Deep on Goldbach's conjecture

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Abstract

Goldbach's conjecture is one of the most difficult unsolved problems in mathematics. This states that every even natural number greater than 2 is the sum of two prime numbers. In 1973, Chen Jingrun proved that every sufficiently large even number can be written as the sum of either two primes, or a prime and a semiprime (the product of two primes). In 2015, Tomohiro Yamada, using the Chen's theorem, showed that every even number $> \exp\exp 36$ can be represented as the sum of a prime and a product of at most two primes. In 2002, Ying Chun Cai proved that every sufficiently large even integer N is equal to $p + P_2$, where P_2 is an almost prime with at most two prime factors and $p \leq N^{0.95}$ is a prime number. In this note, we prove that for every even number $N \geq 2^{21}$, if there is a prime p and a natural number m such that $n < p < N - 1$, $p + m = N$, $4 \cdot n \geq \left(p - 1 - 2 \cdot \sqrt{n} \cdot \log n\right) \cdot \sigma(m)$ and p is coprime with m, then m is necessarily a prime number when $N = 2 \cdot n$, $\sigma(m)$ is the sum-of-divisors function of m and $\log$ is the natural logarithm. That is also true whenever $\frac{n}{n-2-\sqrt{n}\cdot\log n} \geq \frac{\sigma(m)}{2}$ based on the previously defined prime p. Indeed, this is a trivial and short note very easy to check and understand which is a breakthrough result at the same time.

Keywords: Goldbach's conjecture, Prime numbers, Sum-of-divisors function, Euler's totient function

MSC Classification: 11A41 , 11A25

1 Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d|n} d,$$

where $d \mid n$ means the integer d divides n. Define $s(n)$ as $\frac{\sigma(n)}{n}$. In number theory, the *p-adic* order of an integer n is the exponent of the highest power of the prime number p that divides n. It is denoted $\nu_p(n)$. Equivalently, $\nu_p(n)$ is the exponent to which p appears in the prime factorization of n. We can state the sum-of-divisors function of n as

$$\sigma(n) = \prod_{p|n} \frac{p^{\nu_p(n)+1} - 1}{p - 1}$$

with the product extending over all prime numbers p which divide n. In addition, the well-known Euler's totient function $\varphi(n)$ can be formulated as

$$\varphi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Chen's theorem states that every sufficiently large even number can be written as the sum of either two primes, or a prime and a semiprime (the product of two primes) [1]. Tomohiro Yamada using an explicit version of Chen's theorem showed that every even number greater than $e^{e^{36}} \approx 1.7 \cdot 10^{1872344071119343}$ is the sum of a prime and a product of at most two primes [2]. A natural number is called k-almost prime if it has k prime factors [3]. A natural number is prime if and only if it is 1-almost prime, and semiprime if and only if it is 2-almost prime. Let N be a sufficiently large even integer. Ying Chun Cai proved that the equation

$$N = p + P_2, \quad p \leq N^{0.95},$$

is solvable, where p denotes a prime and P_2 denotes an almost prime with at most two prime factors [3]. In mathematics, two integers a and b are coprime, if the only positive integer that is a divisor of both of them is 1. Putting all together yields the proof of the main theorem. We denote log as the natural logarithm.

Theorem 1 *For every even number $N \geq 2^{21}$, if there is a prime p and a natural number m such that $n < p < N - 1$, $p + m = N$, $4 \cdot n \geq (p - 1 - 2 \cdot \sqrt{n} \cdot \log n) \cdot \sigma(m)$ and p is coprime with m, then m is necessarily a prime number when $N = 2 \cdot n$. That is also true whenever $\frac{n}{n - 2 - \sqrt{n} \cdot \log n} \geq \frac{\sigma(m)}{2}$ based on the previously defined prime p.*

2 Proof of Theorem 1

Proof Suppose that there is an even number $N \geq 2^{21}$ which is not a sum of two distinct prime numbers. We consider all the pairs of positive integers $(n-k, n+k)$ where $n = \frac{N}{2}$, $k < n-1$ is a natural number, $n+k$ and $n-k$ are coprime integers and $n+k$ is prime. By definition of the functions $\sigma(x)$ and $\varphi(x)$, we know that

$$2 \cdot N = \sigma((n-k) \cdot (n+k)) - \varphi((n-k) \cdot (n+k))$$

when $n-k$ is also prime. We notice that

$$2 \cdot N < \sigma((n-k) \cdot (n+k)) - \varphi((n-k) \cdot (n+k))$$

when $n-k$ is not a prime. Certainly, we see that $(n-k)+(n+k) = N$ and thus, the inequality

$$2 \cdot ((n-k)+(n+k)) + \varphi((n-k) \cdot (n+k)) < \sigma((n-k) \cdot (n+k))$$

holds when $n-k$ is not a prime. That is equivalent to

$$2 \cdot ((n-k)+(n+k)) + \varphi(n-k) \cdot \varphi(n+k) < \sigma(n-k) \cdot \sigma(n+k)$$

since the functions $\sigma(x)$ and $\varphi(x)$ are multiplicative. Let's divide both sides by $(n-k) \cdot (n+k)$ to obtain that

$$2 \cdot \left(\frac{(n-k)+(n+k)}{(n-k) \cdot (n+k)} \right) + \frac{\varphi(n-k)}{n-k} \cdot \frac{\varphi(n+k)}{n+k} < s(n-k) \cdot s(n+k).$$

We know that

$$s(n-k) \cdot s(n+k) > 1$$

since $s(m) > 1$ for every natural number $m > 1$ [4]. Moreover, we could see that

$$2 \cdot \left(\frac{(n-k)+(n+k)}{(n-k) \cdot (n+k)} \right) = \frac{2}{n+k} + \frac{2}{n-k}$$

and therefore,

$$1 > \frac{2}{n+k} + \frac{2}{n-k} + \frac{\varphi(n-k)}{n-k} \cdot \frac{\varphi(n+k)}{n+k}.$$

It is enough to see that

$$1 > \frac{2}{2097143} + \frac{2}{9} + \frac{2}{3} \geq \frac{2}{n+k} + \frac{2}{n-k} + \frac{\varphi(n-k)}{n-k} \cdot \frac{\varphi(n+k)}{n+k}$$

when $n+k$ is prime and $n-k$ is composite for $N \geq 2^{21}$. Under our assumption, every of these pairs of positive integers $(n-k, n+k)$ implies that

$$2 \cdot N < \sigma((n-k) \cdot (n+k)) - \varphi((n-k) \cdot (n+k))$$

holds when $n = \frac{N}{2}$, $k < n-1$ is a natural number, $n+k$ and $n-k$ are coprime integers and $n+k$ is prime. We can see that

$$2 = \sigma(n+k) - \varphi(n+k)$$

when $n+k$ is prime. Hence, we have

$$N < \frac{1}{2} \cdot (\sigma(n-k) \cdot \sigma(n+k) - \varphi(n-k) \cdot \varphi(n+k)).$$

We know that

$$\frac{1}{2} \cdot (\sigma(n-k) \cdot \sigma(n+k) - \varphi(n-k) \cdot \varphi(n+k))$$

$$
\begin{aligned}
&= \frac{\sigma(n-k)}{2}\cdot\left(\sigma(n+k)-\frac{\varphi(n-k)}{\sigma(n-k)}\cdot\varphi(n+k)\right)\\
&= \frac{\sigma(n-k)}{2}\cdot\left(\sigma(n+k)-2+2-\frac{\varphi(n-k)}{\sigma(n-k)}\cdot\varphi(n+k)\right)\\
&= \frac{\sigma(n-k)}{2}\cdot\left(\varphi(n+k)+2-\frac{\varphi(n-k)}{\sigma(n-k)}\cdot\varphi(n+k)\right)\\
&= \frac{\sigma(n-k)}{2}\cdot\left(\varphi(n+k)\cdot\left(1-\frac{\varphi(n-k)}{\sigma(n-k)}\right)+2\right)\\
&= \sigma(n-k)\cdot\left(\varphi(n+k)\cdot\left(\frac{1}{2}-\frac{\varphi(n-k)}{2\cdot\sigma(n-k)}\right)+1\right)\\
&= \sigma(n-k)\cdot\left(\varphi(n+k)\cdot\left(\frac{1}{2}-\frac{\varphi(n-k)}{2\cdot\sigma(n-k)}\right)+\frac{\varphi(n+k)}{\varphi(n+k)}\right)\\
&= \sigma(n-k)\cdot\left(\varphi(n+k)\cdot\left(\frac{1}{2}+\frac{1}{\varphi(n+k)}-\frac{\varphi(n-k)}{2\cdot\sigma(n-k)}\right)\right)\\
&= \sigma(n-k)\cdot\left(\varphi(n+k)\cdot\left(\frac{1}{2}-\frac{\sqrt{n}\cdot\log n}{\varphi(n+k)}+\frac{1+\sqrt{n}\cdot\log n}{\varphi(n+k)}-\frac{\varphi(n-k)}{2\cdot\sigma(n-k)}\right)\right)\\
&< \sigma(n-k)\cdot\left(\varphi(n+k)\cdot\left(\frac{1}{2}-\frac{\sqrt{n}\cdot\log n}{\varphi(n+k)}\right)\right)\\
&= \left(\frac{1}{2}-\frac{\sqrt{n}\cdot\log n}{\varphi(n+k)}\right)\cdot\sigma(n-k)\cdot\varphi(n+k)
\end{aligned}
$$

when $n+k$ is prime since

$$
\begin{aligned}
\frac{\varphi(n+k)}{1+\sqrt{n}\cdot\log n} &= \frac{n+k-1}{1+\sqrt{n}\cdot\log n}\\
&\geq \frac{n}{1+\sqrt{n}\cdot\log n}\\
&\geq 2\cdot\left(e^{\gamma}\cdot\log\log(n-1)+\frac{2.50637}{\log\log(n-1)}\right)^2\\
&\geq 2\cdot\left(e^{\gamma}\cdot\log\log(n-k)+\frac{2.50637}{\log\log(n-k)}\right)^2\\
&> 2\cdot\left(\frac{n-k}{\varphi(n-k)}\right)^2\\
&= \frac{n-k}{\varphi(n-k)}\cdot 2\cdot\prod_{q\mid(n-k)}\left(\frac{q}{q-1}\right)\\
&> s(n-k)\cdot 2\cdot\prod_{q\mid(n-k)}\left(\frac{q}{q-1}\right)\\
&= \frac{2\cdot\sigma(n-k)}{(n-k)\cdot\prod_{q\mid(n-k)}\left(1-\frac{1}{q}\right)}\\
&= \frac{2\cdot\sigma(n-k)}{\varphi(n-k)}
\end{aligned}
$$

where we know that $\frac{b}{\varphi(b)} < e^{\gamma}\cdot\log\log(b)+\frac{2.50637}{\log\log(b)}$ for every natural number $b\geq 3$ [5] ($\gamma\approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm).

Moreover, we have

$$\frac{n}{1+\sqrt{n}\cdot\log n} \geq 2\cdot\left(e^{\gamma}\cdot\log\log(n-1)+\frac{2.50637}{\log\log(n-1)}\right)^2$$

for every $n \geq 2^{20}$ since $N \geq 2^{21}$. Furthermore, it is known that $\prod_{q\mid b}\left(\frac{q}{q-1}\right) = \frac{b}{\varphi(b)} > s(b) = \frac{\sigma(b)}{b}$ for every natural number $b \geq 2$ [4]. Finally, we would have that

$$N < \left(\frac{1}{2} - \frac{\sqrt{n}\cdot\log n}{\varphi(n+k)}\right)\cdot\sigma(n-k)\cdot\varphi(n+k)$$

and so,

$$4\cdot n\cdot\varphi(n+k) = 2\cdot\varphi(n+k)\cdot N < \left(\varphi(n+k) - 2\cdot\sqrt{n}\cdot\log n\right)\cdot\sigma(n-k)\cdot\varphi(n+k)$$

which is

$$4\cdot n < \left(\varphi(n+k) - 2\cdot\sqrt{n}\cdot\log n\right)\cdot\sigma(n-k).$$

In this way, we obtain a contradiction when we assume that $4\cdot n \geq (\varphi(n+k) - 2\cdot\sqrt{n}\cdot\log n)\cdot\sigma(n-k)$. By reductio ad absurdum, the natural number $n-k$ is necessarily prime when $4\cdot n \geq (\varphi(n+k) - 2\cdot\sqrt{n}\cdot\log n)\cdot\sigma(n-k)$. That inequality is satisfied whenever

$$4\cdot n \geq \left(2\cdot n - 4 - 2\cdot\sqrt{n}\cdot\log n\right)\cdot\sigma(n-k)$$

that is

$$\frac{n}{n-2-\sqrt{n}\cdot\log n} \geq \frac{\sigma(n-k)}{2}$$

since $2\cdot n - 4 \geq \varphi(n+k)$ when $n+k$ is prime. □

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