On Joint Unknown Input and Sliding Mode Estimation

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***Abstract*— We address state estimation in the presence of faults and unknown disturbances combining unknown input observers (UIOs) and sliding mode observers. We consider a well-established UIO design for linear time-invariant systems and augment it with a nonlinear sliding mode action. This latter term deals with matched disturbances affecting the actuation channels, such as actuator faults, while the UIO provides geometric decoupling from the remaining exogenous inputs. We thoroughly present the analysis of the proposed observer, together with existence conditions stemming from the joint design. We also investigate how our design geometrically relates with other known results in the field of unknown-input state estimation, and discuss its benefits and pitfalls. An advantage of our design is that it allows reconstruction of the fault in finite time, under just boundedness assumptions, while other disturbances are rejected by the UIO. Numerical simulations show the effectiveness of the proposed method.**

***Index Terms*— Observers for linear systems, fault estimation, uncertain systems.**

I. Introduction

Unknown input observers (UIOs) are useful tools for estimating the state of a system affected by disturbances, and they have found particular interest in several areas, ranging from fault detection and isolation [1], [2] to cyber-security [3] in a wide spectrum of applications [4], [5]. Sliding mode observers [6], on the other hand, have also acquired popularity for fault detection and reconstruction in several applications [7]–[10] and even for control system security [11]. Both unknown-input and sliding mode observers allow for the reconstruction of the disturbances that are being rejected [7], [10], [12], however their feasibility involves satisfaction of certain rank conditions (cf. [2], [6]).

In this paper, we distinguish between *matched* and *unmatched* disturbances, namely entering the system dynamics through the input (e.g., actuator faults) and some other disjoint channels, respectively. This leads to our motivating question for this work: *is it possible to devise an alternative design based on UIO and sliding mode observers such that the geometric conditions for decoupling can be relaxed?* To this end, we propose a novel design of a UIO endowed with a sliding mode nonlinear action, with the aim to achieve decoupling from unmatched disturbances through the linear gains, while the nonlinear term guarantees rejection of matched disturbances. We investigate the design of the UIO in the coordinate frame induced by the transformations needed for the sliding mode design. As for the nonlinear term, we choose the unit vector approach [6], whose need for a Lyapunov candidate leads us to the formulation of a linear matrix inequality (LMI) condition to obtain the observer gains. Error analysis is conducted and the relationship between observer gains in the two frames is discussed. This coupling between the two observer strategies entails intertwined existence conditions that we also investigate and reformulate in terms of the original system matrices. We reach the conclusion that the necessary geometric conditions needed for our proposed observer are the same as those for a UIO for an equivalent system with lumped disturbances.

Sliding mode approaches to the problem of estimation with unknown inputs have been considered in [13] and [14]. However, the main contribution of this paper is distinct from these works. Notably, in [13] a sliding mode design to deal with the unknown inputs under weaker rank conditions is proposed, whereas dealing with unmatched disturbance is not addressed.

In terms of benefits, our proposed method comes with the advantages of the two observer strategies it implements: no assumptions regarding the bounds of the unmatched disturbances need to be made (unlike for pure sliding mode strategy). As for the matched disturbance and for the purpose of fault reconstruction, we benefit from the sliding mode, i.e., finite-time convergence, robustness, and milder conditions on the fault signal to be reconstructed [15]. For instance, sliding mode reconstruction does not require state augmentation [12], [16] or explicit derivatives of the measurements [17].

Throughout the paper, the following notations are considered. I_n stands for the $n \times n$ identity matrix. $\mathbf{0}_{n\times m}$ is an $n \times m$ all-zeros matrix. $|\cdot|$ denotes the standard 2-norm. For a matrix $A \in \mathbb{R}^{n\times m}$, $A^\dagger$ stands for the pseudo inverse of A such that if A is full row rank, $A^\dagger = A^\top(AA^\top)^{-1}$ and if A is full column rank, $A^\dagger = (A^\top A)^{-1}A^\top$. We denote a positive (negative) definite matrix M as $M \succ (\prec)\ 0$. Im and Ker are the image and the kernel (or null space) of a matrix, respectively. $\dim(\mathscr{V})$ is the dimension of the space $\mathscr{V}$, and $\oplus$ denotes the direct sum of two linear subspaces.

The rest of the paper is organized as follows. In Section II, we state the problem and lay some basic assumptions necessary for the robust UIO design. In Section III, we obtain the

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robust UIO representation and provide convergence analysis. In Section IV, we analyze algebraic conditions involved in the design of the robust UIO. A numerical simulation of an aircraft is given in Section V, and finally the conclusions are drawn in Section VI. The paper is supplemented with an appendix with detailed proofs and simulation parameters.

II. Problem Formulation and Objectives

We consider a linear time-invariant system described by the following dynamical equation:

$$\begin{aligned} \dot{x} &= Ax + B(u+\phi) + Dw, \\ y &= Cx, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input, $\phi \in \mathbb{R}^m$ is a *matched* disturbance (for instance an actuator fault), $w \in \mathbb{R}^q$ is an unknown (and *unmatched*) external disturbance, and $y \in \mathbb{R}^p$ is the measurement/output vector. The vector dimensions are such that $m \leq p \leq n$. The fact that we distinguish between matched and unmatched disturbances translates into the condition

$$\operatorname{Im} B \cap \operatorname{Im} D = 0. \tag{2}$$

The following assumptions are made for the dynamical system (1).

Assumption 1: The disturbance ϕ is bounded, i.e.,

$$|\phi(t)| \leq \phi^*,$$

for all t, where ϕ^* is a known scalar. ◁

Assumption 2: The matrix D is full column rank. ◁

Assumption 3: The following condition holds:

$$\operatorname{rank} CD = \operatorname{rank} D. \tag{3}$$

◁

We remark that Assumption 2 does not harm generality, as it is always possible to obtain a full column rank representation by means of singular value decomposition and by grouping the linearly dependent components in a new disturbance w'. These last three assumptions are functional towards the existence of a UIO for (1). In particular, Assumption 3 is a necessary condition for the existence of the standard UIO [2]. Detectability of a certain matrix pair is also needed but it is embedded in the feasibility of an LMI in our presentation, as we discuss later. As in the standard UIO design [2], to decouple w it is needed that

$$\begin{aligned} (I_n - HC)D &= \mathbf{0}_{n\times 1}, \\ M &= I_n - HC, \end{aligned} \tag{4}$$

which under Assumption 3 has the following solution:

$$H = D(CD)^{\dagger}.$$

We aim to design an observer $\mathcal{O}$ for (1) producing an estimate $\hat{x}$ of x, that satisfies the following observer definition.

Definition 1: Let $e = x - \hat{x}$ be the estimation error. We say $\mathcal{O}$ is an observer for (1) if

$$\lim_{t\to\infty} e(t) = \mathbf{0}_{n\times 1},$$

for all initial conditions x_0 and disturbances w and ϕ. ◁

Based on the estimation error and by using the equivalent output injection method [7], it will be possible to estimate the value of the fault signal ϕ, that will be discussed later.

In the next section, we propose a UIO design whose error is decoupled with w and which uses a nonlinear robustifying term in order to deal with the fault ϕ.

III. Robust UIO Design

In this section, we provide design details and convergence properties of the robust UIO. Before doing so, we go through several changes of coordinates, closely following the presentation of sliding mode theory [6]. In this new coordinate frame, the observer is capable to reject the matched disturbances thanks to the nonlinear term, whereas the geometric decoupling guaranteed by the UIO structure still holds. To this end, we make the following additional assumption [6], under which the aforementioned transformation is feasible.

Assumption 4: The following holds:

$$\operatorname{rank} CMB = m. \tag{5}$$

◁

We first consider a similarity transformation

$$U = \begin{bmatrix} \operatorname{Ker} C & C^\top \end{bmatrix}^\top,$$

which takes the system in the form (cf. [6, Ch. 5.4])

$$\begin{aligned} \dot{\zeta} &= \mathcal{A}\zeta + \mathcal{B}(u+\phi) + \mathcal{D}w, \\ y &= \mathcal{C}\zeta. \end{aligned} \tag{6}$$

In particular, $\mathcal{D} = UD$ and $\mathcal{C} = CU^{-1}$ where $\mathcal{C} = \begin{bmatrix} \mathbf{0}_{p\times(n-p)} & I_p \end{bmatrix}$. The observer to be designed is intended for the system (6), however, since U defines an isomorphism, convergence of the estimation error holds irrespective of the coordinate frame.

Proceeding with a typical UIO design for (6), the decoupling condition (4) becomes

$$(I_n - \mathcal{H}\mathcal{C})\mathcal{D} = \mathbf{0}_{n\times 1}, \tag{7a}$$

$$\mathcal{M} = I_n - \mathcal{H}\mathcal{C}. \tag{7b}$$

It can be immediately verified that $\operatorname{rank} \mathcal{C}\mathcal{D} = \operatorname{rank} CD$. Moreover, since U is full rank, one can observe that $\operatorname{rank} \mathcal{D} = \operatorname{rank} D$ and as a result, condition (3) also holds in this frame, i.e., $\operatorname{rank}(\mathcal{C}\mathcal{D}) = \operatorname{rank} \mathcal{D}$. Therefore, the following solution for $\mathcal{H}$ can be obtained [2]:

$$\mathcal{H} = \mathcal{D}(\mathcal{C}\mathcal{D})^{\dagger} = UH.$$

As far as (5) is concerned, noting that $\mathcal{M} = UMU^{-1}$, it also holds that $\operatorname{rank} \mathcal{C}\mathcal{M}\mathcal{B} = m$. Let now $S = \mathcal{M}\mathcal{B}$ be partitioned as follows:

$$S = \mathcal{M}\mathcal{B} = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix},$$

with $S_1 \in \mathbb{R}^{(n-p)\times m}$ and $S_2 \in \mathbb{R}^{p\times m}$. Then, since $\operatorname{rank} \mathcal{C}\mathcal{M}\mathcal{B} = m$, there exists a similarity transformation T

given by [6, Ch. 5.4]

$$T = \begin{bmatrix} I_{n-p} & -S_1 S_2^{\dagger} \\ \mathbf{0}_{p\times(n-p)} & \mathcal{T}^\top \end{bmatrix},$$

where $S_2^{\dagger}$ is the left pseudo-inverse of S_2, and $\mathcal{T} \in \mathbb{R}^{p\times p}$ is orthogonal and satisfies

$$\mathcal{T}^\top S_2 = \begin{bmatrix} \mathbf{0}_{(p-m)\times m} \\ \bar{S} \end{bmatrix}, \tag{8}$$

with $\bar{S} \in \mathbb{R}^{m\times m}$ nonsingular. Moreover,

$$TS = \begin{bmatrix} \mathbf{0}_{(n-p)\times m} \\ \mathcal{T}^\top S_2 \end{bmatrix}. \tag{9}$$

Through T, we define a new vector $\xi = T\zeta$ applied to (6) and the composite transformation $R = TU$. Hence, in the new coordinates, the robust UIO has the following form:

$$\begin{aligned} \dot{z} &= \bar{N}z + TSu + \bar{L}y + \begin{bmatrix} \mathbf{0}_{(n-p)\times p} \\ I_p \end{bmatrix} \nu, \\ \hat{\xi} &= z + \bar{H}y, \end{aligned} \tag{10}$$

with $\bar{N} = TNT^{-1}$, $N = \mathcal{M}\mathcal{A} - K\mathcal{C}$, $\bar{L} = T(K + N\mathcal{H})$, and $\bar{H} = T\mathcal{H}$, where K is the observer gain to be designed. Moreover, ν is a robustifying term designed as follows:

$$\nu = \begin{cases} \rho|\bar{S}|\dfrac{P_2 e_2}{|P_2 e_2|} & \text{if } e_2 \neq \mathbf{0}_{p\times 1}, \\ \mathbf{0}_{p\times 1} & \text{otherwise,} \end{cases} \tag{11}$$

where ρ and P_2 are design parameters and

$$e_2 = \mathcal{T}^\top (y - CR^{-1}\hat{\xi}). \tag{12}$$

It can be verified (see Appendix B) that the vector e_2 collects the last p components of the estimation error, i.e.,

$$e_2 = \begin{bmatrix} \mathbf{0}_{p\times(n-p)} & I_p \end{bmatrix} e. \tag{13}$$

They are the components in the image of C and those affected by the nonlinear term.

We now present the main result of the paper, showing the stability of the proposed observer following standard Lyapunov arguments.

Theorem 1: System (10) is an observer for (1) in the sense of Definition 1 if $\rho > \phi^*$ and if there exist a block-diagonal matrix $P = \operatorname{diag}(P_1, P_2) \succ 0$, with $P_1 \in \mathbb{R}^{(n-p)\times(n-p)} \succ 0$ and $P_2 \in \mathbb{R}^{p\times p} \succ 0$, and a matrix $\bar{K}$ such that

$$\begin{aligned} &PT\bar{\mathcal{A}}T^{-1} + T^{-\top}\bar{\mathcal{A}}^\top T^\top P \\ &- \bar{K}\mathcal{C}T^{-1} - T^{-\top}\mathcal{C}^\top \bar{K}^\top \prec 0, \end{aligned} \tag{14}$$

where $\bar{\mathcal{A}} = \mathcal{M}\mathcal{A}$. In this condition, if such a solution exists, then the gain of the UIO is given by $K = T^{-1}P^{-1}\bar{K}$.

Proof: Let $e = \xi - \hat{\xi}$. By considering (1) and (10) and from (7), it can be shown that (see Appendix A) the evolution of the estimation error e is given by

$$\dot{e} = \bar{N}e + TS\phi - \begin{bmatrix} \mathbf{0}_{(n-p)\times p} \\ I_p \end{bmatrix} \nu. \tag{15}$$

To prove the convergence of the estimation errors to zero, we choose the Lyapunov function $V = e^\top Pe$. Evaluating $\dot{V}$ along the trajectories of (15) gives

$$\begin{aligned} \dot{V} &= e^\top (P\bar{N} + \bar{N}^\top P)e \\ &+ 2e^\top \left(PTS\phi - P \begin{bmatrix} \mathbf{0}_{(n-p)\times p} \\ I_p \end{bmatrix} \nu \right). \end{aligned} \tag{16}$$

From (9) and (13) and thanks to the block-diagonal structure of P, one obtains

$$\begin{aligned} e^\top PTS\phi &= e_2^\top P_2 \mathcal{T}^\top S_2 \phi, \\ e^\top P \begin{bmatrix} \mathbf{0}_{(n-p)\times p} \\ I_p \end{bmatrix} \nu &= e_2^\top P_2 \nu. \end{aligned} \tag{17}$$

According to (17) and by substituting (11) into it, one gets

$$\begin{aligned} &e^\top \left(PTS\phi - P \begin{bmatrix} \mathbf{0}_{(n-p)\times p} \\ I_p \end{bmatrix} \nu \right) \\ &\leq |e_2||P_2||\mathcal{T}^\top S_2||\phi| - \rho|\bar{S}|e_2^\top P_2 \frac{P_2 e_2}{|P_2 e_2|}. \end{aligned} \tag{18}$$

By considering Assumption 1 and (8), from (18) we have

$$\begin{aligned} &e^\top \left(PTS\phi - P \begin{bmatrix} \mathbf{0}_{(n-p)\times p} \\ I_p \end{bmatrix} \nu \right) \\ &\leq -(\rho - \phi^*)|\bar{S}||P_2||e_2|. \end{aligned} \tag{19}$$

Now, since $\rho > \phi^*$, by considering (19), (16) becomes

$$\dot{V} \leq e^\top (P\bar{N} + \bar{N}^\top P)e. \tag{20}$$

According to the definition of $\bar{N}$ and $\bar{\mathcal{A}}$, by expanding $P\bar{N} + \bar{N}^\top P$, we obtain

$$\begin{aligned} P\bar{N} + \bar{N}^\top P &= PT\bar{\mathcal{A}}T^{-1} + (T\bar{\mathcal{A}}T^{-1})^\top P \\ &- PTK\mathcal{C}T^{-1} - (TK\mathcal{C}T^{-1})^\top P. \end{aligned} \tag{21}$$

Finally, by defining $\bar{K} = PTK$, from (14) and (21), it follows that $\dot{V}$ in (20) is negative definite, then the estimation error converges to zero (regardless of initial conditions, ϕ, and w), which concludes the proof. ■

Based on the results of Theorem 1, an estimate of the state vector x has been obtained. Now, based on the term e_2, we can reconstruct the fault signal ϕ by using the equivalent output injection method as follows [7]:

$$\hat{\phi} = \rho|S_2|S_2^{\dagger} \frac{P_2 e_2}{|P_2 e_2| + \delta}, \tag{22}$$

where $\hat{\phi}$ is the estimate of the fault ϕ and δ is a positive scalar parameter in the continuous approximation of ν.

Remark 1: Since $\bar{K} = PTK$, the LMI (14) can be restated as follows:

$$\begin{aligned} &PT\bar{\mathcal{A}}T^{-1} + T^{-\top}\bar{\mathcal{A}}^\top T^\top P \\ &- PTK\mathcal{C}T^{-1} - T^{-\top}\mathcal{C}^\top K^\top T^\top P \prec 0. \end{aligned} \tag{23}$$

According to (23) and the Lyapunov stability criterion, the necessary condition for the LMI (14) to have a solution is that $T\bar{\mathcal{A}}T^{-1} + TK\mathcal{C}T^{-1}$ should be Hurwitz stable for some K, that is, $\bar{\mathcal{A}} + K\mathcal{C}$ should be Hurwitz stable for some K. Therefore, the necessary condition for solvability of the LMI (14) is that the pair $(\mathcal{C}, \bar{\mathcal{A}})$ should be detectable. Notice that this condition is equivalent – in the transformed coordinates – to that in [2, Thm. 1]; however, given the particular block-diagonal structure of P in Theorem 1, it is only necessary in our case. ◁

The design presented in this section relies on the condition (5), which in turn requires computing the matrix M, whose characterization depends on the decoupling conditions (4). A simpler characterization in terms of the original system (1) matrices is given in the next section.

IV. Geometric Characterization

In this section, we mainly address our initial question whether the geometric condition needed for the design of the proposed robust UIO are any weaker than the ones needed to reject both the input and the disturbance using a UIO (or equivalently a sliding mode observer). As anticipated, we prove here that the proposed design is in fact equivalent in terms of geometric conditions (see Remark 1 for a short discussion on the detectability condition).

To prove this, consider the "lumped disturbance" system derived from (1):

$$\begin{aligned} \dot{x} &= Ax + \bar{D}v, \\ y &= Cx, \end{aligned} \tag{24}$$

where $\bar{D} = \begin{bmatrix} B & D \end{bmatrix}$ and $v = \begin{bmatrix} u + \phi^\top & w^\top \end{bmatrix}^\top$. In the next theorem, we show that the rank conditions necessary for our proposed mixed design are in fact equivalent to the necessary rank conditions for the existence of a UIO for (24).

Theorem 2: Assumptions 3 and 4 are equivalent to the rank condition for the existence of a UIO for (24), that is,

$$\begin{cases} \operatorname{rank} CD = \operatorname{rank} D \\ \operatorname{rank} CMB = m \end{cases} \iff \operatorname{rank} C\bar{D} = \operatorname{rank} \bar{D}.$$

Proof: We prove the theorem in two parts. As a common step for both parts, due to Sylvester's rank formula [18, Ch. 3], the following equivalence holds:

$$\operatorname{rank} \bar{D} = \operatorname{rank} C\bar{D} \iff \operatorname{Ker} C \cap \operatorname{Im} \begin{bmatrix} B & D \end{bmatrix} = 0. \tag{25}$$

The right-hand side of (25) amounts to satisfying at the same time

$$\begin{cases} \operatorname{Ker} C \cap \operatorname{Im} D = 0, & \text{(26a)} \\ \operatorname{Ker} C \cap \operatorname{Im} B = 0. & \text{(26b)} \end{cases}$$

The following identity is also needed for the rest of the proof:

$$\begin{aligned} CM &= C(I_n - HC) = (I_p - CH)C \\ &= (I_p - QQ^\dagger)C, \end{aligned} \tag{27}$$

where $Q = CD$.

(Sufficiency) Condition (26a) is trivially satisfied by hypothesis. For (26b), we need to show that

$$\operatorname{rank} CMB = m \implies \operatorname{Ker} C \cap \operatorname{Im} B = 0. \tag{28}$$

We prove by contrapositive: assume that there exists a nonzero $v \in \operatorname{Ker} C \cap \operatorname{Im} B$, then it is immediately seen that

$$CMBv = (I_p - QQ^\dagger)CBv = \mathbf{0}_{p\times 1},$$

which contradicts the fact that $\operatorname{rank} CMB = m$. Therefore, it must be that $\operatorname{Ker} C \cap \operatorname{Im} B = 0$. Having verified both conditions (26), by equivalence with (25), we have proved sufficiency.

(Necessity) Assume now that $\operatorname{rank} C\bar{D} = \operatorname{rank} \bar{D}$. By reversing our previous argument, (26a) is equivalent to $\operatorname{rank} CD = \operatorname{rank} D$, which completes the first part of the necessity proof. Now we prove that if (26a) holds, then

$$\operatorname{Ker} C \cap \operatorname{Im} B = 0 \implies \operatorname{rank} CMB = m. \tag{29}$$

Assume that $\operatorname{Ker} C \cap \operatorname{Im} B = 0$ but $\operatorname{rank} CMB < m$. From (27), we have that

$$CMB = (I_p - QQ^\dagger)CB,$$

then we can find a nonzero vector $w \in \operatorname{Im} CB$ such that $(I_p - QQ^\dagger)w = \mathbf{0}_{p\times 1}$, or equivalently $w = QQ^\dagger w$, i.e, $w \in \operatorname{Im} CD$. We have found a nonzero vector that is both $\operatorname{Im} CD$ and $\operatorname{Im} CB$, which contradicts (25) under (2) and (26a) (it is easy to see that in this case (25) is equivalent to $\operatorname{Im} CD \cap \operatorname{Im} CB = 0$). Therefore, it must be that $\operatorname{rank} CMB = m$ as well, which concludes the proof. ■

Remark 2: Notice that (28) and (29) cannot be combined in an equivalence statement, because (29) depends on (26a) holding. ◁

The proof of Theorem 2 also gives us further geometric insight on system *design*. In fact, although checking condition (5) is straightforward, its link with the system matrices is not as obvious. The problem is then the following: when (5) is not satisfied, what adjustments could the system engineer make to the sensing and actuation equipment in order to verify it?

Since the necessary geometric conditions are equivalent for both designs, conditions (26a) and (26b) are in fact as well *necessary* for the feasibility of the proposed design. Hence, for a certain matrix D, a system designer might have the freedom to choose B, or at least C, so that the proposed observer structure is possible to find through the transformations described in Sec. III.

To conclude the section, as stressed in the Introduction, the proposed method has the advantages of the sliding mode approach when rejecting and reconstructing matched disturbances, i.e., robustness, convergence in finite time, and only requiring boundedness of ϕ for fault reconstruction (state augmentation methods assume vanishing higher order derivatives [12]), while imposing no constraints on w.

V. Simulation Results

In this section, the effectiveness of the proposed robust UIO is evaluated via a numerical example. In particular, we consider the aircraft model presented in [6, Ch. 4.5], whose system parameters are reported for completeness in Appendix C. For the reader's convenience, we recall that the components of the state vector $x \in \mathbb{R}^5$ are, in order, pitch angle (rad), pitch rate (rad/s), angle of attack (rad), elevator deflection (rad), and flap deflection (rad). The control variable $u \in \mathbb{R}^2$, instead, consists of the elevator command (rad)

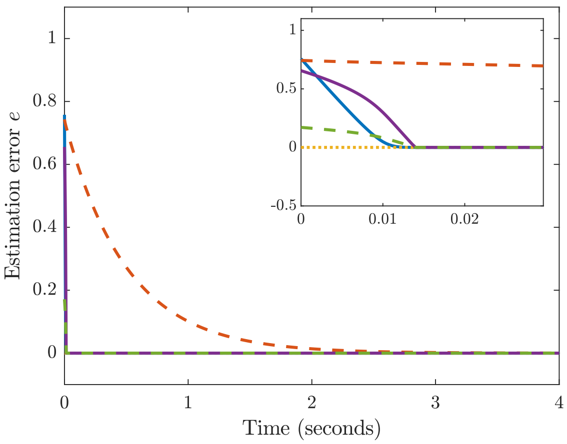

Fig. 1. Estimation error for each state component of the state. In the top right corner, an expanded inset better displays the initial transient.

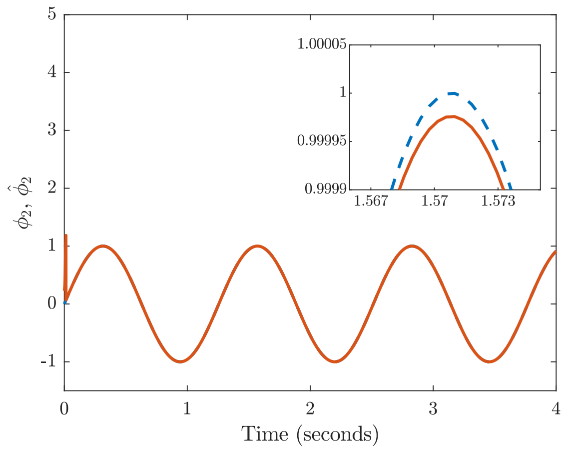

Fig. 2. Comparison of the fault signal (dashed line) and its estimate (solid line).

and the flap command (rad). In addition, we also include the unmatched disturbance w into the design affecting the angle of attack, and that we model it as a random signal uniformly distributed in the interval $[-1, 1]$.

We consider ϕ affecting the second component of the control u to be a sine wave with amplitude 1 and angular frequency 5 rad/s, hence $\phi^* = 1$. We choose this function as an example of a fault signal that has nonzero derivatives of any order, for which an augmented observer would perform poorly (cf. [15]). The initial condition for the system is randomly chosen in the interval $[0, 1]$ for each component of the state. Moreover, the nonlinear gain is set to $\rho = 4$.

Without loss of generality, the control input is selected as $u = -Fx$, where $F \in \mathbb{R}^{2\times 5}$ (see Appendix C) is obtained by pole assignment. The closed-loop eigenvalues are selected as in [6] to be $\{-5.6 \pm i4.2, -1.0, -20.0, -20.0\}$. The UIO is built based on the proposed scheme in Theorem 1 where P_2 and $\bar{K}$ are obtained by solving (14). The other parameters are derived by the transformation described in Sec. III.

The system is simulated using Simulink® over a time horizon of 4 s, using a Bogacki–Shampine variable step solver with tolerance 10^{-4}. The estimation error is shown in Fig. 1. By considering (22) with $\delta = 10^{-3}$ and by using the estimated states, we also provide an estimate $\hat{\phi}$ of the fault ϕ. The reconstructed fault is shown in Fig. 2, verifying the accuracy of the proposed method.

VI. Concluding Remarks

A UIO augmented with a nonlinear sliding mode term was presented in this paper. While both methodologies are precious tools available to the FDI practitioner, their joint design and stability properties were investigated here. Furthermore, we studied the relationship between our mixed approach and a classic UIO for a "lumped disturbance" system, showing that the geometric conditions enabling our design are in fact equivalent to those for the UIO design. Nevertheless, the proposed observer has the advantages of sliding mode estimation, and can be useful for reconstructing certain classes of faults where other methods may not perform as well. Alternative designs that achieve this result with relaxed constraints are object of future research.

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APPENDIX

A. Derivation of (15)

Since $e = \xi - \hat{\xi}$, from (10), it follows that

$$e = \xi - z - \bar{H}y. \tag{30}$$

Since $y = \mathcal{C}T^{-1}\xi$ and $\bar{H} = T\mathcal{H}$, (30) yields

$$e = T(I - \mathcal{H}\mathcal{C})T^{-1}\xi - z,$$

which by considering (7b), can be restated as

$$e = T\mathcal{M}T^{-1}\xi - z. \tag{31}$$

According to (6) and since $\zeta = T^{-1}\xi$, one gets

$$\dot{\xi} = T\mathcal{A}T^{-1}\xi + T\mathcal{B}(u + \phi) + T\mathcal{D}w. \tag{32}$$

According to the definition of $\bar{\mathcal{A}}$, by taking the time derivative of (31) along (10) and (32), we have

$$\begin{aligned}\dot{e} =& T\bar{\mathcal{A}}T^{-1}\xi + T\mathcal{M}\mathcal{B}(u + \phi) + T\mathcal{M}\mathcal{D}w \\ &- \bar{N}z - TSu - \bar{L}y - \begin{bmatrix}\mathbf{0}_{(n-p)\times p} \\ I_p\end{bmatrix}\nu.\end{aligned} \tag{33}$$

By adding and subtracting $T\bar{\mathcal{A}}T^{-1}\hat{\xi}$ to the right-hand side of (33), by considering (7) and according to the definition of S, we obtain

$$\begin{aligned}\dot{e} =& T\bar{\mathcal{A}}T^{-1}e + T\bar{\mathcal{A}}T^{-1}\hat{\xi} + T\mathcal{S}\phi \\ &- \bar{N}z - \bar{L}y - \begin{bmatrix}\mathbf{0}_{(n-p)\times p} \\ I_p\end{bmatrix}\nu.\end{aligned} \tag{34}$$

Recall that $\bar{N} = TNT^{-1}$, $N = \bar{\mathcal{A}} - K\mathcal{C}$, $\bar{L} = T(K + N\mathcal{H})$, and $\bar{H} = T\mathcal{H}$. These relationships lead to the following:

$$\begin{aligned}T\bar{\mathcal{A}}T^{-1}\hat{\xi} - \bar{N}z - \bar{L}y =& T\bar{\mathcal{A}}T^{-1}\hat{\xi} - T(\bar{\mathcal{A}} - K\mathcal{C})T^{-1}z \\ &- T\big(K + (\bar{\mathcal{A}} - K\mathcal{C})T^{-1}\bar{H}\big)y,\end{aligned}$$

which since $z = \hat{\xi} - \bar{H}y$, can be simplified as follows:

$$T\bar{\mathcal{A}}T^{-1}\hat{\xi} - \bar{N}z - \bar{L}y = TK\mathcal{C}T^{-1}\hat{\xi} - TKy. \tag{35}$$

Moreover, since $y = \mathcal{C}T^{-1}\xi$, by the definition of error, it follows from (35) that

$$T\bar{\mathcal{A}}T^{-1}\hat{\xi} - \bar{N}z - \bar{L}y = -TK\mathcal{C}T^{-1}e. \tag{36}$$

Finally, by substituting (36) into (34) and according to the definition of N, we finally obtain (15).

B. Proof that $e_2 = \begin{bmatrix}\mathbf{0}_{p\times(n-p)} & I_p\end{bmatrix} e$

From (12) and since $R^{-1} = U^{-1}T^{-1}$ and $x = U^{-1}T^{-1}\xi$, we have

$$e_2 = \mathcal{T}^\top CU^{-1}T^{-1}(\xi - \hat{\xi}). \tag{37}$$

We recall that $CU^{-1} = \mathcal{C} = \begin{bmatrix}\mathbf{0}_{p\times(n-p)} & I_p\end{bmatrix}$. Therefore, according to the definition of T one gets

$$CU^{-1}T^{-1} = \begin{bmatrix}\mathbf{0}_{p\times(n-p)} & \mathcal{T}\end{bmatrix}. \tag{38}$$

From (37) and (38) it follows that

$$e_2 = \mathcal{T}^\top \begin{bmatrix}\mathbf{0}_{p\times(n-p)} & \mathcal{T}\end{bmatrix}(\xi - \hat{\xi}). \tag{39}$$

By considering (39) and according to the definition of e, one gets

$$e_2 = \mathcal{T}^\top\mathcal{T}\begin{bmatrix}\mathbf{0}_{p\times(n-p)} & I_p\end{bmatrix} e,$$

and since $\mathcal{T}^\top\mathcal{T} = I$, we finally obtain

$$e_2 = \begin{bmatrix}\mathbf{0}_{p\times(n-p)} & I_p\end{bmatrix} e.$$

C. Simulation Parameters in Section V

In this section, the parameters used for simulation are presented. We keep 4 significant digits for noninteger elements in all matrices except for A (using 2 significant digits).

$$A = \begin{bmatrix}0 & 1.00 & 0 & 0 & 0 \\ 0 & -1.99 & -13.41 & -18.95 & -3.60 \\ 0 & 1.00 & -1.74 & -0.08 & -0.59 \\ 0 & 0 & 0 & -20.00 & 0 \\ 0 & 0 & 0 & 0 & -20.00\end{bmatrix},$$

$$B = \begin{bmatrix}0 & 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 0 & 20\end{bmatrix}^\top, \quad C = \begin{bmatrix}1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1\end{bmatrix},$$

$$D = \begin{bmatrix}0 & 0 & 1 & 0 & 0\end{bmatrix}^\top,$$

$$F = \begin{bmatrix}-1.2640 & -0.4285 & 0.3995 & 0.3909 & 0.0704 \\ -0.7854 & -0.1397 & -0.5342 & 0.0745 & 0.0326\end{bmatrix},$$

$$P = \begin{bmatrix}16.8938 & 0 & 0 & 0 & 0 \\ 0 & 20.3096 & 0 & 0 & 0 \\ 0 & 0 & 20.3096 & 0 & 0 \\ 0 & 0 & 0 & 5.0353 & 0 \\ 0 & 0 & 0 & 0 & 5.0353\end{bmatrix},$$

$$T = \begin{bmatrix}1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0\end{bmatrix}, H = \begin{bmatrix}0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$\bar{K} = \begin{bmatrix}-20.3096 & 226.5455 & 320.1370 & 60.8176 \\ 0 & -35.4989 & 0 & 0 \\ -35.4989 & 0 & 0 & 0 \\ 0 & 0 & 0 & 64.0927 \\ 0 & 0 & 64.0927 & 0\end{bmatrix},$$

$$\bar{L} = \begin{bmatrix}-1.2022 & 13.4100 & 18.9500 & 3.6000 \\ 0 & 0 & 0 & 0 \\ -1.7479 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12.7288 \\ 0 & 0 & 12.7288 & 0\end{bmatrix}.$$