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ON ONE INITIAL-BOUNDARY VALUE PROBLEM FOR A HIGH-ORDER PARTIAL DIFFERENTIAL EQUATION IN THE MULTIDIMENSIONAL CASE

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Abstract

In this paper, we study a problem with initial and boundary conditions for one class of high-order partial differential equations in several variables. The solution to the initial boundary value problem is constructed as the sum of a series in the system of eigenfunctions of the multidimensional spectral problem. The eigenvalues of the spectral problem are found and the corresponding system of eigenfunctions is constructed. It is shown that this system of eigenfunctions is complete and forms a Riesz basis in the Sobolev space. Based on the completeness of the system of eigenfunctions, a uniqueness theorem for the solution of the problem is proved. In the Sobolev classes, the existence of a regular solution to the stated initial-boundary value problem is proved.

Keywords: high-order partial differential equation, fractional time derivative, initial-boundary value problem, spectral method, eigenvalues, eigenfunctions, completeness, Riesz basis, uniqueness, existence, series.

Many problems of vibrations of beams and plates, which are of great importance in structural mechanics, lead to differential equations of a higher order [1, p. 141-143], [2, p. 278-280], [3, ch.3], [4, p.45], [5, p.35], [6. ch.4]. We also note that the beam vibration equation is used in many problems when calculating the stability of rotating shafts and studying ship vibrations [7, ch.2].

In this paper, we consider a differential equation with a fractional derivative

$$D_{0t}^{\alpha}u(x,y,t)+\sum_{j=1}^{N}(-1)^{s}a_j^2\frac{\partial^{2s}u(x,y,t)}{\partial x_j^{2s}}+\sum_{j=1}^{M}b_j^2\frac{\partial^{4m}u(x,y,t)}{\partial y_j^{4m}}=f(x,y,t), \tag{1}$$

$$(x,y,t)\in\Pi_1\times\Pi_2\times(0,T),\quad p-1<\alpha\le p$$

with initial and boundary conditions

$$D_{0t}^{\alpha-i}u(x,y,t)\Big|_{t=0}=\varphi_i(x,y),\quad i=1,2,...,p, \tag{2}$$

$$\alpha_j\frac{\partial^{2i}u}{\partial x_j^{2i}}\Big|_{x_j=0}+\beta_j\frac{\partial^{2i}u}{\partial x_j^{2i}}\Big|_{x_j=\pi}=0,\quad 1\le j\le N,\quad i=0,1,\dots,s-1,$$

$$\beta_j\frac{\partial^{2i+1}u}{\partial x_j^{2i+1}}\Big|_{x_j=0}+\alpha_j\frac{\partial^{2i+1}u}{\partial x_j^{2i+1}}\Big|_{x_j=\pi}=0,\quad 1\le j\le N,\ i=0,1,\dots,s-1,$$

$$\left.\frac{\partial^{4k}u}{\partial y_j^{4k}}\right|_{y_j=0}=0,\qquad \left.\frac{\partial^{4k+1}u}{\partial y_j^{4k+1}}\right|_{y_j=0}=0,\ k=\overline{0,m-1},\ j=\overline{1,M},$$

$$\left.\frac{\partial^{4k}u}{\partial y_j^{4k}}\right|_{y_j=l}=0,\qquad \left.\frac{\partial^{4k+1}u}{\partial y_j^{4k+1}}\right|_{y_j=l}=0,\ k=\overline{0,m-1},\ j=\overline{1,M}, \tag{3}$$

which has applications in structural mechanics, here $(x,y,t)=$, $=(x_1,...,x_N,y_1,...,y_N,t)\in\Pi_1\times\Pi_2\times(0,T)$ $\Pi_1=(0,\pi)\times...\times(0,\pi)$, $\Pi_2=(0,l)\times...\times(0,l)$, $N,M,\ m,\ p\in\mathbb{N},\ l,\ T>0$ – are given positive numbers and $f(x,y,t)$, $\varphi_i(x,y),\quad i=1,2,...,p$ – are sufficiently smooth functions expanded in terms of eigenfunctions $\{\psi_n(x)v_m(y),n\in\mathbb{N}^N,\ m\in\mathbb{N}^M\}$ of the spectral problem:

$$\sum_{j=1}^{N}(-1)^s a_j^2 \frac{\partial^{2s}\psi(x)}{\partial x_j^{2s}} - \lambda\psi(x) = 0 \tag{4}$$

$$\begin{cases} \alpha_j \dfrac{\partial^{2i}\psi(x)}{\partial x_j^{2i}}\Big|_{x_j=0} + \beta_j \dfrac{\partial^{2i}\psi(x)}{\partial x_j^{2i}}\Big|_{x_j=\pi} = 0, \quad 1 \le j \le N, \quad i = 0,1,\ldots,s-1, \\ \beta_j \dfrac{\partial^{2i+1}\psi(x)}{\partial x_j^{2i+1}}\Big|_{x_j=0} + \alpha_j \dfrac{\partial^{2i+1}\psi(x)}{\partial x_j^{2i+1}}\Big|_{x_j=\pi} = 0, \quad 1 \le j \le N, \ i = 0,1,\ldots,s-1, \end{cases} \tag{5}$$

and

$$\sum_{j=1}^{N} \frac{\partial^{4m}v(y)}{\partial y_j^{4m}} - \lambda v(y) = 0 \tag{6}$$

$$\begin{cases} \left.\dfrac{\partial^{4k}v(y)}{\partial y_j^{4k}}\right|_{y_j=0} = 0, & \left.\dfrac{\partial^{4k+1}v(y)}{\partial y_j^{4k+1}}\right|_{y_j=0} = 0, \\ \left.\dfrac{\partial^{4k}v(y)}{\partial y_j^{4k}}\right|_{y_j=l} = 0, & \left.\dfrac{\partial^{4k+1}v(y)}{\partial y_j^{4k+1}}\right|_{y_j=l} = 0, \ k = \overline{0, m-1}, \ j = \overline{1, N}, \end{cases} \tag{7}$$

λ – spectral parameter.

Here, for $\alpha < 0$, the fractional integral has the form

$$D_{at}^{\alpha}u(x,y,t) = \frac{sign(t-a)}{\Gamma(-\alpha)} \int_a^t \frac{u(x,y,\tau)\cdot d\tau}{|t-\tau|^{\alpha+1}},$$

for $\alpha = 0$, then $D_{at}^{\alpha}u(x,y,t) = u(x,y,t)$, and for $p-1 < \alpha \le p, \quad p \in \mathbb{N}$, the fractional derivative is determined by the formula

$$D_{at}^{\alpha}u(x,y,t) = sign^p(t-a)\frac{d^p}{dt^p} D_{at}^{\alpha-p}u(x,y,t) = \frac{sign^{p+1}(t-a)}{\Gamma(l-\alpha)}\frac{d^p}{dt^p}\int_a^t \frac{u(x,y,\tau)\cdot d\tau}{|t-\tau|^{\alpha-p+1}}.$$

In works [8-10] for the beam equation, i.e. for equation (1) at $\alpha = 2$, $m = 1$, $N = 0, M = 0$, initial-boundary problems are studied. In this paper, on the basis of papers [8,11,12], we obtain a uniqueness and existence theorem for a solution to problem (1) – (3) in the Sobolev space. The solution is constructed as the sum of a series in the system of eigenfunctions of the multidimensional spectral problem (4), (5) and (6), (7).

Completeness of the system of eigenfunctions in Sobolev classes.

The scalar product in the space $W_2^{s_1,s_2,\ldots,s_N}(\Pi)$, is introduced as follows:

$$\begin{aligned}(f(x),g(x))_{W_2^{s_1,s_2,\ldots,s_N}(\Pi)} &= (f(x),g(x))_{L_2(\Pi)} + \sum_{j_1=1}^{N}(D_{x_{j_1}}^{s_{j_1}}f(x), D_{x_{j_1}}^{s_{j_1}}g(x))_{L_2(\Pi)} + \\ &+ \sum_{1\le j_1<j_2\le N}^{N}\left(D_{x_{j_1}}^{s_{j_1}}D_{x_{j_2}}^{s_{j_2}}f(x), D_{x_{j_1}}^{s_{j_1}}D_{x_{j_2}}^{s_{j_2}}g(x)\right)_{L_2(\Pi)} + \ldots + \\ &+ \sum_{1\le j_1<j_2<\ldots<j_N\le N}^{N}\left(D_{x_{j_1}}^{s_{j_1}}D_{x_{j_2}}^{s_{j_2}}\ldots D_{x_{j_N}}^{s_{j_N}}f(x), \quad D_{x_{j_1}}^{s_{j_1}}D_{x_{j_2}}^{s_{j_2}}\ldots D_{x_{j_N}}^{s_{j_N}}g(x)\right)_{L_2(\Pi)}.\end{aligned}$$

Accordingly, the norm in these spaces $W_2^{s_1,s_2,\ldots,s_N}(\Pi)$ is introduced as follows:

$$\begin{aligned}\|f(x)\|^2_{W_2^{s_1,s_2,\ldots,s_N}(\Pi)} &= \|f(x)\|^2_{L_2(\Pi)} + \sum_{j_1=1}^{N}\left\|D_{x_{j_1}}^{s_{j_1}}f(x)\right\|^2_{L_2(\Pi)} + \\ &+ \sum_{1\le j_1<j_2\le N}\left\|D_{x_{j_1}}^{s_{j_1}}D_{x_{j_2}}^{s_{j_2}}f(x)\right\|^2_{L_2(\Pi)} + \ldots + \sum_{1\le j_1<j_2<\ldots<j_N\le N}\left\|D_{x_{j_1}}^{s_{j_1}}D_{x_{j_2}}^{s_{j_2}}\ldots D_{x_{j_N}}^{s_{j_N}}f(x)\right\|^2_{L_2(\Pi)}.\end{aligned}$$

Denote by $\overset{0}{W}{}_2^{s_1,s_2,\ldots,s_N}(\Pi)$ the class of functions belonging to $W_2^{s_1,s_2,\ldots,s_N}(\Pi)$ and satisfying the boundary conditions

$$\begin{cases} \alpha_j \dfrac{\partial^{2k}\psi(x)}{\partial x_j^{2k}}|_{x_j=0} + \beta_j \dfrac{\partial^{2k}\psi(x)}{\partial x_j^{2k}}|_{x_j=\pi} = 0, \quad 1 \le j \le N, \quad 0 \le 2k < s - \dfrac{N}{2}, \\ \beta_j \dfrac{\partial^{2k+1}\psi(x)}{\partial x_j^{2k+1}}|_{x_j=0} + \alpha_j \dfrac{\partial^{2k+1}\psi(x)}{\partial x_j^{2k+1}}|_{x_j=\pi} = 0, 1 \le j \le N, \ 0 \le 2k+1 < s - \dfrac{N}{2}, \end{cases}$$

In the space $\overset{0}{W}{}_2^{s_1,s_2,...,s_N}(\Pi)$ of functions of N –variables $f(x) = f(x_1, x_2, ..., x_N)$, the complete orthonormal system is formed by all products

$$\psi_{m_1,...,m_N}(x_1, x_2, ..., x_N) = \overline{\overline{y}}_{m_1}(x_1)\overline{\overline{y}}_{m_2}(x_2)\cdot ... \cdot \overline{\overline{y}}_{m_N}(x_N),$$

where

$$\overline{\overline{y}}_{m_j}(x_j) = \sqrt{\frac{2}{\pi}} \cdot \frac{\beta_j \cos\lambda_{m_j} x_j + \varepsilon_{m_j} \operatorname{sign}(\beta_j^2 - \alpha_j^2)\alpha_j \sin\lambda_{m_j} x_j}{\sqrt{\alpha_j^2 + \beta_j^2} \cdot \sqrt{1 + \left|\lambda_{m_j}\right|^{2s_j}}}, \quad m_j \in \mathbb{Z},$$

for $1 \le j \le N$. See the proof in [21].

Fairly next

Theorem 1. Let $\alpha_j \ne 0, \ \beta_j \ne 0, \ \ \left|\alpha_j\right| \ne \left|\beta_j\right|$ be real numbers for each $1 \le j \le N$ and

$$\rho = \max_{1 \le j \le p}\left(\sqrt{\theta_j^2 + 2(\frac{\theta_j}{\sqrt{2}} + (\varphi_j + 1)^{s_j} - 1)^2} \cdot \sigma(s_j)\right) < 1,$$

where $\sigma(0) = \dfrac{1}{\sqrt{2}}, \quad \sigma(s_j) = 1$ for $s_j > 0, \ \ \theta_j = \sqrt{2}\max\limits_{x\in[0,\pi]}\left|e^{i\varphi_j x} - 1\right|, \ \ \lambda_{m_j} = 2m_j + \varepsilon_{m_j} \cdot \varphi_j,$

$\varphi_j = \dfrac{1}{\pi}\arccos\dfrac{-2\alpha_j\beta_j}{\alpha_j^2 + \beta_j^2}, \ \ \varepsilon_{m_j} = \varepsilon_{-m_j} = \pm 1$, for $m_j \in \mathbb{Z}$.

Then the system of eigenfunctions

$$\left\{\psi_{m_1,...,m_N}(x_1, x_2, ..., x_N)\right\}_{(m_1,...,m_N)\in\mathbb{Z}^N} =$$

$$= \left\{\prod_{j=1}^{N}\sqrt{\frac{2}{\pi}} \cdot \frac{\beta_j \cos\lambda_{m_j} x + \varepsilon_{m_j} \cdot \operatorname{sign}(\beta_j^2 - \alpha_j^2)\alpha_j \sin\lambda_{m_j} x_j}{\sqrt{\alpha_j^2 + \beta_j^2} \cdot \sqrt{1 + \left|\lambda_{m_j}\right|^{2s_j}}}\right\}_{(m_1,...,m_p)\in\mathbb{Z}^p} \tag{8}$$

spectral problem (4) – (5) forms a complete orthonormal system in Sobolev classes $\overset{0}{W}{}_2^{s_1,s_2,...,s_N}(\Pi)$.

We will look for eigenfunctions of problem (6), (7) in the form of a product $v(y) = \prod\limits_{i=1}^{N} X_i(y_i)$ Then, to determine each $X_i(y_i), \ i = \overline{1, N}$, we get a one-dimensional spectral problem of the form:

$$X^{(4m)}(x) - \lambda X(x) = 0, \ 0 < x < l, \tag{9}$$

$$X^{(4k)}(0) = X^{(4k+1)}(0) = X^{(4k)}(l) = X^{(4k+1)}(l) = 0, \ \ k = \overline{0, m-1}. \tag{10}$$

Here, for simplicity, $X_i(y_i)$ are denoted by $X(x)$.

We introduce the space $W_2^s(0,l)$ with the norm

$$\| f \|^2_{W_2^s(0,l)} = \| f \|^2_{L_2(0,l)} + \left\| D^s f \right\|^2_{L_2(0,l)},$$

where S is an arbitrary natural number, while $W_2^0(0,l) = L_2(0,l)$.

Denote by L the differential operator generated by the differential expression $l(X)=X^{(4m)}(x)$ on the space $W_2^{4m}(0,l)$ of functions $X(x)$ satisfying the boundary conditions (10). We denote such a space by $\overset{\circ}{V}{}_2^{4m}(0,l)$.

The following

Lemma 1. $LX=X^{(4m)}(x)$ operator with domain

$$D(L)=\left\{X(x):\, X(x)\in W_2^{4m}(0,l),\right.$$
$$\left.X^{(4k)}(0)=X^{(4k+1)}(0)=X^{(4k)}(l)=X^{(4k+1)}(l)=0,\quad k=0,1,\ldots,m-1\right\},$$

is a positive and symmetric operator in the space $L_2(0,l)$.

See the proof in [20].

Let $\lambda=d^{4m},\ \ d>0$. Then for (6) the characteristic equation has the form:

$$\mu^{4m}-d^{4m}=0.$$

The roots of this equation are determined by the formula

$$\mu_j=de^{i\frac{j\pi}{2m}},\ \ j=\overline{0,\,4m-1}.$$

The following equality holds true for the operator L:

$$L-d^{4m}I=\frac{d^{4m}}{dx^{4m}}-d^{4m}I=\prod_{j=0}^{4m-1}\left(\frac{d}{dx}-\mu_jI\right)=\prod_{j=0}^{2m-1}\left(\frac{d^2}{dx^2}-\mu_j^2I\right)=$$
$$=\prod_{j=0}^{m-1}\left(\frac{d^4}{dx^4}-\mu_j^4I\right)=\left(\frac{d^4}{dx^4}-d^4I\right)\prod_{j=1}^{m-1}\left(\frac{d^4}{dx^4}-\mu_j^4I\right). \tag{11}$$

Here $\mu_j^4=d^4\cdot e^{i\frac{2j\pi}{m}},\ j=1,\ldots,m-1$, are not positive numbers. It follows from equality It follows from equality (11) that the operator $LX=X^{(4m)}(x)$ with domain $D(L)$ has its own function $X=X(x)$ if and only if the function $X(x)$ is a non-trivial solution to the problem of the following form:

$$X^{(4)}(x)=d^4X(x),\ \ 0<x<l, \tag{12}$$

$$X(0)=X'(0)=X(l)=X'(l)=0. \tag{13}$$

Indeed, let $X^{(4m)}(x)=d^{4m}X(x),\ X(x)\in D(L)$ and $X(x)\neq 0$ not be a solution to problem (12), (13). Then

$$\prod_{j=0}^{m-1}\left(\frac{d^4}{dx^4}-\mu_j^4I\right)X=0,\ X(x)\in D(L),\ \ X(x)\neq 0$$

or

$$\prod_{j=1}^{m-1}\left(\frac{d^4}{dx^4}-\mu_j^4I\right)\left(\frac{d^4}{dx^4}-d^4I\right)X=0,\ X(x)\in D(L),\ \ X(x)\neq 0.$$

Since the spectral problem (12), (13) has only positive eigenvalues, we have

$$\prod_{j=2}^{m-1}\left(\frac{d^4}{dx^4}-\mu_j^4I\right)\left(\frac{d^4}{dx^4}-d^4I\right)X\equiv 0,\ \ X(x)\in D(L),\ X(x)\neq 0.$$

Similarly, we get

$$\prod_{j=3}^{m-1}\left(\frac{d^4}{dx^4}-\mu_j^4I\right)\left(\frac{d^4}{dx^4}-d^4I\right)X\equiv 0,\ X(x)\in D(L),\ \ X(x)\neq 0.$$

Continuing this process, we get

$$\left(\frac{d^4}{dx^4}-d^4I\right)X\equiv 0,\ \ X(x)\in D(L),\ \ X(x)\neq 0.$$

This contradiction proves our assertion. Thus, the operator $LX = X^{(4m)}(x)$ with the domain of definition $D(L)$ has its own function $X(x)$ only in the case when the function $X(x)$ is a non-trivial solution to problem (12), (13).

Based on the works [17, p. 220-222], [8] and [20] the numbers are positive roots of the transcendental equation

$$chdl \cdot \cos dl = 1. \quad (14)$$

Using a graphical method, it is not difficult to show the existence of a countable set of positive roots of this equation:

$$0 < d_1 < d_2 < ... < d_n < ... ,$$

moreover, for large n the asymptotic formula

$$d_n = \frac{\pi n}{l} + \frac{\pi}{2l} + O\left(e^{-2\pi n}\right). \quad (15)$$

The corresponding system of eigenfunctions has the form

$$\bar{\bar{X}}_n(x) = -\frac{|\sin d_n l| - \sin d_n l \cos d_n l}{1-\cos^2 d_n l}\left(chd_n x - \cos d_n x\right) + shd_n x - \sin d_n x. \quad (16)$$

From formula (16), depending on the evenness and oddness of the number n, we obtain the corresponding system of eigenfunctions orthonormal in $L_2(0,l)$

$$\tilde{X}_n(x) = \begin{cases} \dfrac{1}{\sqrt{l}\left|tg\dfrac{d_n l}{2}\right|}\left(\dfrac{shd_n(x-\frac{l}{2})}{ch\dfrac{d_n l}{2}} - \dfrac{\sin d_n(x-\frac{l}{2})}{\cos\dfrac{d_n l}{2}}\right), & n = 2i,\ i = 1,2,..., \\ \dfrac{1}{\sqrt{l}\left|ctg\dfrac{d_n l}{2}\right|}\left(\dfrac{chd_n(x-\frac{l}{2})}{sh\dfrac{d_n l}{2}} + \dfrac{\cos d_n(x-\frac{l}{2})}{\sin\dfrac{d_n l}{2}}\right), & n = 2i-1,\ i = 1,2,\dots. \end{cases} \quad (17)$$

Since the operator $LX = X^{(4m)}(x)$ with the domain $D(L)$ has its own function $X(x)$ only if the function $X(x)$ is a non-trivial solution to problem (12), (13).

Therefore, we obtain the eigenvalues of problem (9), (10) by the formula $\lambda_n = d_n^{4m},\ n = 1,\ 2,\ \dots$, where d_n is the root of equation (14), and the eigenfunctions by formula (17). Indeed, this can be seen directly by calculating the derivatives of the functions $\tilde{X}_n(x)$, i.e. $\tilde{X}_n^{(4m)}(x) = d_n^{4m}\tilde{X}_n(x),\quad 0 < x < l$ and

$$\tilde{X}_n^{(4k)}(0) = d_n^4\tilde{X}_n^{(4k-4)}(0) = \ldots = d_n^{4k}\tilde{X}_n(0) = 0,\quad k = \overline{1,\ m-1},$$

$$\tilde{X}_n^{(4k+1)}(0) = d_n^4\tilde{X}_n^{(4k-3)}(0) = \ldots = d_n^{4k}\tilde{X}_n'(0) = 0,\quad k = \overline{1,\ m-1},$$

$$\tilde{X}_n^{(4k)}(l) = d_n^4\tilde{X}_n^{(4k-4)}(l) = \ldots = d_n^{4k}\tilde{X}_n(l) = 0,\quad k = \overline{1,\ m-1},$$

$$\tilde{X}_n^{(4k+1)}(l) = d_n^4\tilde{X}_n^{(4k-3)}(l) = \ldots = d_n^{4k}\tilde{X}_n'(l) = 0,\quad k = \overline{1,\ m-1}.$$

Let

$$X_n(x) = \frac{1}{\sqrt{1+d_n^{4s}}}\begin{cases} \dfrac{1}{\sqrt{l}\left|tg\dfrac{d_n l}{2}\right|}\left(\dfrac{shd_n(x-\frac{l}{2})}{ch\dfrac{d_n l}{2}} - \dfrac{\sin d_n(x-\frac{l}{2})}{\cos\dfrac{d_n l}{2}}\right), & n = 2i,\ i = 1,2,..., \\ \dfrac{1}{\sqrt{l}\left|ctg\dfrac{d_n l}{2}\right|}\left(\dfrac{chd_n(x-\frac{l}{2})}{sh\dfrac{d_n l}{2}} + \dfrac{\cos d_n(x-\frac{l}{2})}{\sin\dfrac{d_n l}{2}}\right), & n = 2i-1,\ i = 1,2,\dots \end{cases} \quad (18)$$

the corresponding system of eigenfunctions of problem (9), (10).

The following

Lemma 2. The eigenfunctions $X_n(x)$ of the operator L corresponding to different eigenvalues $\lambda_n = d_n^{4m}, \quad n = 1,2,\dots$ are orthonormal in the class $W_2^{2s}(0,l), \ s = 1, 2, \dots$.

See the proof in [20].

Denote by $\overset{\circ}{V}{}_2^{2s}(0,l)$ the set of all functions $f(x) \in W_2^{2s}(0,l)$ satisfying the boundary conditions $f^{(4k)}(0) = f^{(4k+1)}(0) = f^{(4k)}(l) = f^{(4k+1)}(l) = 0$ as $k = 0, 1, \dots, \left[\frac{s+1}{2}\right] - 1.$

The following

Theorem 2. *The system of eigenfunctions (18) of the spectral problem (9) and (10) is a complete orthonormal system in the Sobolev class* $\overset{\circ}{V}{}_2^{2s}(0,l)$.

The scalar product in the space $W_2^{s_1,s_2,\dots,s_N}(\Pi)$ is introduced as follows:

$$\left(f(x), g(x)\right)_{W_2^{s_1,s_2,\dots,s_N}(\Pi)} = \left(f(x), g(x)\right)_{L_2(\Pi)} +$$
$$+\sum_{j_1=1}^{N}\left(D_{x_{j_1}}^{s_{j_1}} f(x), D_{x_{j_1}}^{s_{j_1}} g(x)\right)_{L_2(\Pi)} + \sum_{1\le j_1<j_2\le N}\left(D_{x_{j_1}}^{s_{j_1}} D_{x_{j_2}}^{s_{j_2}} f(x), D_{x_{j_1}}^{s_{j_1}} D_{x_{j_2}}^{s_{j_2}} g(x)\right)_{L_2(\Pi)} +$$
$$+\dots+\sum_{1\le j_1<j_2<\dots<j_N\le N}\left(D_{x_{j_1}}^{s_{j_1}} D_{x_{j_2}}^{s_{j_2}}\dots D_{x_{j_N}}^{s_{j_N}} f(x), D_{x_{j_1}}^{s_{j_1}} D_{x_{j_2}}^{s_{j_2}}\dots D_{x_{j_N}}^{s_{j_N}} g(x)\right)_{L_2(\Pi)}.$$

Then, respectively, the norm in the space $W_2^{s_1,s_2,\dots,s_N}(\Pi)$ is defined by the formula

$$\left\| f(x)\right\|^2_{W_2^{s_1,s_2,\dots,s_N}(\Pi)} = \left\| f(x)\right\|^2_{L_2(\Pi)} + \sum_{j_1=1}^{N}\left\| D_{x_{j_1}}^{s_{j_1}} f(x)\right\|^2_{L_2(\Pi)} +$$
$$+\sum_{1\le j_1<j_2\le N}\left\| D_{x_{j_1}}^{s_{j_1}} D_{x_{j_2}}^{s_{j_2}} f(x)\right\|^2_{L_2(\Pi)} + \dots + \sum_{1\le j_1<j_2<\dots<j_N\le N}\left\| D_{x_{j_1}}^{s_{j_1}} D_{x_{j_2}}^{s_{j_2}}\dots D_{x_{j_N}}^{s_{j_N}} f(x)\right\|^2_{L_2(\Pi)}.$$

Denote by $\overset{\circ}{V}{}_2^{2s_1,2s_2,\dots,2s_N}(\Pi)$ the set of all functions $f(x) \in W_2^{2s_1,2s_2,\dots,2s_N}(\Pi)$ satisfying the boundary conditions

$$\left.\frac{\partial^{4k_j} f(x)}{\partial x_j^{4k_j}}\right|_{x_j=0} = 0, \quad \left.\frac{\partial^{4k_j+1} f(x)}{\partial x_j^{4k_j+1}}\right|_{x_j=0} = 0, \quad \left.\frac{\partial^{4k_j} f(x)}{\partial x_j^{4k_j}}\right|_{x_j=l} = 0, \quad \left.\frac{\partial^{4k_j+1} f(x)}{\partial x_j^{4k_j+1}}\right|_{x_j=l} = 0$$

at $k_j = \overline{0, \ \left[\frac{s_j+1}{2}\right] - 1}, \quad j = \overline{1, \ N}$.

In the space of $\overset{\circ}{V}{}_2^{2s_1,2s_2,\dots,2s_N}(\Pi)$ functions of N –variables $f(x) = f(x_1,\dots,x_N)$, the complete orthonormal system is formed from all products

$v_{m_1,\dots,m_N}(x_1,\dots,x_N) = X_{m_1}(x_1)\cdot\dots\cdot X_{m_N}(x_N)$,

where

$$X_{m_j}(x_j) = \frac{1}{\sqrt{1+d_{m_j}^{4s_j}}} \times$$

$$\times\begin{cases}\dfrac{1}{\sqrt{l}\left|tg\dfrac{d_{m_j}l}{2}\right|}\left(\dfrac{shd_{m_j}(x_j-\frac{l}{2})}{ch\dfrac{d_{m_j}l}{2}}-\dfrac{\sin d_{m_j}(x_j-\frac{l}{2})}{\cos\dfrac{d_{m_j}l}{2}}\right), & m_j=2k_j,\ k_j=1,2,...,\\ \dfrac{1}{\sqrt{l}\left|ctg\dfrac{d_{m_j}l}{2}\right|}\left(\dfrac{chd_{m_j}(x_j-\frac{l}{2})}{sh\dfrac{d_{m_j}l}{2}}+\dfrac{\cos d_{m_j}(x_j-\frac{l}{2})}{\sin\dfrac{d_{m_j}l}{2}}\right), & m_j=2k_j-1,\ k_j=1,2,\dots,\end{cases} \tag{19}$$

d_{m_j} – is the root of equation (14).

Thus, the following is true.

Theorem 3. System of eigenfunctions

$$\left\{v_{m_1,...,m_N}\left(x_1,...,x_N\right)\right\}_{(m_1,...,m_N)\in\mathbb{N}^N}=\left\{\prod_{j=1}^{N}X_{m_j}(x_j)\right\}_{(m_1,...,m_N)\in\mathbb{N}^N} \tag{20}$$

spectral problem (6), (7) is a complete orthonormal system in the Sobolev class $\overset{\circ}{V}{}_2^{2s_1,2s_2,...,2s_N}(\Pi)$.

Existence and uniqueness of the solution of the initial-boundary problem.

A regular solution of equation (1) in the region $Q=\Pi_1\times\Pi_2\times(0,T)$ is a function $u(x,y,t)$ from the class $u(x,y,t)\in C(\bar{Q})$, $D_{0t}^{\alpha}u(x,y,t)\in C(Q)$, $\dfrac{\partial^{2s-1}u(x,y,t)}{\partial x_j^{2s-1}}\in C(\bar{Q})$, $\dfrac{\partial^{2s}u(x,y,t)}{\partial x_j^{2s}}\in C(Q)$, $\dfrac{\partial^{4m-3}u(x,y,t)}{\partial y_j^{4m-3}}\in C(\bar{Q})$, $\dfrac{\partial^{4m}u(x,y,t)}{\partial y_j^{4m}}\in C(Q)$, $j=1,2,...,N$, and satisfying equation (1) at all points $(x,y,t)\in Q$.

Denote by $\overset{\circ}{V}{}_2^{s_1,s_2,...,s_N;\theta}(Q)$ the set of all functions $u(x,y,t)\in W_2^{s_1,s_2,...,s_N;\theta}(Q)$ satisfying the boundary conditions

$$\left.\frac{\partial^{4k_j}u(x,y,t)}{\partial y_j^{4k_j}}\right|_{y_j=0}=0,\quad \left.\frac{\partial^{4k_j+1}u(x,y,t)}{\partial y_j^{4k_j+1}}\right|_{y_j=0}=0,$$

$$\left.\frac{\partial^{4k_j}u(x,y,t)}{\partial y_j^{4k_j}}\right|_{y_j=l}=0,\quad \left.\frac{\partial^{4k_j+1}u(x,y,t)}{\partial y_j^{4k_j+1}}\right|_{y_j=l}=0$$

at $k_j=\overline{0,\ \left[\dfrac{s_j+3}{4}\right]-1}$, $\quad j=\overline{1,\ N}$.

The function $u(x,y,t)$ is called a regular solution to problem (1)–(3) in the domain ttt if the function $Q=\Pi_1\times\Pi_2\times(0,T)$ is a regular solution to the equation (1) in the domain $Q=\Pi_1\times\Pi_2\times(0,T)$ and satisfies the initial and boundary conditions (2) and (3).

Let the function $u(x,y,t)\in W_2^{s_1,s_2,...,s_N;\,m_1,m_2,...,m_M;\,\theta}(Q)$ with exponent $s_1=s_2=...=s_N\geq 2s+\dfrac{N+M}{2}$, $m_1=m_2=...=m_N\geq 4m+\dfrac{N+M}{2}$, $\theta=-[-\alpha]$, satisfy equation (1) at all points $(x,y,t)\in Q$ and satisfy the initial and boundary conditions (2) and (3).

Then the function $u(x,y,t)$ is a regular solution to problem (1)–(3) in the domain Q.

We introduce the functions

$$T_{n_1,...,n_N;\, m_1,...,m_N}(t) = \int_{\Pi_1} dx \int_{\Pi_2} u(x,y,t)\tilde{\psi}_{n_1,...,n_N}(x)\tilde{v}_{m_1,...,m_N}(y)dy\,, \tag{21}$$

where

$$\left\{\tilde{\psi}_{n_1,...,n_N}(x_1,x_2,...,x_N)\right\}_{(n_1,...,n_N)\in\mathbb{N}^N} =$$

$$=\left\{\prod_{j=1}^{N}\sqrt{\frac{2}{\pi}}\cdot\frac{\beta_j\cos\lambda_{n_j}x+\varepsilon_{n_j}\cdot sign(\beta_j^2-\alpha_j^2)\alpha_j\sin\lambda_{n_j}x_j}{\sqrt{\alpha_j^2+\beta_j^2}}\right\}_{(n_1,...,n_p)\in\mathbb{N}^p} \tag{22}$$

and

$$\tilde{v}_{m_1,...,m_N}(y) = \prod_{j=1}^{N}\tilde{X}_{m_j}(y_j)\,.$$

By virtue of (1)–(3), the unknown functions $T_m(t) = T_{n_1,...,n_N;\, m_1,...,m_N}(t)$ satisfy the equations

$$D_{0t}^{\alpha}T_{n_1,...,n_N;\, m_1,...,m_N}(t) + \lambda_{n_1,...,n_N;\, m_1,...,m_N}T_{n_1,...,n_N;\, m_1,...,m_N}(t) = f_{n_1,...,n_N;\, m_1,...,m_N}(t),\ \ p-1<\alpha\le p,\ \ p\in\mathbb{N}\,, \tag{23}$$

and initial conditions

$$\lim_{t\to 0}D_{0t}^{\alpha-i}T_{n_1,...,n_N;\, m_1,...,m_N}(t) = (\varphi_i)_{n_1,...,n_N;\, m_1,...,m_N}\,,\ \ i=1,2,...,p,\ \ n_j\in\mathbb{N},\ m_j\in\mathbb{N}\,. \tag{24}$$

where

$$f_{n_1,...,n_N;\, m_1,...,m_N}(t) = \int_{\Pi_1} dx \int_{\Pi_2} f(x,y,t)\tilde{\psi}_{n_1,...,n_N}(x)\tilde{v}_{m_1,...,m_N}(y)dy,$$

$$(\varphi_j)_{n_1,...,n_N;\, m_1,...,m_N}(t) = \int_{\Pi_1} dx \int_{\Pi_2} \varphi_j(x,y,t)\tilde{\psi}_{n_1,...,n_N}(x)\tilde{v}_{m_1,...,m_N}(y)dy\,.$$

The solution of the Cauchy problem (22), (23) is known (see, for example, [14, pp. 601–602], [15, pp. 221–223], [16, pp. 16–17]) and it has the form

$$T_{n_1,...,n_N;\, m_1,...,m_N}(t) = \sum_{j=1}^{p}(\varphi_j)_{n_1,...,n_N;\, m_1,...,m_N}\cdot t^{\alpha-j}\cdot E_{\alpha,\alpha-j+1}\left(\mu_{n_1,...,n_N;\, m_1,...,m_N}\cdot t^{\alpha}\right)+$$

$$+\int_0^t (t-\tau)^{\alpha-1}\cdot E_{\alpha,\alpha}\left[\mu_{n_1,...,n_N;\, m_1,...,m_N}\cdot(t-\tau)^{\alpha}\right] f_{n_1,...,n_N;\, m_1,...,m_N}(\tau)\,d\tau\,, \tag{25}$$

where

$$\mu_{n_1,...,n_N;\, m_1,...,m_N} = -\lambda_{n_1,...,n_N;\, m_1,...,m_N} = -\sum_{j=1}^{N} a_j^2(2n_j+\varepsilon_{n_j}\cdot\varphi_j)^2 - \sum_{j=1}^{M} b_j^2 d_{m_j}^{4m}\,, \tag{26}$$

$$E_{\alpha,\alpha-j+1}\left(\mu_{n_1,...,n_N;\, m_1,...,m_N}\cdot t^{\alpha}\right) = \sum_{q=0}^{\infty}\frac{\left(\mu_{n_1,...,n_N;\, m_1,...,m_N}t^{\alpha}\right)^q}{\Gamma(\alpha q+\alpha-j+1)}\,, \tag{27}$$

$$E_{\alpha,\alpha}\left(\mu_{n_1,...,n_N;\, m_1,...,m_N}\cdot(t-\tau)^{\alpha}\right) = \sum_{q=1}^{\infty}\frac{\left(\mu_{n_1,...,n_N;\, m_1,...,m_N}\right)^{q-1}(t-\tau)^{\alpha(q-1)}}{\Gamma(\alpha q)}\,. \tag{28}$$

Since the functions (21) are constructed explicitly (24), then, based on the completeness of the system of eigenfunctions (20) and (22) in $L_2(\Pi)$, it is easy to prove the uniqueness of the solution to problem (1) - (3).

Let $f(x,y,t)\equiv 0$ and $\varphi_i(t)\equiv 0,\ i=\overline{1,p}$. Then formulas (25) and (21) imply that $\int_{\Pi_1} dx \int_{\Pi_2} u(x,y,t)\tilde{\psi}_{n_1,...,n_N}(x)\tilde{v}_{m_1,...,m_N}(y)dy$ for all $n_1,...,n_N\in Z$, $m_1,...,m_N\in\mathbb{N}$ and any $t\in[0,T]$. Hence, due to the completeness of the system of eigenfunctions (20) and (22) in $L_2(\Pi_1\times\Pi_2)$, it follows that $u(x,y,t)=0$ is almost everywhere in the domain $\Pi_1\times\Pi_2$ for any $t\in[0,T]$. As is known, by the Sobolev embedding theorem, the function $u(x,y,t)$ is continuous on $\overline{Q}$, then $u(x,y,t)\equiv 0$ in $\overline{Q}$. This proves the uniqueness of the solution of problem (1) – (3).

For each $t>0$ and $x\in\Pi_1$ the function qqq with respect to the variable $u(x,y,t)$ is a function from the class $u(x,y,t)\in \overset{\circ}{V}{}_2^{m_1,m_2,\dots,m_N}(\Pi_2)$. And also, for each $t>0$ and $y\in\Pi_2$, the function $u(x,y,t)$ with respect to the variable x is a function from the class $u(x,y,t)\in \overset{0}{W}{}_2^{s_1,s_2,\dots,s_N}(\Pi_1)$.

Therefore, considering $t>0$ as a parameter, we will seek the solution of problem (1) – (3) from the class $\overset{0}{W}{}_2^{s_1,s_2,\dots,s_N}(\Pi_1)\times \overset{\circ}{V}{}_2^{m_1,m_2,\dots,m_N}(\Pi_2)$ as the sum of a series in the system of eigenfunctions (22) and (20) of the spectral problem (4), (5) and (6), (7):

$$u(x,y,t)=\sum_{n_1=1}^{\infty}\dots\sum_{n_N=1}^{\infty}\sum_{m_1=1}^{\infty}\dots\sum_{m_N=1}^{\infty}T_{n_1,\dots,n_N;\,m_1,\dots,m_N}(t)\tilde{\psi}_{n_1,\dots,n_N}(x)\tilde{v}_{m_1,\dots,m_N}(y),\qquad(29)$$

where $T_{n_1,\dots,n_N;\,m_1,\dots,m_N}(t)$ –is determined by formula (25).

After substituting (25) into (29), we obtain a unique solution to problem (1) – (3) in the form of a series

$$u(x,y,t)=\sum_{n_1=1}^{\infty}\dots\sum_{n_N=1}^{\infty}\sum_{m_1=1}^{\infty}\dots\sum_{m_N=1}^{\infty}\Bigg[\sum_{j=1}^{p}(\varphi_j)_{n_1,\dots,n_N;\,m_1,\dots,m_N}\,t^{\alpha-j}\cdot E_{\alpha,\alpha-j+1}\left(\mu_{n_1,\dots,n_N;\,m_1,\dots,m_N}\cdot t^{\alpha}\right)+$$

$$+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left[\mu_{n_1,\dots,n_N;\,m_1,\dots,m_N}\cdot(t-\tau)^{\alpha}\right]f_{n_1,\dots,n_N;\,m_1,\dots,m_N}(\tau)\,d\tau\Bigg]\cdot\tilde{\psi}_{n_1,\dots,n_N}(x)\tilde{v}_{m_1,\dots,m_N}(y)\qquad(30)$$

Let us find conditions for the existence of a solution from the class $\overset{0}{W}{}_2^{s_1,s_2,\dots,s_N}(\Pi_1)\times \overset{\circ}{V}{}_2^{m_1,m_2,\dots,m_N;\,\theta}(\Pi_2)$. The following

Theorem 4. Let $\alpha_j\neq 0,\ \beta_j\neq 0,\ \ |\alpha_j|\neq|\beta_j|$ be real numbers for each $1\le j\le N$ and

$$\rho=\max_{1\le j\le p}\left(\sqrt{\theta_j^2+2\Big(\frac{\theta_j}{\sqrt{2}}+(\varphi_j+1)^{s_j}-1\Big)^2}\cdot\sigma(s_j)\right)<1,$$

where $\sigma(0)=\dfrac{1}{\sqrt{2}},\quad \sigma(s_j)=1$ for $s_j>0$, $\theta_j=\sqrt{2}\max\limits_{x\in[0,\pi]}\left|e^{i\varphi_j x}-1\right|$, $\lambda_{m_j}=2m_j+\varepsilon_{m_j}\cdot\varphi_j$,

$\varphi_j=\dfrac{1}{\pi}\arccos\dfrac{-2\alpha_j\beta_j}{\alpha_j^2+\beta_j^2}$, $\varepsilon_{m_j}=\varepsilon_{-m_j}=\pm1$, for $m_j\in\square$, $s_j\ge 2+\dfrac{N}{2}$ and let the initial functions $\varphi_i(x,y),\ \ i=1,\,2,\,\dots,\,p$ and the right side $f(x,y,t)$ satisfy the condition

$$\sum_{n_1=1}^{\infty}\dots\sum_{n_N=1}^{\infty}\sum_{m_1=1}^{\infty}\dots\dots\sum_{m_M=1}^{\infty}\Bigg|\sum_{j=1}^{p}(\varphi_j)_{n_1,\dots,n_N;\,m_1,\dots,m_N}\,t^{\alpha-j}\cdot E_{\alpha,\alpha-j+1}\left(\mu_{n_1,\dots,n_N;\,m_1,\dots,m_N}\cdot t^{\alpha}\right)+$$

$$+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left[\mu_{n_1,\dots,n_N;\,m_1,\dots,m_N}\cdot(t-\tau)^{\alpha}\right]f_{n_1,\dots,n_N;\,m_1,\dots,m_N}(\tau)\,d\tau\cdot\prod_{k=1}^{N}\left(1+\lambda_{n_k}^{2s_k}\right)\Bigg|^2\cdot\prod_{k=1}^{M}\left(1+d_{m_k}^{2m_k}\right)<\infty.\ (31)$$

with indicators

$$s_1=s_2=\dots=s_N>2s+\frac{N}{2},\quad m_1=m_2=\dots=m_N>4m+\frac{N}{2},\qquad \theta=-[-\alpha]$$

at every $t>0$. Then a regular solution to problem (1) – (3) from the class $\overset{0}{W}{}_2^{s_1,s_2,\dots,s_N}(\Pi_1)\times \overset{\circ}{V}{}_2^{m_1,m_2,\dots,m_N;\,\theta}(\Pi_2)$ exists, is unique, and is represented as a series (30).

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