International Journal of Electrical and Electronics Research ISSN 2348-6988 (online)
Vol. 10, Issue 4, pp: (33-37), Month: October - December 2022, Available at: www.researchpublish.com

Research on Two Types of Fractional Integrals

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DOI: https://doi.org/10.5281/zenodo.7483642

Published Date: 26-December-2022

Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus, we solve two types of fractional integrals. Complex power of fractional analytic function, product rule for fractional derivatives and a new multiplication of fractional analytic functions play important roles in this paper. In fact, our results are generalizations of the results in traditional calculus.**

Keywords: **Jumarie type of R-L fractional calculus, fractional integrals, complex power of fractional analytic function, product rule, new multiplication.**

I. INTRODUCTION

Fractional calculus is a natural extension of the traditional calculus. In fact, since the beginning of the theory of differential and integral calculus, some mathematicians have studied their ideas on the calculation of non-integer order derivatives and integrals. However, the application of fractional derivatives and integrals has been scarce until recently. In the last decade, fractional calculus are widely used in physics, mechanics, viscoelasticity, control theory, biology, electrical engineering, and economics [1-9].

However, the definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie type of R-L fractional derivative [10-13]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with ordinary calculus.

In this article, based on Jumarie's modified R-L fractional calculus, we evaluate the following two types of fractional integrals:

$$\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}I_x^{\alpha}\right)\left[cos_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)\right], \tag{1}$$

$$\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}I_x^{\alpha}\right)\left[sin_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)\right], \tag{2}$$

where $0 < \alpha \leq 1$. Complex power of fractional analytic function, a new multiplication of fractional analytic functions and product rule for fractional derivatives play important roles in this paper. In fact, our results are generalizations of the results in classical calculus.

II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper.

Definition 2.1 ([14])**:** Assume that $0 < \alpha \leq 1$, and x_0 is a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\,, \tag{3}$$

And the Jumarie type of R-L α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\,, \tag{4}$$

where $\Gamma(\ )$ is the gamma function.

In the following, we provide some properties of Jumarie's fractional derivative.

Proposition 2.2 ([15])**:** *Let α, β, x_0, C be real numbers and $\beta \geq \alpha > 0$, then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{5}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{6}$$

Next, the definition of fractional analytic function is introduced.

Definition 2.3([16])**:** Let x, x_0, and a_k be real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on$[a,b]$.

Next, we introduce a new multiplication of fractional analytic function.

Definition 2.4 ([17])**:** If $0 < \alpha \leq 1$, and x_0 is a real number. Suppose that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{7}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{8}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{9}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \otimes \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \end{aligned} \tag{10}$$

Definition 2.5 ([17])**:** Suppose that $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{11}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{12}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(g_\alpha(x^\alpha)\right)^{\otimes k}, \tag{13}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{b_k}{k!} (f_\alpha(x^\alpha))^{\otimes k}. \tag{14}$$

Definition 2.6 ([17])**:** Let $0 < \alpha \leq 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)} x^\alpha. \tag{15}$$

Then these two fractional analytic functions are called inverse to each other.

Definition 2.7 ([18])**:** If $0 < \alpha \leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}. \tag{16}$$

The α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. In addition, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 2k}, \tag{17}$$

and

$$sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes (2k+1)}. \tag{18}$$

In the following, the complex power of fractional analytic function is defined.

Definition 2.8 ([14])**:** Let $0 < \alpha \leq 1$ and z be a complex number. The z-th power of the α-fractional analytic function $f_\alpha(x^\alpha)$ is defined by

$$[f_\alpha(x^\alpha)]^{\otimes z} = E_\alpha\left(zLn_\alpha(f_\alpha(x^\alpha))\right). \tag{19}$$

Proposition 2.9 (fractional Euler's formula)**:** *Let* $0 < \alpha \leq 1$, x *be a real number, then*

$$E_\alpha(ix^\alpha) = cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha). \tag{20}$$

Theorem 2.10 (product rule for fractional derivatives) ([19])**:** *Let* $0 < \alpha \leq 1$, *and* $f_\alpha(x^\alpha)$ *and* $g_\alpha(x^\alpha)$ *be* α*-fractional analytic at* $x = x_0$, *then*

$$\left({}_{x_0}D_x^\alpha\right)[f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha)] = \left({}_{x_0}D_x^\alpha\right)[f_\alpha(x^\alpha)] \otimes g_\alpha(x^\alpha) + f_\alpha(x^\alpha) \otimes \left({}_{x_0}D_x^\alpha\right)[g_\alpha(x^\alpha)]. \tag{21}$$

III. MAIN RESULTS

In this section, two major results in this paper are introduced. At first, we need a lemma.

Lemma 3.1: *Let* $0 < \alpha \leq 1$, $i = \sqrt{-1}$, *then the complex power of* α*-fractional analytic function*

$$\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes i} = cos_\alpha\left(Ln_\alpha(x^\alpha)\right) + isin_\alpha\left(Ln_\alpha(x^\alpha)\right). \tag{22}$$

Proof $\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes i}$

$= E_\alpha\left(iLn_\alpha(x^\alpha)\right)$

$= cos_\alpha\left(Ln_\alpha(x^\alpha)\right) + isin_\alpha\left(Ln_\alpha(x^\alpha)\right)$ (by fractional Euler's formula) Q.e.d.

Theorem 3.2: *Let* $0 < \alpha \leq 1$, *then*

$$\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}I_x^\alpha\right)\left[cos_\alpha\left(Ln_\alpha(x^\alpha)\right)\right] = \frac{1}{2}\left[\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right) \otimes \left[sin_\alpha\left(Ln_\alpha(x^\alpha)\right) + cos_\alpha\left(Ln_\alpha(x^\alpha)\right)\right] - 1\right], \tag{23}$$

and

$$\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}I_x^\alpha\right)\left[sin_\alpha\left(Ln_\alpha(x^\alpha)\right)\right] = \frac{1}{2}\left[\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right) \otimes \left[sin_\alpha\left(Ln_\alpha(x^\alpha)\right) - cos_\alpha\left(Ln_\alpha(x^\alpha)\right)\right] + 1\right]. \tag{24}$$

Proof Since $\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes i}\right]$

$$=\frac{1}{i+1}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes(i+1)}-\frac{1}{i+1}$$

$$=\frac{1-i}{2}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes i}\right]-\frac{1-i}{2}$$

$$=\frac{1-i}{2}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes\left[cos_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)+isin_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)\right]\right]-\frac{1-i}{2}\quad\text{(by Lemma 3.1)}$$

$$=\frac{1}{2}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes\left[sin_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)+cos_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)\right]-1\right]$$
$$+i\frac{1}{2}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes\left[sin_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)-cos_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)\right]+1\right].\tag{25}$$

And

$$\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes i}\right]$$
$$=\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}I_x^{\alpha}\right)\left[cos_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)+isin_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)\right]$$
$$=\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}I_x^{\alpha}\right)\left[cos_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)\right]+i\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}I_x^{\alpha}\right)\left[sin_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)\right].\tag{26}$$

It follows that

$$\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}I_x^{\alpha}\right)\left[cos_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)\right]=\frac{1}{2}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes\left[sin_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)+cos_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)\right]-1\right],$$

and

$$\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}I_x^{\alpha}\right)\left[sin_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)\right]=\frac{1}{2}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes\left[sin_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)-cos_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)\right]+1\right].$$

Q.e.d.

Remark 3.3: By product rule for fractional derivatives, we have

$$\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}D_x^{\alpha}\right)\left[\frac{1}{2}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes\left[sin_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)+cos_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)\right]-1\right]\right]$$
$$=\frac{1}{2}\left[sin_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)+cos_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)\right]+\frac{1}{2}\left[cos_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)-sin_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)\right]$$
$$=cos_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right).\tag{27}$$

And

$$\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}D_x^{\alpha}\right)\left[\frac{1}{2}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes\left[sin_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)-cos_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)\right]+1\right]\right]$$
$$=\frac{1}{2}\left[sin_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)-cos_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)\right]+\frac{1}{2}\left[cos_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)+sin_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right)\right]$$
$$=sin_{\alpha}\left(Ln_{\alpha}(x^{\alpha})\right).\tag{28}$$

Therefore, we can easily know that Theorem 3.2 holds.

IV. CONCLUSION

In this paper, we solve two types of fractional integrals based on Jumarie's modified R-L fractional integral. A new multiplication of fractional analytic functions, complex power of fractional analytic function, and product rule for fractional derivatives play important roles in this article. In fact, our results are generalizations of the traditional calculus results. In the future, we will continue to use these methods to expand the research fields to fractional differential equations and engineering mathematics.

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