Computers and Mathematics with Applications 77 (2019) 2980–2988

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Numerical analysis and simulations of contact problem with wear

Michal Jureczka, Anna Ochal *

Jagiellonian University in Krakow, Faculty of Mathematics and Computer Science, Lojasiewicza 6, 30-348 Krakow, Poland

ARTICLE INFO

Article history:
Available online 14 September 2018

Dedicated to Professor Mircea Sofonea on the occasion of his 60th birthday

Keywords:
Friction contact
Wear
Error estimate
Finite element method
Numerical simulations

ABSTRACT

This paper presents a quasistatic problem of an elastic body in frictional contact with a moving foundation. The model takes into account wear of the contact surface of the body caused by the friction. We recall existence and uniqueness results obtained in Sofonea et al. (2017). The main aim of this paper is to present a fully discrete scheme for numerical approximation together with an error estimation of a solution to this problem. Finally, computational simulations are performed to illustrate the mathematical model.

1. Introduction

Frictional contact between a deformable body and a foundation is a phenomenon that occurs in various forms in different physical settings. In any particular case different factors can influence the behavior of the body. Such factors include for example constitutive law of the body, friction law that describes the contact with the foundation, the influence of the temperature or piezoelectricity effects. This is why different models have been developed in the field of contact mechanics.

From mathematical point of view we are interested in obtaining existence and uniqueness of a solution to a given problem. The classical approach to obtain these results requires the use of the theory of variational or hemivariational inequalities. It was developed and presented in [1]. The adaptation of this theory to solve quasistatic or dynamic contact problems with friction and wear can be found for example in [2–5]. Good examples of monographs summarizing progress in this field are [6,7].

In many cases in the field of contact mechanics the proof of existence and uniqueness of the solution is not constructive. The next step in dealing with these cases is usually presenting the discrete numerical scheme and estimation of finite element method error. There are some examples of scientific works that provide the error estimation for particular cases, see for example [8–11] and the reference therein.

In this article we focus on mechanical contact problem with wear modeled by Archard's law of surface wear. One of the cases in which this factor can play important role is contact between brake pads and rotors used in automotive industry. In this model the friction between body and the foundation can cause the contact surface of the body to wear over time. The proof of existence and uniqueness of a solution to such a problem has been already conducted and presented in [12], whereas our paper introduces numerical treatment of this model. We provide numerical error estimation and present computational simulations for an example of stated problem.

* Corresponding author.
E-mail addresses: michal.jureczka@ii.uj.edu.pl (M. Jureczka), ochal@ii.uj.edu.pl (A. Ochal).

https://doi.org/10.1016/j.camwa.2018.08.044
0898-1221/

The remainder of this paper is organized as follows. In Section 2 we introduce the physical setting and mechanical contact problem. In Section 3 we present the variational formulation of this problem and recall the existence and uniqueness result. A fully discrete numerical scheme and its error estimation are derived in Section 4. Finally, in Section 5, we present computational simulations showing the evolution of displacement of the body and wear of a layer of soft material that covers the body for a set of sample data.

2. Mechanical contact problem

An elastic body occupies a domain $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$ in application. We assume that its boundary Γ is divided into three disjoint measurable parts Γ_D, Γ_C, Γ_N, where the part Γ_D has a positive measure. Additionally Γ is Lipschitz continuous, and therefore the outward normal vector $\boldsymbol{\nu}$ to Γ exists a.e. on the boundary. The body is clamped on Γ_D with a displacement equal to 0. The surface transactions of density $\boldsymbol{f}_N$ act on the boundary Γ_N and the volume forces of density $\boldsymbol{f}_0$ act in Ω. These forces are time dependent.

The contact phenomenon on Γ_C is modeled using normal compliance condition with unilateral constraint and Coulomb's law of dry friction. The body is in contact with a moving obstacle, the so-called foundation. We assume that the foundation is made of a hard perfectly rigid material and the contact surface of the body Γ_C is covered by a layer of soft material. This layer is deformable and the foundation may penetrate it. Frictional contact with the foundation may cause this layer to wear over time.

We assume that the acceleration of the body is close to zero, so our problem is quasistatic. In our model framework of the small strain theory is employed. We are interested in the body displacement and foundation wear in the time interval $[0, T]$, with $T > 0$.

Let us denote by "$\cdot$" and $\|\cdot\|$ the scalar product and the Euclidean norm in $\mathbb{R}^d$ or $\mathbb{S}^d$, respectively, where $\mathbb{S}^d = \mathbb{R}^{d\times d}_{sym}$. Indices i and j run from 1 to d and the index after a comma represents the partial derivative with respect to the corresponding component of the independent variable. Summation over repeated indices is implied. We denote the divergence operator by $\mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j})$. We also use the standard notions of Lebesgue and Sobolev spaces $(L^2(\Omega))^d = L^2(\Omega;\mathbb{R}^d)$ and $H^1(\Omega)^d = H^1(\Omega;\mathbb{R}^d)$). Let us define the linearized (small) strain tensor with dependence on displacement for all $\boldsymbol{u} \in H^1(\Omega)^d$

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

Let $u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}$ and $\sigma_\nu = \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{\nu}$ be the normal components of $\boldsymbol{u}$ and $\boldsymbol{\sigma}$, respectively, and let $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$ be their tangential components, respectively. In what follows, for simplicity, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x}$.

Now let us introduce the classical formulation of described mechanical contact problem with wear.

Problem *P*. Find a displacement field $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$ and a wear function $w : \Gamma_C \times [0, T] \to \mathbb{R}_+$ such that for all $t \in [0, T]$

$$\boldsymbol{\sigma}(t) = \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \quad \text{in } \Omega \tag{1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in } \Omega \tag{2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Gamma_D \tag{3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_N(t) \quad \text{on } \Gamma_N \tag{4}$$

$$\left.\begin{aligned} u_\nu(t) \le g,\ \sigma_\nu(t) + p(u_\nu(t) - w(t)) \le 0,\\ (u_\nu(t) - g)\Big(\sigma_\nu(t) + p(u_\nu(t) - w(t))\Big) = 0 \end{aligned}\right\} \quad \text{on } \Gamma_C \tag{5}$$

$$-\boldsymbol{\sigma}_{\boldsymbol{\tau}}(t) = \mu\, p(u_\nu(t) - w(t))\boldsymbol{n}^*(t) \quad \text{on } \Gamma_C \tag{6}$$

$$w'(t) = \alpha(t)\, p(u_\nu(t) - w(t)) \quad \text{on } \Gamma_C \tag{7}$$

$$w(0) = 0 \quad \text{on } \Gamma_C \tag{8}$$

where $\boldsymbol{n}^*(t) = -\frac{\boldsymbol{v}^*(t)}{\|\boldsymbol{v}^*(t)\|}$, $\alpha(t) = \kappa\|\boldsymbol{v}^*(t)\|$ and $\boldsymbol{v}^*$ denotes velocity of the foundation.

Here, Eq. (1) represents an elastic constitutive law and $\mathcal{F}$ is an elasticity operator. Equilibrium equation (2) corresponds to the assumption that acceleration of the body is negligibly small. Eq. (3) describes the fact that body is clamped on Γ_D and (4) represents outside forces acting on Γ_N. Eq. (5) describes the damping response of the foundation with $g > 0$ being the thickness of a soft layer covering Γ_C. The friction is modeled by Eq. (6). Here, $\boldsymbol{v}^*$ is assumed to be significantly larger than tangential body velocity $\boldsymbol{u}'_{\boldsymbol{\tau}}$, κ represents the wear coefficient and μ is the friction coefficient. Eqs. (7) and (8) govern the evolution of the wear function. Detailed derivation of this model is presented in paper [12].

3. Variational formulation

In order to obtain variational formulation for the stated problem we introduce additional notation. We recall that $C([0,T];X)$ is the space of continuous functions from $[0,T]$ to X. We consider the following Hilbert spaces:

$$\mathcal{H} = L^2(\Omega;\mathbb{S}^d),\ \ V = \{\boldsymbol{v} \in H^1(\Omega)^d \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D\},$$

endowed with the inner scalar products

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_{\Omega} \sigma_{ij}\tau_{ij}\,dx,\ \ (\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},$$

respectively, and corresponding norms $\|\cdot\|_X$ with X being $\mathcal{H}$ and V. The completeness of the norm $\|\cdot\|_V$ follows from Korn's inequality, and its application is allowed because of the assumption $meas(\Gamma_D) > 0$. Moreover, we denote by $\langle\cdot,\cdot\rangle_{V^*\times V}$ the duality pairing between a dual space V^* and V.

When the trace of $\boldsymbol{v} \in V$ on the boundary Γ is considered, we still write $\boldsymbol{v}$ for simplicity. By the Sobolev trace theorem, there exists a constant $c_0 > 0$ depending only on Ω, Γ_D and Γ_C such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_C)^d} \le c_0\|\boldsymbol{v}\|_V \quad \text{for all } \boldsymbol{v} \in V. \tag{9}$$

We introduce the set of admissible displacements, namely

$$U = \{\boldsymbol{v} \in V \mid v_\nu \le g \text{ on } \Gamma_C\}.$$

Now we present the hypotheses on the data of Problem P.
$\underline{H(\mathcal{F})}$: The elasticity operator $\mathcal{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

(a) $\mathcal{F}(\cdot,\boldsymbol{\varepsilon})$ is measurable on Ω for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$, $\mathcal{F}(\cdot,\boldsymbol{0}) \in \mathcal{H}$,
(b) $\exists L_{\mathcal{F}} > 0$ s.t. $\|\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{F}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|$ for all $\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega$,
(c) $\exists m_{\mathcal{F}} > 0$ s.t. $(\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{F}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2$ for all $\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega$.

$\underline{H(p)}$: The normal compliance function $p : \Gamma_C \times \mathbb{R} \to \mathbb{R}_+$ satisfies

(a) $p(\cdot,r)$ is measurable on Γ_C for all $r \in \mathbb{R}$,
(b) $\exists L_p > 0$ s.t. $|p(\boldsymbol{x},r_1) - p(\boldsymbol{x},r_2)| \le L_p|r_1 - r_2|$ for all $r_1,r_2 \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_C$,
(c) $(p(\boldsymbol{x},r_1) - p(\boldsymbol{x},r_2))(r_1 - r_2) \ge 0$ for all $r_1,r_2 \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_C$,
(d) $p(\boldsymbol{x},r) = 0$ for all $r \le 0$, a.e. $\boldsymbol{x} \in \Gamma_C$.

Densities of body and traction forces satisfy
$\underline{H(f)}$: $\boldsymbol{f}_0 \in C([0,T];L^2(\Omega)^d)$, $\boldsymbol{f}_N \in C([0,T];L^2(\Gamma_N)^d)$.
$\underline{(H_0)}$: The friction and wear coefficients, and the foundation velocity satisfy

(a) $\mu \in L^\infty(\Gamma_C)$, $\quad \mu(\boldsymbol{x}) \ge 0$, a.e. $\boldsymbol{x} \in \Gamma_C$,
(b) $\kappa \in L^\infty(\Gamma_C)$, $\quad \kappa(\boldsymbol{x}) \ge 0$, a.e. $\boldsymbol{x} \in \Gamma_C$,
(c) $\boldsymbol{v}^* \in C([0,T];\mathbb{R}^d)$, $\quad \|\boldsymbol{v}^*(t)\| \ge \nu > 0$ for all $t \in [0,T]$.

We notice that hypotheses (H_0) indicate the following regularities:

$$\boldsymbol{n}^* \in C([0,T];\mathbb{R}^d), \quad \alpha \in C([0,T];L^\infty(\Gamma_C)). \tag{10}$$

Finally, we present the following smallness assumption:
$\underline{(H_s)}$: $\quad c_0^2 L_p\|\mu\|_{L^\infty(\Gamma_C)} < m_{\mathcal{F}}$, where a constant c_0 is introduced in (9).

Now we define some operators and functions in order to present variational formulation of Problem P. Let $F : V \to V^*$, $\boldsymbol{f} : [0,T] \to V^*$ and $\varphi : [0,T] \times L^2(\Gamma_C) \times V \times V \to \mathbb{R}$ be defined for all $\boldsymbol{u},\boldsymbol{v} \in V$, $w \in L^2(\Gamma_C)$, $t \in [0,T]$ as follows:

$$\langle F\boldsymbol{u},\boldsymbol{v}\rangle_{V^*\times V} = (\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},$$

$$\langle \boldsymbol{f}(t),\boldsymbol{v}\rangle_{V^*\times V} = \int_{\Omega} \boldsymbol{f_0}(t)\cdot\boldsymbol{v}\,dx + \int_{\Gamma_N} \boldsymbol{f_N}(t)\cdot\boldsymbol{v}\,da,$$

$$\varphi(t,w,\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_C} p(u_\nu - w)\,v_\nu\,da + \int_{\Gamma_C} \mu\,p(u_\nu - w)\,\boldsymbol{n}^*(t)\cdot\boldsymbol{v}_\tau\,da.$$

Using the standard procedure and Green's formula we obtain the variational formulation of Problem P in the following form.

Problem P_V. Find $\boldsymbol{u} : [0, T] \to U$ and $w : [0, T] \to L^2(\Gamma_C)$ such that for all $t \in [0, T]$

$$\langle F\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{V^*\times V} + \varphi(t, w(t), \boldsymbol{u}(t), \boldsymbol{v}) - \varphi(t, w(t), \boldsymbol{u}(t), \boldsymbol{u}(t)) \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{V^*\times V} \quad \text{for all } \boldsymbol{v} \in U, \tag{11}$$

$$w(t) = \int_0^t \alpha(s)\, p(u_\nu(s) - w(s))\, ds. \tag{12}$$

Let us now recall the following existence and uniqueness result for Problem P_V (for details see Theorem 4.1 in [12]).

Theorem 1. *Assume $H(\mathcal{F})$, $H(p)$, $H(f)$, (H_0) and (H_s). Then Problem P_V has the unique solution with the regularity*

$$\boldsymbol{\sigma} \in C([0, T]; \mathcal{H}), \quad \boldsymbol{u} \in C([0, T]; V), \quad w \in C^1([0, T]; L^2(\Gamma_C)),$$

and, in addition,

$$w(t) \geq 0 \quad \text{for all } t \in [0, T], \text{ a.e. on } \Gamma_C.$$

4. Numerical scheme

Here and below we denote $W = L^2(\Gamma_C)$. Let $V^h \subset V$ and $W^h \subset W$ be two families of finite dimensional subspaces with a discretization parameter $h > 0$. Let then $U^h = U \cap V^h$. We introduce the time step $k = T/N$ for $N \in \mathbb{N}, N > 0$ and $t_n = nk$, $n = 0, 1, \ldots, N$. We also use notation $g_j = g(t_j)$ for any $g \in C([0, T]; X)$, where X is any introduced function space.

We make the following additional assumptions on the solution $\boldsymbol{u}$ to Problem P and the velocity of the foundation $\boldsymbol{v}^*$.

$\underline{(H_1)}$: $\boldsymbol{u} \in H^1(0, T; V)$, $\boldsymbol{v}^* \in W^{1,\infty}(0, T; \mathbb{R}^d)$.

Assumptions (H_1) and (H_0)(b) indicate that

$$\alpha \in W^{1,\infty}(0, T; L^\infty(\Gamma_C)). \tag{13}$$

We now present the following fully discrete scheme.

Problem P^{hk}. Find $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_j^{hk}\}_{j=0}^N \subset U^h$ and $w^{hk} = \{w_j^{hk}\}_{j=0}^N \subset W^h$ such that

$$\langle F\boldsymbol{u}_j^{hk}, \boldsymbol{v}^h - \boldsymbol{u}_j^{hk}\rangle_{V^*\times V} + \varphi_j(w_j^{hk}, \boldsymbol{u}_j^{hk}, \boldsymbol{v}^h) - \varphi_j(w_j^{hk}, \boldsymbol{u}_j^{hk}, \boldsymbol{u}_j^{hk}) \geq \langle \boldsymbol{f}_j, \boldsymbol{v}^h - \boldsymbol{u}_j^{hk}\rangle_{V^*\times V} \quad \text{for all } \boldsymbol{v}^h \in U^h, j \in \{0, \ldots, N\}, \tag{14}$$

$$w_j^{hk} = k \sum_{m=0}^{j-1} \alpha_m\, p((u_m^{hk})_\nu - w_m^{hk}) \qquad \text{for } j \in \{1, \ldots, N\}, \tag{15}$$

$$w_0^{hk} = 0.$$

We remark that existence of the unique solution to Problem P^{hk} follows from application of discrete version of Theorem 1. Now we introduce some preliminary material, namely we recall a special case of the Jensen inequality.

Lemma 2. *Let $I \subset \mathbb{R}$ be a set of positive measure and let $f : I \to \mathbb{R}$ be an integrable function. Then*

$$\left(\frac{1}{|I|}\int_I f(x)\, dx\right)^2 \leq \frac{1}{|I|}\int_I (f(x))^2\, dx.$$

In a particular case, we have

$$\left(\int_{t_m}^{t_{m+1}} f(x)\, dx\right)^2 \leq k \int_{t_m}^{t_{m+1}} (f(x))^2\, dx. \tag{16}$$

The next lemma presents the Gronwall inequality in the following form (see [9], Lemma 7.26).

Lemma 3. *Let T be given. For $N > 0$ we define $k = T/N$. Let $\{g_n\}_{n=1}^N$, $\{e_n\}_{n=1}^N$ be two nonnegative sequences satisfying for $c > 0$ and for all $n \in \{1, \ldots, N\}$*

$$e_n \leq c g_n + c \sum_{j=1}^n k e_j.$$

Then

$$\max_{1\leq n\leq N} e_n \leq \hat{c} \max_{1\leq n\leq N} g_n \quad \text{with } \hat{c} > 0.$$

Now we present the following theorem concerning error estimation of introduced numerical scheme.

Theorem 4. *Under the assumptions of Theorem 1 and additional hypothesis (H_1), for the unique solution $(\boldsymbol{u}, w) \in C([0,T];V) \times C^1([0,T];W)$ of Problem P_V and the unique solution $(\boldsymbol{u}^{hk}, w^{hk}) \subset U^h \times W^h$ of Problem P^{hk} there exists a constant $\tilde{c} > 0$ such that*

$$\begin{aligned} &k\sum_{j=1}^{N}(\|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_V^2 + \|w_j - w_j^{hk}\|_W^2) \\ &\le \tilde{c} \inf_{\boldsymbol{v}^h \in U^h}\Big\{k^2 + k\,\|\boldsymbol{u}_0 - \boldsymbol{u}_0^{hk}\|_V^2 + k\sum_{j=1}^{N}\|\boldsymbol{u}_j - \boldsymbol{v}_j^h\|_V^2 + k\sum_{j=1}^{N}|R_j(w_j, \boldsymbol{u}_j, \boldsymbol{v}_j^h)|\Big\}, \end{aligned} \tag{17}$$

where

$$R_j(w_j, \boldsymbol{u}_j, \boldsymbol{v}_j^h) = \langle F\boldsymbol{u}_j, \boldsymbol{v}_j^h - \boldsymbol{u}_j\rangle_{V^*\times V} + \varphi_j(w_j, \boldsymbol{u}_j, \boldsymbol{v}_j^h) - \varphi_j(w_j, \boldsymbol{u}_j, \boldsymbol{u}_j) - \langle \boldsymbol{f}_j, \boldsymbol{v}_j^h - \boldsymbol{u}_j\rangle_{V^*\times V}. \tag{18}$$

Proof. Throughout this proof we denote as $c > 0$ a generic constant (value of c may differ in different equations) and numerical solution errors for any $j \in \{1, \dots, N\}$ by

$$\boldsymbol{e}_j = \boldsymbol{u}_j - \boldsymbol{u}_j^{hk}, \qquad \varepsilon_j = w_j - w_j^{hk}.$$

Let us fix $j \in \{1, \dots, N\}$. We start with the estimate of ε_j. From (12) and (15) after using an elementary inequality $\left(\sum_{m=0}^{l} a_m\right)^2 \le c\, l \sum_{m=0}^{l} a_m^2$ for $a_m \in \mathbb{R}_+$ we obtain

$$\begin{aligned} \|\varepsilon_j\|_W^2 &= \int_{\Gamma_C}\Big|\sum_{m=0}^{j-1}\int_{t_m}^{t_{m+1}} \alpha(s)\,p(u_\nu(s) - w(s)) - \alpha_m\, p((u_m^{hk})_\nu - w_m^{hk})\,ds\Big|^2\,da \\ &\le cN\sum_{m=0}^{j-1}\int_{\Gamma_C}\Big|\int_{t_m}^{t_{m+1}} \alpha(s)\,p(u_\nu(s) - w(s)) - \alpha_m\, p((u_m^{hk})_\nu - w_m^{hk})\,ds\Big|^2\,da. \end{aligned} \tag{19}$$

Since p is Lipschitz continuous and $p(0) = 0$ we have

$$\begin{aligned} &\Big|\int_{t_m}^{t_{m+1}} \alpha(s)\,p(u_\nu(s) - w(s)) - \alpha_m\, p((u_m^{hk})_\nu - w_m^{hk})\,ds\Big| \\ &\le\Big|\int_{t_m}^{t_{m+1}}\Big(\alpha(s)\,(p(u_\nu(s) - w(s)) - p(0)) - \alpha_m\,(p(u_\nu(s) - w(s)) - p(0)) \\ &\quad + \alpha_m\, p(u_\nu(s) - w(s)) - \alpha_m\, p((u_m^{hk})_\nu - w_m^{hk})\Big)\,ds\Big| \\ &\le c\int_{t_m}^{t_{m+1}}|\alpha(s) - \alpha_m|\,(|u_\nu(s)| + |w(s)|) + \alpha_m\,(|u_\nu(s) - (u_m^{hk})_\nu| + |w(s) - w_m^{hk}|)\,ds. \end{aligned}$$

Returning to (19) and using Lemma 2 with $I = [t_m, t_{m+1}]$ (cf. (16)), since $kN = T$, we obtain

$$\begin{aligned} \|\varepsilon_j\|_W^2 \le cT\sum_{m=0}^{j-1}\int_{t_m}^{t_{m+1}}\int_{\Gamma_C}\Big(&|\alpha(s) - \alpha_m|^2\,(|u_\nu(s)|^2 + |w(s)|^2) \\ &+\alpha_m^2\,(|u_\nu(s) - (u_m)_\nu|^2 + |(u_m)_\nu - (u_m^{hk})_\nu|^2 \\ &+|w(s) - w_m|^2 + |w_m - w_m^{hk}|^2)\Big)\,da\,ds. \end{aligned} \tag{20}$$

Since $\alpha \in W^{1,\infty}(0,T;L^\infty(\Gamma_C))$ (cf. (13)), we obtain for all $m \in \{1, \dots, N\}$, $s \in (t_m, t_{m+1}]$ and almost every $\boldsymbol{x} \in \Gamma_C$

$$\begin{aligned} |\alpha(s,\boldsymbol{x}) - \alpha_m(\boldsymbol{x})| &= \Big|\int_{t_m}^{s}\alpha'(\tau,\boldsymbol{x})\,d\tau\Big| \le \int_{t_m}^{t_{m+1}}|\alpha'(\tau,\boldsymbol{x})|\,d\tau \\ &\le k\,\|\alpha'\|_{L^\infty(0,T;L^\infty(\Gamma_C))} \le k\,\|\alpha\|_{W^{1,\infty}(0,T;L^\infty(\Gamma_C))}. \end{aligned} \tag{21}$$

Similarly, using Lemma 2, from assumption (H_1) and the fact that $w \in C^1([0,T];W)$ (cf. Theorem 1) we have for all $m \in \{1,\dots,N\}$ and $s \in (t_m, t_{m+1}]$

$$\int_{\Gamma_C} |u_\nu(s) - (u_m)_\nu|^2\, da \le k \int_{\Gamma_C} \int_{t_m}^{t_{m+1}} |u'_\nu(\tau)|^2\, d\tau\, da \le k \int_{t_m}^{t_{m+1}} \|u'_\nu(\tau)\|_W^2\, d\tau, \tag{22}$$

$$\int_{\Gamma_C} |w(s) - w_m|^2\, da \le k \int_{t_m}^{t_{m+1}} \|w'(\tau)\|_W^2\, d\tau. \tag{23}$$

We return to (20), using (21)–(23) and the fact that $\alpha_m(\boldsymbol{x}) \le c$ for all $m \in \{1,\dots,N\}$ and almost every $\boldsymbol{x} \in \Gamma_C$ (cf. (10)), we obtain

$$\begin{aligned}\|\varepsilon_j\|_W^2 \le{}& c\, k^2 \int_0^T \|u_\nu(s)\|_W^2 + \|w(s)\|_W^2\, ds \\ &+ c\, k \sum_{m=0}^{j-1} \int_{t_m}^{t_{m+1}} \int_{t_m}^{t_{m+1}} \|u'_\nu(\tau)\|_W^2 + \|w'(\tau)\|_W^2\, d\tau\, ds \\ &+ c \sum_{m=0}^{j-1} \int_{t_m}^{t_{m+1}} \int_{\Gamma_C} |(u_m)_\nu - (u_m^{hk})_\nu|^2 + |w_m - w_m^{hk}|^2\, da\, ds.\end{aligned} \tag{24}$$

Finally, using (9) and the fact that $w \in C^1([0,T];W)$, we get

$$\|\varepsilon_j\|_W^2 \le c\, k^2 + c\, k \sum_{m=0}^{j-1} \big(\|\boldsymbol{e}_m\|_V^2 + \|\varepsilon_m\|_W^2\big). \tag{25}$$

Let us now fix $j \in \{1,\dots,N\}$ and estimate $\boldsymbol{e}_j$. We set $t = t_j$ and $\boldsymbol{v} = \boldsymbol{u}_j^{hk}$ in (11), and $\boldsymbol{v}^h = \boldsymbol{v}_j^h$ in (14), we obtain

$$\begin{aligned}&\langle F\boldsymbol{u}_j - F\boldsymbol{u}_j^{hk}, \boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\rangle_{V^*\times V} \le \langle F\boldsymbol{u}_j^{hk} - F\boldsymbol{u}_j, \boldsymbol{v}_j^h - \boldsymbol{u}_j\rangle_{V^*\times V} \\ &+ \varphi_j(w_j^{hk}, \boldsymbol{u}_j^{hk}, \boldsymbol{v}_j^h) - \varphi_j(w_j, \boldsymbol{u}_j, \boldsymbol{v}_j^h) - \varphi_j(w_j^{hk}, \boldsymbol{u}_j^{hk}, \boldsymbol{u}_j^{hk}) + \varphi_j(w_j, \boldsymbol{u}_j, \boldsymbol{u}_j^{hk}) \\ &+ R_j(w_j, \boldsymbol{u}_j, \boldsymbol{v}_j^h),\end{aligned} \tag{26}$$

where R_j is given by (18). Since p is monotone and Lipschitz continuous, using (H_0) and (9), we estimate for all $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ and $w_1, w_2 \in L^2(\Gamma_C)$

$$\begin{aligned}&\varphi_j(w_1, \boldsymbol{u}_1, \boldsymbol{v}_2) - \varphi_j(w_2, \boldsymbol{u}_2, \boldsymbol{v}_2) - \varphi_j(w_1, \boldsymbol{u}_1, \boldsymbol{v}_1) + \varphi_j(w_2, \boldsymbol{u}_2, \boldsymbol{v}_1) \\ \le{}& \int_{\Gamma_C} [p(u_{1\nu} - w_1) - p(u_{2\nu} - w_2)]\,[(v_{2\nu} - w_2) - (v_{1\nu} - w_1)]\, da \\ &+ \int_{\Gamma_C} [p(u_{1\nu} - w_1) - p(u_{2\nu} - w_2)]\,(w_2 - w_1)\, da \\ &+ \int_{\Gamma_C} \mu\, [p(u_{1\nu} - w_1) - p(u_{2\nu} - w_2)]\, \boldsymbol{n}_j^* \cdot (\boldsymbol{v}_{2\tau} - \boldsymbol{v}_{1\tau})\, da \\ \le{}& L_p \int_{\Gamma_C} (|u_{1\nu} - u_{2\nu}| + |w_1 - w_2|)\, |w_1 - w_2|\, da \\ &+ L_p\, \|\mu\|_{L^\infty(\Gamma_C)} \int_{\Gamma_C} (|u_{1\nu} - u_{2\nu}| + |w_1 - w_2|)\, \|\boldsymbol{v}_{1\tau} - \boldsymbol{v}_{2\tau}\|\, da \\ \le{}& L_p(c_0\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V + \|w_1 - w_2\|_W)\|w_1 - w_2\|_W \\ &+ c_0\, L_p\, \|\mu\|_{L^\infty(\Gamma_C)} (c_0\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V + \|w_1 - w_2\|_W)\, \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V.\end{aligned} \tag{27}$$

Hence, using this inequality into (26) and the triangle inequality

$$\|\boldsymbol{u}_j^{hk} - \boldsymbol{v}_j^h\|_V \le \|\boldsymbol{u}_j - \boldsymbol{v}_j^h\|_V + \|\boldsymbol{e}_j\|_V$$

we obtain

$$\begin{aligned}(m_{\mathcal{F}} - c_0^2\, L_p\, \|\mu\|_{L^\infty(\Gamma_C)})\|\boldsymbol{e}_j\|_V^2 \le{}& c\|\boldsymbol{e}_j\|_V \|\boldsymbol{u}_j - \boldsymbol{v}_j^h\|_V + |R_j(w_j, \boldsymbol{u}_j, \boldsymbol{v}_j^h)| \\ &+ c\, \|\boldsymbol{e}_j\|_V\, \|\varepsilon_j\|_W + c\, \|\varepsilon_j\|_W\, \|\boldsymbol{u}_j - \boldsymbol{v}_j^h\|_V + c\|\varepsilon_j\|_W^2.\end{aligned}$$

Defining $A := m_{\mathcal{F}} - c_0^2\, L_p\, \|\mu\|_{L^\infty(\Gamma_C)} > 0$ (by (H_s)), we get

$$A\|\boldsymbol{e}_j\|_V^2 \le \sqrt{\frac{A}{2}}\|\boldsymbol{e}_j\|_V \frac{\sqrt{2}}{\sqrt{A}} c\|\boldsymbol{u}_j - \boldsymbol{v}_j^h\|_V + |R_j(w_j, \boldsymbol{u}_j, \boldsymbol{v}_j^h)| + \sqrt{\frac{A}{2}}\|\boldsymbol{e}_j\|_V \frac{\sqrt{2}}{\sqrt{A}} c\|\varepsilon_j\|_W + c\|\boldsymbol{u}_j - \boldsymbol{v}_j^h\|_V^2 + c\|\varepsilon_j\|_W^2.$$

Using inequality $ab \le \frac{a^2+b^2}{2}$, we obtain

$$\frac{A}{2}\|\boldsymbol{e}_j\|_V^2 \le c\|\boldsymbol{u}_j - \boldsymbol{v}_j^h\|_V^2 + |R_j(w_j, \boldsymbol{u}_j, \boldsymbol{v}_j^h)| + c\,\|\varepsilon_j\|_W^2.$$

Using (25) to estimate the last term on the right hand side of this and adding obtained inequality to (25), we get

$$\|\boldsymbol{e}_j\|_V^2 + \|\varepsilon_j\|_W^2 \le c\,\|\boldsymbol{u}_j - \boldsymbol{v}_j^h\|_V^2 + |R_j(w_j, \boldsymbol{u}_j, \boldsymbol{v}_j^h)| + ck^2 + ck\,\|\boldsymbol{e}_0\|_V^2 + ck\sum_{m=1}^{j-1}(\|\boldsymbol{e}_m\|_V^2 + \|\varepsilon_m\|_W^2).$$

Summing this inequality from $j = 1$ to n for $n \in \{1, \dots, N\}$ and multiplying it by k, we obtain

$$\begin{aligned} k\sum_{j=1}^{n}(\|\boldsymbol{e}_j\|_V^2 + \|\varepsilon_j\|_W^2) \le & ck\sum_{j=1}^{n}\|\boldsymbol{u}_j - \boldsymbol{v}_j^h\|_V^2 + k\sum_{j=1}^{n}|R_j(w_j, \boldsymbol{u}_j, \boldsymbol{v}_j^h)| \\ & + ck^2 + ck\,\|\boldsymbol{e}_0\|_V^2 + c\sum_{j=1}^{n}k^2\sum_{m=1}^{j}(\|\boldsymbol{e}_m\|_V^2 + \|\varepsilon_m\|_W^2). \end{aligned}$$

Applying Lemma 3 with $e_n = k\sum_{j=1}^{n}(\|\boldsymbol{e}_j\|_V^2 + \|\varepsilon_j\|_W^2)$ proves the theorem.

We finish this section by providing a sample error estimate under additional assumption on the solution regularity. We consider a polygonal domain Ω and a space of continuous piecewise affine functions V^h.

Theorem 5. *Under hypotheses of Theorem 4 and assuming the solution regularity $\boldsymbol{u} \in C([0, T]; H^2(\Omega)^d)$, we have the following error estimate:*

$$k\sum_{j=1}^{N}(\|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_V^2 + \|w_j - w_j^{hk}\|_W^2) \le \tilde{c}\,(k^2 + kh^2 + h), \ \text{with} \ \tilde{c} > 0.$$

Proof. We fix any $t = t_j, j \in \{0, 1, \dots, N\}$ and denote by $\Pi^h\boldsymbol{u}_j \in U^h$ the finite element interpolant of $\boldsymbol{u}_j$. By the standard finite element interpolation error bounds (see for example [13]) we have

$$\|\boldsymbol{\eta} - \Pi^h\boldsymbol{\eta}\|_V \le c\,h\,\|\boldsymbol{\eta}\|_{H^2(\Omega)^d} \quad \text{for all } \boldsymbol{\eta} \in H^2(\Omega)^d.$$

By similar calculations to (27) we obtain

$$|R_j(w_j, \boldsymbol{u}_j, \boldsymbol{v}_j^h)| \le c\,\|\boldsymbol{u}_j - \boldsymbol{v}_j^h\|_V.$$

Using inequality (17), we get

$$\begin{aligned} & k\sum_{j=1}^{N}(\|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_V^2 + \|w_j - w_j^{hk}\|_W^2) \\ & \le c\Big(k^2 + k\,\|\boldsymbol{u}_0 - \Pi^h\boldsymbol{u}_0\|_V^2 + k\sum_{j=1}^{N}\|\boldsymbol{u}_j - \Pi^h\boldsymbol{u}_j\|_V^2 + k\sum_{j=1}^{N}|R_j(w_j, \boldsymbol{u}_j, \Pi^h\boldsymbol{u}_j)|\Big) \\ & \le ck^2 + ckh^2 + ckNh^2 + ckNh. \end{aligned}$$

Since $kN = T$, we obtain the desired conclusion.

5. Simulations

In this section we present results of our simulations. We take $d = 2$ and we consider a rectangular set $\Omega = [0, 3] \times [0, 1]$ with following parts of the boundary:

$$\Gamma_D = \{0\} \times [0, 1], \quad \Gamma_N = ([0, 3] \times \{1\}) \cup (\{3\} \times [0, 1]), \quad \Gamma_C = [0, 3] \times \{0\}.$$

The elasticity operator $\mathcal{F}$ is defined by

$$\mathcal{F}(\boldsymbol{\tau}) = 2\eta\boldsymbol{\tau} + \lambda\mathrm{tr}(\boldsymbol{\tau})I, \qquad \boldsymbol{\tau} \in \mathbb{S}^2.$$

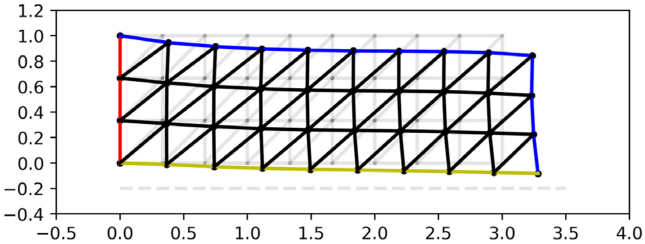

Fig. 1. Body's shape after 25 iterations.

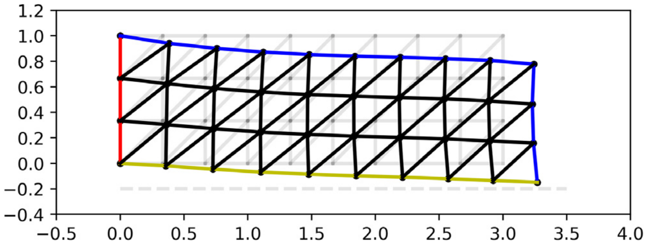

Fig. 2. Body's shape after 50 iterations.

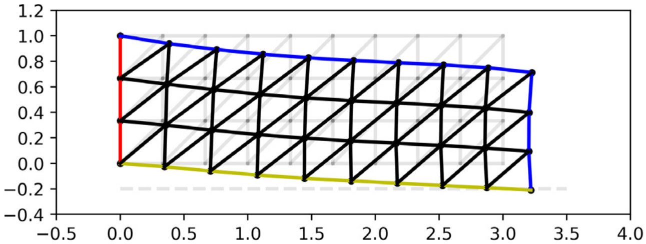

Fig. 3. Body's shape after 75 iterations.

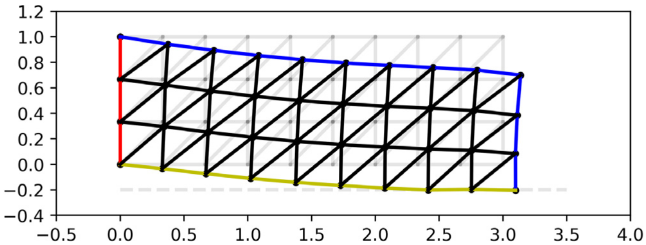

Fig. 4. Body's shape after 100 iterations.

Here I denotes the identity matrix, tr denotes the trace of the matrix, λ and η are the Lame coefficients, $\lambda, \eta > 0$. In our simulations we take the following data:

$$\lambda = \eta = 1, \quad T = 1$$

$$\boldsymbol{u}_0(\boldsymbol{x}) = (0, 0), \quad \boldsymbol{x} \in \Omega$$

$$p(r) = \begin{cases} 10\,r, & r \in [0, \infty) \\ 0, & r \in (-\infty, 0) \end{cases}$$

$$\mu(\boldsymbol{x}) = 1, \quad \boldsymbol{x} \in \Omega$$

$$\kappa(\boldsymbol{x}) = 1.5, \quad \boldsymbol{x} \in \Omega$$

$$\boldsymbol{v}^*(\boldsymbol{x}, t) = (1, 0), \quad \boldsymbol{x} \in \Gamma_C,\ t \in [0, T]$$

$$\boldsymbol{f}_N(\boldsymbol{x}, t) = (0, -0.1), \quad \boldsymbol{x} \in \Omega,\ t \in [0, T]$$
$$\boldsymbol{f}_0(\boldsymbol{x}, t) = (0, 0), \quad \boldsymbol{x} \in \Omega,\ t \in [0, T]$$
$$g = 0.2.$$

We use space V^h of continuous piecewise affine functions as a family of approximating subspaces and we set time step equal to 0.1. In Figs. 1–4 we present outputs of finite element method algorithm after 25, 50, 75 and 100 iterations, respectively. We can observe that the body is squeezed and the foundation penetrates the soft layer covering Γ_C. This phenomenon is caused by the downward force $\boldsymbol{f}_N$. At the final time (Fig. 4) part of the upper layer of the foundation is worn out. The penetration is not greater than 0.2 because of the prescribed thickness g of a wearable layer.

Acknowledgments

The research was supported by the National Science Center of Poland under the Maestro Project No. 2012/06/A/ST1/00262, the International Project co-financed by the Ministry of Science and Higher Education of Republic of Poland under Grant No. 3792/GGPJ/H2020/2017/0, and the H2020-MSCA-RISE-2018 Research and Innovation Staff Exchange Scheme Fellowship within the Project No. 823731 CONMECH.

References

[1] P.D. Panagiotopoulos, Hemivariational Inequalities, Applications in Mechanics and Engineering, Springer-Verlag, 1993.
[2] K. Bartosz, Hemivariational inequalities approach to the dynamic viscoelastic sliding contact problem with wear, Nonlinear Anal. TMA 65 (2006) 546–566.
[3] L. Gasinski, A. Ochal, M. Shillor, Quasistatic thermoviscoelastic problem with normal compliance, multivalued friction and wear diffusion, Nonlinear Anal. RWA 27 (2016) 183–202.
[4] K.L. Kutler, M. Shillor, Dynamic contact normal compliance wear and discontinuous friction coefficient, SIAM J. Math. Anal. 34 (1) (2002) 1–27.
[5] J.M. Viaño, A. Rodriguez-Arós, M. Sofonea, Asymptotic derivation of quasistatic frictional contact models with wear for elastic rods, J. Math. Anal. Appl. 401 (2) (2013) 641–653.
[6] S. Migorski, A. Ochal, M. Sofonea, Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, in: Advances in Mechanics and Mathematics, vol. 26, Springer, 2013.
[7] M. Shillor, M. Sofonea, J.J. Telega, Models and Analysis of Quasistatic Contact, Springer-Verlag, 2004.
[8] J. Chen, W. Han, M. Sofonea, Numerical analysis of a quasistatic problem of sliding frictional contact with wear, Methods Appl. Anal. 7 (2000) 687–704.
[9] W. Han, M. Sofonea, Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity, American Mathematical Society and International Press, 2002.
[10] A. Ochal, M. Jureczka, Numerical treatment of contact problems with thermal effect, Discrete Contin. Dyn. Syst. Ser. B 23 (1) (2018) 387–400.
[11] J. Rojek, J.J. Telega, S. Stupkiewicz, Contact problems with friction, adhesion and wear in orthopaedic biomechanics. Part II-numerical implementation and application to implanted knee joints, J. Theoret. Appl. Mech. 39 (2001) 679–706.
[12] M. Sofonea, F. Pătrulescu, Y. Souleiman, Analysis of a contact problem with wear and unilateral constraint, Appl. Anal. 95 (11) (2017) 2590–2607.
[13] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North Holland, Amsterdam, 1978.