

# Deep on Goldbach's conjecture

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## Abstract

Goldbach's conjecture is one of the most difficult unsolved problems in mathematics. This states that every even natural number greater than 2 is the sum of two prime numbers. In 1973, Chen Jingrun proved that every sufficiently large even number can be written as the sum of either two primes, or a prime and a semiprime (the product of two primes). In 2015, Tomohiro Yamada, using the Chen's theorem, showed that every even number  $> \mathbf{exp\ exp\ 36}$  can be represented as the sum of a prime and a product of at most two primes. In 2002, Ying Chun Cai proved that every sufficiently large even integer  $N$  is equal to  $p + P_2$ , where  $P_2$  is an almost prime with at most two prime factors and  $p \leq N^{0.95}$  is a prime number. In this note, we prove that for every even number  $N \geq 2^{21}$ , if there is a prime  $p$  and a natural number  $m$  such that  $n < p < N - 1$ ,  $p + m = N$ ,  $4 \cdot n \geq (p - 1 - 2 \cdot n^{0.7}) \cdot \sigma(m)$  and  $p$  is coprime with  $m$ , then  $m$  is necessarily a prime number when  $N = 2 \cdot n$  and  $\sigma(m)$  is the sum-of-divisors function of  $m$ . Indeed, this is a trivial and short note very easy to check and understand which is a breakthrough result at the same time.

**Keywords:** Goldbach's conjecture, Prime numbers, Sum-of-divisors function, Euler's totient function

**MSC Classification:** 11A41 , 11A25

# 1 Introduction

As usual  $\sigma(n)$  is the sum-of-divisors function of  $n$

$$\sum_{d|n} d,$$

where  $d \mid n$  means the integer  $d$  divides  $n$ . Define  $s(n)$  as  $\frac{\sigma(n)}{n}$ . In number theory, the  $p$ -adic order of an integer  $n$  is the exponent of the highest power of the prime number  $p$  that divides  $n$ . It is denoted  $\nu_p(n)$ . Equivalently,  $\nu_p(n)$  is the exponent to which  $p$  appears in the prime factorization of  $n$ . We can state the sum-of-divisors function of  $n$  as

$$\sigma(n) = \prod_{p|n} \frac{p^{\nu_p(n)+1} - 1}{p - 1}$$

with the product extending over all prime numbers  $p$  which divide  $n$ . In addition, the well-known Euler's totient function  $\varphi(n)$  can be formulated as

$$\varphi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Chen's theorem states that every sufficiently large even number can be written as the sum of either two primes, or a prime and a semiprime (the product of two primes) [1]. Tomohiro Yamada using an explicit version of Chen's theorem showed that every even number greater than  $e^{e^{36}} \approx 1.7 \cdot 10^{1872344071119343}$  is the sum of a prime and a product of at most two primes [2]. A natural number is called  $k$ -almost prime if it has  $k$  prime factors [3]. A natural number is prime if and only if it is 1-almost prime, and semiprime if and only if it is 2-almost prime. Let  $N$  be a sufficiently large even integer. Ying Chun Cai proved that the equation

$$N = p + P_2, \quad p \leq N^{0.95},$$

is solvable, where  $p$  denotes a prime and  $P_2$  denotes an almost prime with at most two prime factors [3]. In mathematics, two integers  $a$  and  $b$  are coprime, if the only positive integer that is a divisor of both of them is 1. Putting all together yields the proof of the main theorem.

**Theorem 1** *For every even number  $N \geq 2^{21}$ , if there is a prime  $p$  and a natural number  $m$  such that  $n < p < N - 1$ ,  $p + m = N$ ,  $4 \cdot n \geq (p - 1 - 2 \cdot n^{0.7}) \cdot \sigma(m)$  and  $p$  is coprime with  $m$ , then  $m$  is necessarily a prime number when  $N = 2 \cdot n$ .*

## 2 Proof of Theorem 1

*Proof* Suppose that there is an even number  $N \geq 2^{21}$  which is not a sum of two distinct prime numbers. We consider all the pairs of positive integers  $(n-k, n+k)$  where  $n = \frac{N}{2}$ ,  $k < n-1$  is a natural number,  $n+k$  and  $n-k$  are coprime integers and  $n+k$  is prime. By definition of the functions  $\sigma(x)$  and  $\varphi(x)$ , we know that

$$2 \cdot N = \sigma((n-k) \cdot (n+k)) - \varphi((n-k) \cdot (n+k))$$

when  $n-k$  is also prime. We notice that

$$2 \cdot N < \sigma((n-k) \cdot (n+k)) - \varphi((n-k) \cdot (n+k))$$

when  $n-k$  is not a prime. Certainly, we see that  $(n-k) + (n+k) = N$  and thus, the inequality

$$2 \cdot ((n-k) + (n+k)) + \varphi((n-k) \cdot (n+k)) < \sigma((n-k) \cdot (n+k))$$

holds when  $n-k$  is not a prime. That is equivalent to

$$2 \cdot ((n-k) + (n+k)) + \varphi(n-k) \cdot \varphi(n+k) < \sigma(n-k) \cdot \sigma(n+k)$$

since the functions  $\sigma(x)$  and  $\varphi(x)$  are multiplicative. Let's divide both sides by  $(n-k) \cdot (n+k)$  to obtain that

$$2 \cdot \left( \frac{(n-k) + (n+k)}{(n-k) \cdot (n+k)} \right) + \frac{\varphi(n-k)}{n-k} \cdot \frac{\varphi(n+k)}{n+k} < s(n-k) \cdot s(n+k).$$

We know that

$$s(n-k) \cdot s(n+k) > 1$$

since  $s(m) > 1$  for every natural number  $m > 1$  [4]. Moreover, we could see that

$$2 \cdot \left( \frac{(n-k) + (n+k)}{(n-k) \cdot (n+k)} \right) = \frac{2}{n+k} + \frac{2}{n-k}$$

and therefore,

$$1 > \frac{2}{n+k} + \frac{2}{n-k} + \frac{\varphi(n-k)}{n-k} \cdot \frac{\varphi(n+k)}{n+k}.$$

It is enough to see that

$$1 > \frac{2}{2097143} + \frac{2}{9} + \frac{2}{3} \geq \frac{2}{n+k} + \frac{2}{n-k} + \frac{\varphi(n-k)}{n-k} \cdot \frac{\varphi(n+k)}{n+k}$$

when  $n+k$  is prime and  $n-k$  is composite for  $N \geq 2^{21}$ . Under our assumption, every of these pairs of positive integers  $(n-k, n+k)$  implies that

$$2 \cdot N < \sigma((n-k) \cdot (n+k)) - \varphi((n-k) \cdot (n+k))$$

holds when  $n = \frac{N}{2}$ ,  $k < n-1$  is a natural number,  $n+k$  and  $n-k$  are coprime integers and  $n+k$  is prime. We can see that

$$2 = \sigma(n+k) - \varphi(n+k)$$

when  $n+k$  is prime. Hence, we have

$$N < \frac{1}{2} \cdot (\sigma(n-k) \cdot \sigma(n+k) - \varphi(n-k) \cdot \varphi(n+k)).$$

We know that

$$\frac{1}{2} \cdot (\sigma(n-k) \cdot \sigma(n+k) - \varphi(n-k) \cdot \varphi(n+k))$$

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$$\begin{aligned}
 &= \frac{\sigma(n-k)}{2} \cdot \left( \sigma(n+k) - \frac{\varphi(n-k)}{\sigma(n-k)} \cdot \varphi(n+k) \right) \\
 &= \frac{\sigma(n-k)}{2} \cdot \left( \sigma(n+k) - 2 + 2 - \frac{\varphi(n-k)}{\sigma(n-k)} \cdot \varphi(n+k) \right) \\
 &= \frac{\sigma(n-k)}{2} \cdot \left( \varphi(n+k) + 2 - \frac{\varphi(n-k)}{\sigma(n-k)} \cdot \varphi(n+k) \right) \\
 &= \frac{\sigma(n-k)}{2} \cdot \left( \varphi(n+k) \cdot \left( 1 - \frac{\varphi(n-k)}{\sigma(n-k)} \right) + 2 \right) \\
 &= \sigma(n-k) \cdot \left( \varphi(n+k) \cdot \left( \frac{1}{2} - \frac{\varphi(n-k)}{2 \cdot \sigma(n-k)} \right) + 1 \right) \\
 &= \sigma(n-k) \cdot \left( \varphi(n+k) \cdot \left( \frac{1}{2} - \frac{\varphi(n-k)}{2 \cdot \sigma(n-k)} \right) + \frac{\varphi(n+k)}{\varphi(n+k)} \right) \\
 &= \sigma(n-k) \cdot \left( \varphi(n+k) \cdot \left( \frac{1}{2} + \frac{1}{\varphi(n+k)} - \frac{\varphi(n-k)}{2 \cdot \sigma(n-k)} \right) \right) \\
 &= \sigma(n-k) \cdot \left( \varphi(n+k) \cdot \left( \frac{1}{2} - \frac{n^{0.7}}{\varphi(n+k)} + \frac{1+n^{0.7}}{\varphi(n+k)} - \frac{\varphi(n-k)}{2 \cdot \sigma(n-k)} \right) \right) \\
 &< \sigma(n-k) \cdot \left( \varphi(n+k) \cdot \left( \frac{1}{2} - \frac{n^{0.7}}{\varphi(n+k)} \right) \right) \\
 &= \left( \frac{1}{2} - \frac{n^{0.7}}{\varphi(n+k)} \right) \cdot \sigma(n-k) \cdot \varphi(n+k)
 \end{aligned}$$

when  $n+k$  is prime since

$$\begin{aligned}
 \frac{\varphi(n+k)}{1+n^{0.7}} &= \frac{n+k-1}{1+n^{0.7}} \\
 &\geq \frac{n}{1+n^{0.7}} \\
 &\geq 2 \cdot \left( e^\gamma \cdot \log \log(n-1) + \frac{2.50637}{\log \log(n-1)} \right)^2 \\
 &\geq 2 \cdot \left( e^\gamma \cdot \log \log(n-k) + \frac{2.50637}{\log \log(n-k)} \right)^2 \\
 &> 2 \cdot \left( \frac{n-k}{\varphi(n-k)} \right)^2 \\
 &= \frac{n-k}{\varphi(n-k)} \cdot 2 \cdot \prod_{q|(n-k)} \left( \frac{q}{q-1} \right) \\
 &> s(n-k) \cdot 2 \cdot \prod_{q|(n-k)} \left( \frac{q}{q-1} \right) \\
 &= \frac{2 \cdot \sigma(n-k)}{(n-k) \cdot \prod_{q|(n-k)} \left( 1 - \frac{1}{q} \right)} \\
 &= \frac{2 \cdot \sigma(n-k)}{\varphi(n-k)}
 \end{aligned}$$

where we know that  $\frac{b}{\varphi(b)} < e^\gamma \cdot \log \log(b) + \frac{2.50637}{\log \log(b)}$  for every natural number  $b \geq 3$  [5] ( $\gamma \approx 0.57721$  is the Euler-Mascheroni constant and  $\log$  is the natural logarithm).

Moreover, we have

$$\frac{n}{1+n^{0.7}} \geq 2 \cdot \left( e^\gamma \cdot \log \log(n-1) + \frac{2.50637}{\log \log(n-1)} \right)^2$$

for every natural number  $n \geq 2^{20}$  since  $N \geq 2^{21}$ . Certainly, the function

$$f(x) = \frac{x}{1+x^{0.7}} - 2 \cdot \left( e^\gamma \cdot \log \log(x-1) + \frac{2.50637}{\log \log(x-1)} \right)^2$$

is strictly increasing and positive for every real number  $x \geq 2^{20}$  because of its derivative is greater than 0 for all  $x \geq 2^{20}$  and it is positive in the value of  $2^{20}$ . Furthermore, it is known that  $\prod_{q|b} \left( \frac{q}{q-1} \right) = \frac{b}{\varphi(b)} > s(b) = \frac{\sigma(b)}{b}$  for every natural number  $b \geq 2$  [4]. Finally, we would have that

$$N < \left( \frac{1}{2} - \frac{n^{0.7}}{\varphi(n+k)} \right) \cdot \sigma(n-k) \cdot \varphi(n+k)$$

and so,

$$4 \cdot n \cdot \varphi(n+k) = 2 \cdot \varphi(n+k) \cdot N < (\varphi(n+k) - 2 \cdot n^{0.7}) \cdot \sigma(n-k) \cdot \varphi(n+k)$$

which is

$$4 \cdot n < (\varphi(n+k) - 2 \cdot n^{0.7}) \cdot \sigma(n-k).$$

In this way, we obtain a contradiction when we assume that  $4 \cdot n \geq (\varphi(n+k) - 2 \cdot n^{0.7}) \cdot \sigma(n-k)$ . By reductio ad absurdum, the natural number  $n-k$  is necessarily prime when  $4 \cdot n \geq (\varphi(n+k) - 2 \cdot n^{0.7}) \cdot \sigma(n-k)$ .  $\square$

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