$(x_\rho(t), u_\rho(t)) \to (x(t), u(t)) \text{ as } \rho \to 0,$$

for all $t \in [0,T]$.

Proof: (i) Denote $g_\rho\colon (0,T) \times E \times V \to V^*$ by

$$g_\rho(t,x,u) = g(t,x,u) + \frac{1}{\rho} Pu.$$

Obviously, we can see that g_ρ reads all the conditions of $H(g)$. Therefore, the desired result (i) is a direct consequence of Theorem 3.1.

(ii) To do so, we first introduce an intermediate problem: Find $\widetilde{u}_\rho \in C(0,T;V)$ such that

$$\langle g(t, x(t), \widetilde{u}_\rho(t)), v - \widetilde{u}_\rho(t)\rangle + \frac{1}{\rho}\langle P\widetilde{u}_\rho(t), v - \widetilde{u}_\rho(t)\rangle + \varphi(v) - \varphi(\widetilde{u}_\rho(t)) \ge 0 \tag{7}$$

for all $v \in V$ and $t \in [0,T]$, where x is the mild trajectory of problem (1). However, it is trivial that problem (7) has a unique solution $\widetilde{u}_\rho \in C(0,T;V)$, see the proof of Theorem 3.1 or [3, Theorem 3.3].

Let $u_0 \in K$ be fixed and $\widetilde{u}_\rho \in C(0,T;V)$ be the unique solution of problem (7). So, we have

$$\langle g(t,x(t),\widetilde{u}_\rho(t)), u_0 - \widetilde{u}_\rho(t)\rangle + \frac{1}{\rho}\langle P\widetilde{u}_\rho(t), u_0 - \widetilde{u}_\rho(t)\rangle + \varphi(u_0) - \varphi(\widetilde{u}_\rho(t)) \geq 0$$

for all $t \in [0,T]$. We now use the strong monotonicity of g to find

$$\begin{aligned} m_g\|u_0 - \widetilde{u}_\rho(t)\|_V^2 &\leq \langle g(t,x(t),\widetilde{u}_\rho(t)) - g(t,x(t),u_0), \widetilde{u}_\rho(t) - u_0\rangle \\ &\leq \frac{1}{\rho}\langle P\widetilde{u}_\rho(t), u_0 - \widetilde{u}_\rho(t)\rangle + \varphi(u_0) - \varphi(\widetilde{u}_\rho(t)) + \langle g(t,x(t),u_0), u_0 - \widetilde{u}_\rho(t)\rangle \end{aligned}$$

for all $t \in [0,T]$. In the meantime, remember that $Pv = 0$ for all $v \in K$ and P is monotone, we confess

$$\begin{aligned} m_g\|u_0 - \widetilde{u}_\rho(t)\|_V^2 &\leq \frac{1}{\rho}\langle P\widetilde{u}_\rho(t) - Pu_0, u_0 - \widetilde{u}_\rho(t)\rangle + \langle g(t,x(t),u_0), u_0 - \widetilde{u}_\rho(t)\rangle \\ &\quad + \varphi(u_0) - \varphi(\widetilde{u}_\rho(t)) \\ &\leq \varphi(u_0) - \varphi(\widetilde{u}_\rho(t)) + \langle g(t,x(t),u_0), u_0 - \widetilde{u}_\rho(t)\rangle \end{aligned}$$

for all $t \in [0,T]$. Since φ is convex and lower semicontinuous, then from [41, Proposition 1.29] there exist an element $l \in V^*$ and a constant $\beta \in \mathbb{R}$ such that

$$\varphi(v) \geq \langle l, v\rangle + \beta \quad \text{for all } v \in V.$$

Hence, one has

$$\begin{aligned} m_g\|u_0 - \widetilde{u}_\rho(t)\|_V^2 &\leq \varphi(u_0) - \varphi(\widetilde{u}_\rho(t)) + \langle g(t,x(t),u_0), u_0 - \widetilde{u}_\rho(t)\rangle \\ &\leq \varphi(u_0) - \langle l, \widetilde{u}_\rho(t)\rangle - \beta + \|g(t,x(t),u_0)\|_{V^*}\|\widetilde{u}_\rho(t) - u_0\|_V \\ &\leq \varphi(u_0) + \|l\|_{V^*}\|\widetilde{u}_\rho(t)\|_V + \|g(t,x(t),u_0)\|_{V^*}\|\widetilde{u}_\rho(t) - u_0\|_V + |\beta| \\ &\leq \varphi(u_0) + \left(\|g(t,x(t),u_0)\|_{V^*} + \|l\|_{V^*}\right)\|\widetilde{u}_\rho(t) - u_0\|_V + \|l\|_{V^*}\|u_0\|_V + |\beta| \end{aligned}$$

for all $t \in [0,T]$. Further, we apply the elementary inequality, $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ for all $a, b \in \mathbb{R}$, to get

$$\frac{m_g}{2}\|u_0 - \widetilde{u}_\rho(t)\|_V^2 \leq \varphi(u_0) + \frac{M_1^2}{2m_g} + \|l\|_{V^*}\|u_0\|_V + |\beta|,$$

where $M_1 := \max_{t\in[0,T]}\|g(t,x(t),u_0)\|_{V^*} + \|l\|_{V^*}$. This means that $\{\widetilde{u}_\rho(t) - u_0\}_{\rho>0,t\in[0,T]}$ is bounded, so does $\{\widetilde{u}_\rho(t)\}_{\rho>0,t\in[0,T]}$.

Therefore, for each $t \in [0,T]$ fixed, passing to a relabeled subsequence, we may assume that

$$\widetilde{u}_\rho(t) \to \widetilde{u}(t) \quad \text{weakly in } V \text{ as } \rho \to 0,$$

for some $\widetilde{u}(t) \in V$. We shall show that $\widetilde{u}(t) \in K$. Indeed, the strong monotonicity of g guarantees

$$\begin{aligned} \frac{1}{\rho}\langle P\widetilde{u}_\rho(t), \widetilde{u}_\rho(t) - v\rangle &\leq \langle g(t,x(t),\widetilde{u}_\rho(t)), v - \widetilde{u}_\rho(t)\rangle + \varphi(v) - \varphi(\widetilde{u}_\rho(t)) \\ &\leq \langle g(t,x(t),v), v - \widetilde{u}_\rho(t)\rangle + \varphi(v) - \varphi(\widetilde{u}_\rho(t)) \end{aligned} \tag{8}$$

for all $v \in V$. Taking $v = \widetilde{u}(t)$ into the above inequality, it yields

$$\frac{1}{\rho}\langle P\widetilde{u}_\rho(t), \widetilde{u}_\rho(t) - \widetilde{u}(t)\rangle \leq \langle g(t, x(t), \widetilde{u}(t)), \widetilde{u}(t) - \widetilde{u}_\rho(t)\rangle + \varphi(\widetilde{u}(t)) - \varphi(\widetilde{u}_\rho(t)).$$

Because φ is convex and lower semicontinuous, so it is weakly lower semicontinuous as well. This deduces

$$\limsup_{\rho\to 0}\langle P\widetilde{u}_\rho(t), \widetilde{u}_\rho(t) - \widetilde{u}\rangle \leq 0.$$

Notice that P is bounded, monotone, and continuous, then it is pseudomonotone too, see [36, Theorem 3.74, page 88]. Which combines with (8) to infer

$$\begin{aligned}\langle P\widetilde{u}(t), \widetilde{u}(t) - v\rangle &\leq \liminf_{\rho\to 0}\langle P\widetilde{u}_\rho(t), \widetilde{u}_\rho(t) - v\rangle \\ &\leq \limsup_{\rho\to 0}\langle P\widetilde{u}_\rho(t), \widetilde{u}_\rho(t) - v\rangle \leq 0\end{aligned}$$

for all $v \in V$. This indicates that $P\widetilde{u}(t) = 0$, so, one has $\widetilde{u}(t) \in K$.

Next, we will present $\widetilde{u}(t) = u(t)$ for all $t \in [0, T]$. Now, we use the monotonicity of g and P, and the fact $Pv = 0$ for all $v \in K$ again to conclude

$$\begin{aligned}\langle g(t, x(t), v), \widetilde{u}_\rho(t) - v\rangle &\leq \langle g(t, x(t), \widetilde{u}_\rho(t)), \widetilde{u}_\rho(t) - v\rangle \\ &\leq -\frac{1}{\rho}\langle Pv - P\widetilde{u}_\rho(t), v - \widetilde{u}_\rho(t)\rangle + \varphi(v) - \varphi(\widetilde{u}_\rho(t)) \\ &\leq \varphi(v) - \varphi(\widetilde{u}_\rho(t))\end{aligned} \tag{9}$$

for all $v \in K$. Passing to the upper limit as $\rho \to 0$ for the above inequality, it generates

$$\langle g(t, x(t), v), v - \widetilde{u}(t)\rangle + \varphi(v) - \varphi(\widetilde{u}(t)) \geq 0$$

for all $v \in K$. We now invoke Minty approach (see e.g. [3, Theorem3.3]) to find

$$\langle g(t, x(t), \widetilde{u}(t)), v - \widetilde{u}(t)\rangle + \varphi(v) - \varphi(\widetilde{u}(t)) \geq 0$$

for all $v \in K$. However, $u(t)$ is the unique solution of the above inequality, so, we conclude that $\widetilde{u}(t) = u(t)$ for all $t \in [0, T]$. In the meantime, the latter and [41, Theorem 1.20] deduce that for each $t \in [0, T]$ the whole sequence $\{\widetilde{u}_\rho(t)\}_{\rho>0}$ converges weakly to $u(t)$. On the other side, taking $v = u(t)$ into (9), we pass to the limit as $\rho \to 0$ to conclude

$$\lim_{\rho\to 0}\langle g(t, x(t), \widetilde{u}_\rho(t)), \widetilde{u}_\rho(t) - u(t)\rangle = 0.$$

The above result combines with the convergence $\widetilde{u}_\rho(t) \to u(t)$ weakly in V as $\rho \to 0$ and the strong monotonicity of g to obtain

$$\begin{aligned}\lim_{\rho\to 0} m_g\|u(t) - \widetilde{u}_\rho(t)\|_V^2 &\leq \lim_{\rho\to 0}\langle g(t, x(t), u(t)) - g(t, x(t), \widetilde{u}_\rho(t)), u(t) - \widetilde{u}_\rho(t)\rangle \\ &= \lim_{\rho\to 0}\langle g(t, x(t), u(t)), u(t) - \widetilde{u}_\rho(t)\rangle \\ &\quad - \lim_{\rho\to 0}\langle g(t, x(t), \widetilde{u}_\rho(t)), u(t) - \widetilde{u}_\rho(t)\rangle = 0\end{aligned}$$

for all $t \in [0, T]$. So, we are able to get that for each $t \in [0, T]$,

$$\widetilde{u}_\rho(t) \to u(t) \quad \text{in } V \quad \text{as } \rho \to 0. \tag{10}$$

Let $(x_\rho, u_\rho) \in C(0, T; E) \times C(0, T; V)$ be the unique solution to problem (6). So, we have

$$\langle g(t, x_\rho(t), u_\rho(t)), v - u_\rho(t)\rangle + \frac{1}{\rho}\langle Pu_\rho(t), v - u_\rho(t)\rangle + \varphi(v) - \varphi(u_\rho(t)) \geq 0$$

for all $v \in V$ and $t \in [0, T]$. Notice that P is monotone, so, inserting $v = \widetilde{u}_\rho(t)$ into the above inequality and $v = u_\rho(t)$ to (7), we sum the resulting inequalities to yield

$$\langle g(t, x_\rho(t), u_\rho(t)) - g(t, x(t), \widetilde{u}_\rho(t)), u_\rho(t) - \widetilde{u}_\rho(t)\rangle \leq 0$$

for all $t \in [0, T]$, hence one has

$$\begin{aligned} m_g\|u_\rho(t) - \widetilde{u}_\rho(t)\|_V^2 &\leq \langle g(t, x_\rho(t), u_\rho(t)) - g(t, x_\rho(t), \widetilde{u}_\rho(t)), u_\rho(t) - \widetilde{u}_\rho(t)\rangle \\ &\leq \langle g(t, x(t), \widetilde{u}_\rho(t)) - g(t, x_\rho(t), \widetilde{u}_\rho(t)), u_\rho(t) - \widetilde{u}_\rho(t)\rangle \\ &\leq \|g(t, x(t), \widetilde{u}_\rho(t)) - g(t, x_\rho(t), \widetilde{u}_\rho(t))\|_{V^*}\|u_\rho(t) - \widetilde{u}_\rho(t)\|_V \\ &\leq L_g\|x(t) - x_\rho(t)\|_E\|u_\rho(t) - \widetilde{u}_\rho(t)\|_V \end{aligned}$$

for all $t \in [0, T]$. This means

$$\|u_\rho(t) - u(t)\|_V \leq \|\widetilde{u}_\rho(t) - u(t)\|_V + \frac{L_g}{m_g}\|x(t) - x_\rho(t)\|_E \tag{11}$$

for all $t \in [0, T]$. In the meantime, as before we have done in the proof of Theorem 3.1, we also can obtain

$$\begin{aligned} \|x(t) - x_\rho(t)\|_E \leq M_A &\int_0^t \psi(s)\|\widetilde{u}_\rho(s) - u(s)\|_V \,\mathrm{d}s \\ &+ M_A\int_0^t \psi(s)\left(1 + \frac{L_g}{m_g}\right)\|x(s) - x_\rho(s)\|_E \,\mathrm{d}s \end{aligned}$$

for all $t \in [0, T]$. It follows from Gronwall's inequality that there exists a constant $C_0 > 0$ independent of ρ such that

$$\|x(t) - x_\rho(t)\|_E \leq C_0\int_0^t \psi(s)\|\widetilde{u}_\rho(s) - u(s)\|_V \,\mathrm{d}s$$

for all $t \in [0, T]$. This inequality couples with the convergence (10) and Lebesgue-dominated convergence theorem, see cf. [36, Theorem 1.65], to guarantee

$$\begin{aligned} \lim_{\rho\to 0}\|x(t) - x_\rho(t)\|_E &\leq \lim_{\rho\to 0} C_0\int_0^t \psi(s)\|\widetilde{u}_\rho(s) - u(s)\|_V \,\mathrm{d}s \\ &= C_0\int_0^t \lim_{\rho\to 0}\psi(s)\|\widetilde{u}_\rho(s) - u(s)\|_V \,\mathrm{d}s \to 0 \end{aligned}$$

for all $t \in [0, T]$, which completes the proof of the theorem. ■

5. An obstacle parabolic-elliptic system

Indeed, it is worth mentioning that differential variational inequalities can serve as a powerful mathematical tool to study the complicate dynamic systems, which are parabolic equations governed by elliptic equations, for example, quasistatic contact problem with adhesion or wear effect, see [39–41]. However, in the section, we will apply this main results obtained in Sections 3 and 4 to investigate an obstacle parabolic-elliptic system.

Let $0 < T < \infty$. Also let Ω be a bounded and open domain in $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary $\Gamma = \partial\Omega$, where the boundary Γ is divided two disjoint parts Γ_1 and Γ_2 with $\text{meas}(\Gamma_1) > 0$. Denote n be the outward unit normal on boundary Γ. We now introduce the following function spaces

$$E = L^2(\Omega)$$
$$V = \left\{ v \in W^{1,2}(\Omega) : \quad v = 0 \text{ on } \Gamma_1 \quad (\text{in the sense of trace}) \right\}.$$

It is obvious that the function space V is a Hilbert space with the norm of

$$\|v\|_V := \left(\int_\Omega |\nabla v(z)|^2 \, dz \right)^{1/2} \quad \text{for all } v \in V.$$

We now consider the obstacle parabolic-elliptic system defined as follows:

$$\begin{aligned} & x_t(z,t) - \Delta x(z,t) = e(z,t,x(z,t),u(z,t)) && \text{in } \Omega \times (0,T], \\ & x(z,0) = \xi(z) && \text{in } \Omega, \\ & x(z,t) = 0 && \text{on } \Gamma \times [0,T], \end{aligned} \tag{12}$$

where $u \colon \Omega \times [0,T] \to \mathbb{R}$ is such that

$$\begin{aligned} & -\operatorname{div}(a(z)\nabla u(z,t)) + \beta(z)u(z,t) = h(z,t,x(z,t)) && \text{in } \Omega \times [0,T], \\ & u(z,t) = 0 && \text{on } \Gamma_1 \times [0,T], \\ & \frac{\partial u(z,t)}{\partial n_a} + k(z) = 0 && \text{on } \Gamma_2 \times [0,T], \\ & u(z,t) \geq \Phi(z) && \text{in } \Omega \times [0,T], \end{aligned} \tag{13}$$

with $\Phi \in V$ and $\partial u(z,t)/\partial n_a := (a(z)\nabla u(z,t), n)_{\mathbb{R}^N}$.

<u>$H(e)$</u>: The nonlinear function $e \colon \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(i) for all $(s,r) \in \mathbb{R}^2$ the function $(z,t) \mapsto e(z,t,s,r)$ is measurable on $\Omega \times [0,T]$;

(ii) there exist functions $\vartheta \in L^1_+(Q)$ and $\psi \in L^1_+(0,T)$ such that

$$|e(z,t,0,0)| \leq \vartheta(z,t)$$
$$|e(z,t,s_1,r_1) - e(z,t,s_2,r_2)| \leq \psi(t)\,(|s_1 - s_2| + |r_1 - r_2|)$$

for a.e. $t \in [0,T]$, $z \in \Omega$ and all $(s_1,r_1), (s_2,r_2) \in \mathbb{R}^2$, where $Q = \Omega \times [0,T]$.

<u>$H(h)$</u>: $h \colon \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is such that

(i) the function $z \mapsto h(z,0,0)$ belongs to $L^2(\Omega)$;

(ii) there exists a constant $L_h > 0$ such that for a.e. $z \in \Omega$

$$|h(z,t_1,s_1) - h(z,t_2,s_2)| \leq L_h\,(|t_1 - t_2| + |s_1 - s_2|)$$

for all $t_1, t_2 \in [0,T]$ and $s_1, s_2 \in \mathbb{R}$.

H(0): The functions k, ξ, β and a have the following regularities

$$k \in L^2(\Gamma_2), \quad \xi \in L^2(\Omega),$$
$$a \in L^\infty(\Omega) \text{ such that } a(z) \geq c_a > 0 \text{ for a.e. } z \in \Omega,$$
$$\beta \in L^\infty(\Omega) \text{ such that } \beta(z) \geq 0 \text{ for a.e. } z \in \Omega.$$

Define the operator $A\colon D(A) := H^2(\Omega) \cap H_0^1(\Omega) \subset E \to E$ by $Ax = \Delta x$ for $x \in D(A)$. Obviously, we can see that A generates a C_0-semigroup e^{At} of contractions (i.e. $\sup_{t\in[0,+\infty)} \|e^{At}\| \leq 1$) in E, see [42]. Let $x(t) = x(\cdot, t)$, $x_0 = \xi(\cdot)$ and

$$f(t, x(t), u(t)) = e(\cdot, t, x(\cdot, t), u(\cdot, t)) \quad \text{for } t \in [0, T]. \tag{14}$$

Also, we introduce the functions, $F\colon V \to V^*$, $\varphi\colon V \to \mathbb{R}$ and $g\colon [0, T] \times E \times V \to V^*$ by

$$\langle F(u), v\rangle = \int_\Omega a(z)\,(\nabla u(z), \nabla v(z))_{\mathbb{R}^N}\,\mathrm{d}z + \int_\Omega \beta(z)u(z)v(z)\,\mathrm{d}z, \tag{15}$$

$$\varphi(u) = \int_{\Gamma_2} k(z)u(z)\,\mathrm{d}\Gamma, \tag{16}$$

$$\langle g(t, x, u), v\rangle = \langle F(u), v\rangle - \int_\Omega h(z, t, x(z))v(z)\,\mathrm{d}z \tag{17}$$

for a.e. $t \in [0, T]$, all $u, v \in V$ and $x \in E$, and a constraint set K

$$K := \{v \in V \,:\, v(z) \geq \Phi(z) \quad \text{for a.e. } z \in \Omega\}.$$

Under the above conditions, it is not difficult to find that system (12) and (13) can be restated to the following abstract formulation: Find functions $x\colon [0, T] \to E$ and $u\colon [0, T] \to V$ such that

$$\begin{aligned} &x'(t) = Ax(t) + f(t, x(t), u(t)),\\ &u(t) \in K \quad \langle g(t, x(t), u(t)), v - u(t)\rangle + \varphi(v) - \varphi(u(t)) \geq 0 \quad \text{for all } v \in K,\\ &x(0) = x_0, \end{aligned}$$

for a.e. $t \in [0, T]$.

Theorem 5.1: *Assume that $H(e)$, $H(h)$, and $H(0)$ are fulfilled. Then, the system* (12) *and* (13) *has a unique mild solution* $(x, u) \in C(0, T; E) \times C(0, T; V)$.

Proof: The desired result can be obtained directly by employing Theorem 3.1. So, we only verify that all conditions stated in Theorem 3.1 are valid.

For nonlinear function f defined in (14), $H(e)$(i) ensures $H(f)$(i). In addition, $H(e)$(ii) deduces

$$\int_0^T \int_\Omega |e(t, z, 0, 0)|\,\mathrm{d}z\,\mathrm{d}t \leq \int_Q \vartheta(x, t)\,\mathrm{d}z\,\mathrm{d}t \leq \|\vartheta\|_{L^1(Q)},$$

hence hypothesis $H(f)$ (ii) holds. However, hypothesis $H(e)$ (ii) leads to the following estimates

$$\|f(t, x_1, u_1) - f(t, x_2, u_2)\|_E = \left(\int_\Omega |e(z, t, x_1(z), u_1(z)) - e(z, t, x_2(z), u_2(z))|^2\,\mathrm{d}z\right)^{1/2}$$

$$
\begin{aligned}
&\leq \psi(t)\left(\int_\Omega 2\left(|x_1(z)-x_2(z)|^2+|u_1(z)-u_2(z)|^2\right)\,\mathrm{d}z\right)^{1/2}\\
&\leq c_1\psi(t)\left(\|x_1-x_2\|_E+\|u_1-u_2\|_V\right)
\end{aligned}
$$

for some $c_1>0$, where the last inequality is obtained by using Poincaré's inequality and elementary inequality $\sqrt{c^2+d^2}\leq c+d$ for all $c,d\geq 0$. So, $H(f)$ (iii) is satisfied.

The regular condition $k\in L^2(\Gamma_2)$ guarantees that the functional $\varphi\colon V\to\mathbb{R}$ defined (16), is convex and continuous.

Indeed, by using the definition of g (see (17)), the function $u\mapsto g(t,x,u)$ is continuous. For any $u,v\in V$, we apply Poincaré's inequality again to deliver

$$
\begin{aligned}
\langle g(t,x,u)-g(t,x,v),u-v\rangle &= \langle F(u)-F(v),u-v\rangle\\
&=\int_\Omega a(z)|\nabla u(z)-\nabla v(z)|^2\,\mathrm{d}z\\
&\quad+\int_\Omega \beta(z)|u(z)-v(z)|^2\,\mathrm{d}z\geq c_a\|u-v\|_V^2,
\end{aligned}
$$

for all $u,v\in V$, for a.e. $t\in(0,T)$ and $x\in E$, for some $c_2>0$. This infers that $H(g)$ (i) is available with $m_g=c_a$. For any $v^*\in K$ fixed, from Poincaré's inequality and Hölder's inequality, we are able to find

$$
\begin{aligned}
\langle g(t,x,u),u-v^*\rangle+\varphi(u)-\varphi(v^*) &= \int_\Omega a(z)\left(\nabla u(z),\nabla u(z)-\nabla v^*(z)\right)_{\mathbb{R}^N}\,\mathrm{d}z\\
&\quad+\int_\Omega \beta(z)u(z)\left(u(z)-v^*(z)\right)\,\mathrm{d}z+\int_{\Gamma_2}k(z)u(z)\,\mathrm{d}\Gamma\\
&\quad-\int_{\Gamma_2}k(z)v^*(z)\,\mathrm{d}\Gamma\\
&\quad-\int_\Omega h(z,t,x(z))\left(u(z)-v(z)\right)\,\mathrm{d}z\\
&\geq c_a\|u\|_V^2-\|a\|_{L^\infty(\Omega)}\|u\|_V\|v^*\|_V-c_3\|\beta\|_{L^\infty(\Omega)}\|u\|_V\|v^*\|_V\\
&\quad-c_4\|k\|_{L^2(\Gamma_2)}\left(\|u\|_V+\|v^*\|_V\right)\\
&\quad-c_5\|h(\cdot,t,x(\cdot))\|_E\left(\|u\|_V+\|v^*\|_V\right)\\
&\geq \|u\|_V\big(c_a\|u\|_V-\|a\|_{L^\infty(\Omega)}\|v^*\|_V-c_5\|h(\cdot,t,x(\cdot))\|_E\\
&\quad-c_3\|\beta\|_{L^\infty(\Omega)}\|v^*\|_V\\
&\quad-c_4\|k\|_{L^2(\Gamma_2)}\big)-c_5\|h(\cdot,t,x(\cdot))\|_E\|v^*\|_V-c_4\|k\|_{L^2(\Gamma_2)}\|v^*\|_V
\end{aligned}
$$

for all $t\in[0,T]$, $x\in E$ and $u\in K$, for some $c_3,c_4,c_5>0$. This reveals that condition $H(g)$(ii) is valid. Let $x_1,x_2\in E$ and $t_1,t_2\in[0,T]$. We have

$$
\begin{aligned}
\|g(t_1,x_1,u)-g(t_2,x_2,u)\|_{V^*} &= \sup_{v\in V,\|v\|_V=1}\langle g(t_1,x_1,u)-g(t_2,x_2,u),v\rangle\\
&=\sup_{v\in V,\|v\|_V=1}\int_\Omega \left(h(z,t_2,x_2(z))-h(z,t_1,x_1(z))\right)v(z)\,\mathrm{d}z\\
&\leq \sup_{v\in V,\|v\|_V=1} L_h\left(|t_1-t_2|\int_\Omega v(z)\,\mathrm{d}z+\int_\Omega |x_1(z)-x_2(z)|v(z)\,\mathrm{d}z\right)\\
&\leq L_hc_6\left(|t_1-t_2|+\|x_1-x_2\|_E\right)
\end{aligned}
$$

for all $t \in [0, T], x_1, x_2 \in E$ and $u \in V$, for some $c_6 > 0$, where the last inequality is gotten by applying Hölder's inequality. Therefore, condition $H(g)$ (iii) is verified.

Consequently, we are now in a position to utilize Theorem 3.1 to obtain the desired result. ■

Furthermore, we introduce a penalized problem corresponding to the obstacle parabolic-elliptic system system (12) and (13):

$$\begin{aligned} & x_{\rho,t}(z,t) - \Delta x_\rho(z,t) = e(z,t,x_\rho(z,t),u_\rho(z,t)) && \text{in } \Omega \times (0,T], \\ & x_\rho(z,0) = \xi(z) && \text{in } \Omega, \\ & x_\rho(z,t) = 0 && \text{on } \Gamma \times [0,T], \end{aligned} \tag{18}$$

where $u_\rho \colon \Omega \times [0,T] \to \mathbb{R}$ is such that

$$\begin{aligned} & -\operatorname{div}(a(z)\nabla u_\rho(z,t)) + \frac{1}{\rho}(u_\rho(z,t) - \Phi(z))^+ + \beta(z)u_\rho(z,t) = h(z,t,x_\rho(z,t)) && \text{in } \Omega \times [0,T], \\ & u_\rho(z,t) = 0 && \text{on } \Gamma_1 \times [0,T], \\ & \frac{\partial u_\rho(z,t)}{\partial n_a} + k(z) = 0 && \text{on } \Gamma_2 \times [0,T]. \end{aligned} \tag{19}$$

Moreover, we introduce a penalty operator $P\colon V \to V^*$ of the constraint set K, which is defined by

$$\langle Pu, v\rangle = \int_\Omega (u(z) - \Phi(z))^+ v(z)\,\mathrm{d}z \quad \text{for all } u, v \in V. \tag{20}$$

Obviously, from Theorems 4.1 and 5.1, we have the following theorem.

Theorem 5.2: *Assume that $H(e)$, $H(h)$, and $H(0)$ are satisfied. Then, we have*

(i) *system* (18) *and* (19) *has a unique solution* $(x_\rho, u_\rho) \in C(0,T;E) \times C(0,T;V)$.
(ii) *the following convergence holds*

$$(x_\rho(t), u_\rho(t)) \to (x(t), u(t)) \quad \text{in } E \times V \text{ as } \rho \to 0,$$

for all $t \in [0,T]$, *where* $(x,u) \in C(0,T;E) \times C(0,T;V)$ *is the unique solution of the system* (12) *and* (13).

6. Conclusions

In this paper, a class of abstract differential variational inequalities involving constraints in Banach spaces is introduced and studied. We have established a well-posedness result and a convergence theorem by using the theory of semigroups, penalty method and Minty approach. Furthermore, we apply the abstract results to a comprehensive obstacle parabolic-elliptic system. Finally, it is a non-trivial interesting open question to extend the results of this paper to differential hemivariational inequalities. This extension is important in many applications, see [36] and the references therein, where nonconvex potentials are used to model the physical phenomena, and the variational inequality approach is not possible. We intend to carry out the research in this direction in our future work.

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