Deep on Goldbach's conjecture

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Abstract

Goldbach's conjecture is one of the most difficult unsolved problems in mathematics. This states that every even natural number greater than 2 is the sum of two prime numbers. In 1973, Chen Jingrun proved that every sufficiently large even number can be written as the sum of either two primes, or a prime and a semiprime (the product of two primes). In 2015, Tomohiro Yamada, using the Chen's theorem, showed that every even number $> \exp \exp 36$ can be represented as the sum of a prime and a product of at most two primes. In 2002, Ying Chun Cai proved that every sufficiently large even integer N is equal to $p + P_2$, where P_2 is an almost prime with at most two prime factors and $p \leq N^{0.95}$ is a prime number. This conjecture has been verified for every even number $N \leq 4 \cdot 10^{18}$. In this note, we prove that for every even number $N \geq 4 \cdot 10^{18}$, if there is a prime p and a natural number m such that $n < p < N - 1$, $p + m = N$, $4 \cdot n \geq \left(p - 1 - 2 \cdot n^{0.889}\right) \cdot \sigma(m)$ and p is coprime with m, then m is necessarily a prime number when $N = 2 \cdot n$ and $\sigma(m)$ is the sum-of-divisors function of m.

Keywords: Goldbach's conjecture, Prime numbers, Sum-of-divisors function, Euler's totient function

MSC Classification: 11A41 , 11A25

1 Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d|n} d,$$

where $d \mid n$ means the integer d divides n. Define $s(n)$ as $\frac{\sigma(n)}{n}$. In number theory, the *p-adic* order of an integer n is the exponent of the highest power of the prime number p that divides n. It is denoted $\nu_p(n)$. Equivalently, $\nu_p(n)$ is the exponent to which p appears in the prime factorization of n. We can state the sum-of-divisors function of n as

$$\sigma(n) = \prod_{p|n} \frac{p^{\nu_p(n)+1} - 1}{p - 1}$$

with the product extending over all prime numbers p which divide n. In addition, the well-known Euler's totient function $\varphi(n)$ can be formulated as

$$\varphi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Chen's theorem states that every sufficiently large even number can be written as the sum of either two primes, or a prime and a semiprime (the product of two primes) [1]. Tomohiro Yamada using an explicit version of Chen's theorem showed that every even number greater than $e^{e^{36}} \approx 1.7 \cdot 10^{1872344071119343}$ is the sum of a prime and a product of at most two primes [2]. A natural number is called k-almost prime if it has k prime factors [3]. A natural number is prime if and only if it is 1-almost prime, and semiprime if and only if it is 2-almost prime. Let N be a sufficiently large even integer. Ying Chun Cai proved that the equation

$$N = p + P_2, \quad p \leq N^{0.95},$$

is solvable, where p denotes a prime and P_2 denotes an almost prime with at most two prime factors [3]. This conjecture has been verified for every even number $N \leq 4 \cdot 10^{18}$ [4]. In mathematics, two integers a and b are coprime, if the only positive integer that is a divisor of both of them is 1. Putting all together yields the proof of the main theorem.

Theorem 1 *For every even number $N \geq 4 \cdot 10^{18}$, if there is a prime p and a natural number m such that $n < p < N - 1$, $p + m = N$, $4 \cdot n \geq \left(p - 1 - 2 \cdot n^{0.889}\right) \cdot \sigma(m)$ and p is coprime with m, then m is necessarily a prime number when $N = 2 \cdot n$.*

2 Proof of Theorem 1

Proof Suppose that there is an even number $N \geq 4 \cdot 10^{18}$ which is not a sum of two distinct prime numbers. We consider all the pairs of positive integers $(n-k, n+k)$ where $n = \frac{N}{2}$, $k < n-1$ is a natural number, $n+k$ and $n-k$ are coprime integers and $n+k$ is prime. By definition of the functions $\sigma(x)$ and $\varphi(x)$, we know that

$$2 \cdot N = \sigma((n-k) \cdot (n+k)) - \varphi((n-k) \cdot (n+k))$$

when $n-k$ is also prime. We notice that

$$2 \cdot N < \sigma((n-k) \cdot (n+k)) - \varphi((n-k) \cdot (n+k))$$

when $n-k$ is not a prime. Certainly, we see that $(n-k)+(n+k) = N$ and thus, the inequality

$$2 \cdot ((n-k)+(n+k)) + \varphi((n-k) \cdot (n+k)) < \sigma((n-k) \cdot (n+k))$$

holds when $n-k$ is not a prime. That is equivalent to

$$2 \cdot ((n-k)+(n+k)) + \varphi(n-k) \cdot \varphi(n+k) < \sigma(n-k) \cdot \sigma(n+k)$$

since the functions $\sigma(x)$ and $\varphi(x)$ are multiplicative. Let's divide both sides by $(n-k) \cdot (n+k)$ to obtain that

$$2 \cdot \left(\frac{(n-k)+(n+k)}{(n-k) \cdot (n+k)}\right) + \frac{\varphi(n-k)}{n-k} \cdot \frac{\varphi(n+k)}{n+k} < s(n-k) \cdot s(n+k).$$

We know that

$$s(n-k) \cdot s(n+k) > 1$$

since $s(m) > 1$ for every natural number $m > 1$ [5]. Moreover, we could see that

$$2 \cdot \left(\frac{(n-k)+(n+k)}{(n-k) \cdot (n+k)}\right) = \frac{2}{n+k} + \frac{2}{n-k}$$

and therefore,

$$1 > \frac{2}{n+k} + \frac{2}{n-k} + \frac{\varphi(n-k)}{n-k} \cdot \frac{\varphi(n+k)}{n+k}.$$

It is enough to see that

$$1 > \frac{2}{2 \cdot 10^{18}} + \frac{2}{9} + \frac{2}{3} \geq \frac{2}{n+k} + \frac{2}{n-k} + \frac{\varphi(n-k)}{n-k} \cdot \frac{\varphi(n+k)}{n+k}$$

when $n+k$ is prime and $n-k$ is composite for $N \geq 4 \cdot 10^{18}$. Indeed, when $n+k$ is prime and $n-k$ is composite, then $n+k > 2 \cdot 10^{18}$ and $n-k \geq 9$ for $N \geq 4 \cdot 10^{18}$. Under our assumption, every of these pairs of positive integers $(n-k, n+k)$ implies that

$$2 \cdot N < \sigma((n-k) \cdot (n+k)) - \varphi((n-k) \cdot (n+k))$$

holds when $n = \frac{N}{2}$, $k < n-1$ is a natural number, $n+k$ and $n-k$ are coprime integers and $n+k$ is prime. We can see that

$$2 = \sigma(n+k) - \varphi(n+k)$$

when $n+k$ is prime. Hence, we have

$$N < \frac{1}{2} \cdot (\sigma(n-k) \cdot \sigma(n+k) - \varphi(n-k) \cdot \varphi(n+k)).$$

We know that

$$\frac{1}{2} \cdot (\sigma(n-k) \cdot \sigma(n+k) - \varphi(n-k) \cdot \varphi(n+k))$$

$$
\begin{aligned}
&= \frac{\sigma(n-k)}{2} \cdot \left(\sigma(n+k) - \frac{\varphi(n-k)}{\sigma(n-k)} \cdot \varphi(n+k)\right) \\
&= \frac{\sigma(n-k)}{2} \cdot \left(\sigma(n+k) - 2 + 2 - \frac{\varphi(n-k)}{\sigma(n-k)} \cdot \varphi(n+k)\right) \\
&= \frac{\sigma(n-k)}{2} \cdot \left(\varphi(n+k) + 2 - \frac{\varphi(n-k)}{\sigma(n-k)} \cdot \varphi(n+k)\right) \\
&= \frac{\sigma(n-k)}{2} \cdot \left(\varphi(n+k) \cdot \left(1 - \frac{\varphi(n-k)}{\sigma(n-k)}\right) + 2\right) \\
&= \sigma(n-k) \cdot \left(\varphi(n+k) \cdot \left(\frac{1}{2} - \frac{\varphi(n-k)}{2 \cdot \sigma(n-k)}\right) + 1\right) \\
&= \sigma(n-k) \cdot \left(\varphi(n+k) \cdot \left(\frac{1}{2} - \frac{\varphi(n-k)}{2 \cdot \sigma(n-k)}\right) + \frac{\varphi(n+k)}{\varphi(n+k)}\right) \\
&= \sigma(n-k) \cdot \left(\varphi(n+k) \cdot \left(\frac{1}{2} + \frac{1}{\varphi(n+k)} - \frac{\varphi(n-k)}{2 \cdot \sigma(n-k)}\right)\right) \\
&= \sigma(n-k) \cdot \left(\varphi(n+k) \cdot \left(\frac{1}{2} - \frac{n^{0.889}}{\varphi(n+k)} + \frac{1+n^{0.889}}{\varphi(n+k)} - \frac{\varphi(n-k)}{2 \cdot \sigma(n-k)}\right)\right) \\
&< \sigma(n-k) \cdot \left(\varphi(n+k) \cdot \left(\frac{1}{2} - \frac{n^{0.889}}{\varphi(n+k)}\right)\right) \\
&= \left(\frac{1}{2} - \frac{n^{0.889}}{\varphi(n+k)}\right) \cdot \sigma(n-k) \cdot \varphi(n+k)
\end{aligned}
$$

when $n+k$ is prime since

$$
\begin{aligned}
\frac{\varphi(n+k)}{1+n^{0.889}} &= \frac{n+k-1}{1+n^{0.889}} \\
&\geq \frac{n}{1+n^{0.889}} \\
&\geq 2 \cdot \left(e^{\gamma} \cdot \log\log(n-1) + \frac{2.5}{\log\log(n-1)}\right)^2 \\
&\geq 2 \cdot \left(e^{\gamma} \cdot \log\log(n-k) + \frac{2.5}{\log\log(n-k)}\right)^2 \\
&> 2 \cdot \left(\frac{n-k}{\varphi(n-k)}\right)^2 \\
&= \frac{n-k}{\varphi(n-k)} \cdot 2 \cdot \prod_{q \mid (n-k)} \left(\frac{q}{q-1}\right) \\
&> s(n-k) \cdot 2 \cdot \prod_{q \mid (n-k)} \left(\frac{q}{q-1}\right) \\
&= \frac{2 \cdot \sigma(n-k)}{(n-k) \cdot \prod_{q \mid (n-k)} \left(1 - \frac{1}{q}\right)} \\
&= \frac{2 \cdot \sigma(n-k)}{\varphi(n-k)}
\end{aligned}
$$

where we know that $\frac{b}{\varphi(b)} < e^{\gamma} \cdot \log\log(b) + \frac{2.5}{\log\log(b)}$ for every odd number $b \geq 3$ [6] ($\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm).

Moreover, we have

$$\frac{n}{1+n^{0.889}} \geq 2 \cdot \left(e^{\gamma} \cdot \log\log(n-1) + \frac{2.5}{\log\log(n-1)}\right)^2$$

for every natural number $n \geq 2 \cdot 10^{18}$ since $N \geq 4 \cdot 10^{18}$. Certainly, the function

$$f(x) = \frac{x}{1+x^{0.889}} - 2 \cdot \left(e^{\gamma} \cdot \log\log(x-1) + \frac{2.5}{\log\log(x-1)}\right)^2$$

is strictly increasing and positive for every real number $x \geq 2 \cdot 10^{18}$ because of its derivative is greater than 0 for all $x \geq 2 \cdot 10^{18}$ and it is positive in the value of $2 \cdot 10^{18}$. Furthermore, it is known that $\prod_{q|b} \left(\frac{q}{q-1}\right) = \frac{b}{\varphi(b)} > s(b) = \frac{\sigma(b)}{b}$ for every natural number $b \geq 2$ [5]. Finally, we would have that

$$N < \left(\frac{1}{2} - \frac{n^{0.889}}{\varphi(n+k)}\right) \cdot \sigma(n-k) \cdot \varphi(n+k)$$

and so,

$$4 \cdot n \cdot \varphi(n+k) = 2 \cdot \varphi(n+k) \cdot N < \left(\varphi(n+k) - 2 \cdot n^{0.889}\right) \cdot \sigma(n-k) \cdot \varphi(n+k)$$

which is

$$4 \cdot n < \left(\varphi(n+k) - 2 \cdot n^{0.889}\right) \cdot \sigma(n-k).$$

In this way, we obtain a contradiction when we assume that $4 \cdot n \geq \left(\varphi(n+k) - 2 \cdot n^{0.889}\right) \cdot \sigma(n-k)$. By reductio ad absurdum, the natural number $n-k$ is necessarily prime when $4 \cdot n \geq \left(\varphi(n+k) - 2 \cdot n^{0.889}\right) \cdot \sigma(n-k)$. □

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