Goldbach's Conjecture — A Route to the Inconsistency of Arithmetic

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Abstract. This paper proves an inconsistency in Peano arithmetic (PA). The contradiction we derive is based on two properties of a specific set which we use to reformulate a strengthened form of the strong Goldbach conjecture.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Theorem. *PA is contradictory, i.e. the statement FALSE can be derived*.

Proof. We define the set S_g := { (pk, mk, qk) | k, m $\in \mathbb{N}$; p, q $\in \mathbb{P}_3$, p < q; m = (p + q) / 2 }.

SSGB is equivalent to saying that every integer x ≥ 4 is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers x ≥ 4 appear as m in a middle component mk of S_g. So, by the definition of S_g we have

$$\text{SSGB} \iff \forall x \in \mathbb{N}_4 \quad \exists (pk, mk, qk) \in S_g \quad x = m.$$

$$\neg\text{SSGB} \iff \exists x \in \mathbb{N}_4 \quad \forall (pk, mk, qk) \in S_g \quad x \neq m.$$

The set S_g has the following two properties.

First, the whole range of $\mathbb{N}_3$ can be expressed by the triple components of S_g (”*covering*”), because every integer x ≥ 3 can be written as some pk with k = 1 when x is prime, as some pk with k ≠ 1 when x is composite and not a power of 2, or as (3 + 5)k / 2 when x is a power of 2; p $\in \mathbb{P}_3$, k $\in \mathbb{N}$. So we have

(C) $\forall x \in \mathbb{N}_3 \quad \exists (pk, mk, qk) \in S_g \quad x = pk \ \vee \ x = mk = 4k.$

A few examples of the covering:

x = 19: (**19·1**, 21·1, 23·1), (**19·1**, 60·1, 101·1)

x = 36: (**3·12**, 7·12, 11·12)

x = 38: (**19·2**, 21·2, 23·2)

x = 42: (**3·14**, 5·14, 7·14), (**7·6**, 9·6, 11·6)

x = 64: (3·16, **4·16**, 5·16)

x = 10000: (**5·2000**, 6·2000, 7·2000).

Second, according to the statement SSGB, all pairs (p, q) of distinct odd primes are used in the definition of the set S_g ("*maximality*"). So we have

(M) $\forall$ p, q $\in \mathbb{P}_3$, p < q $\forall$ k $\in \mathbb{N}$ (pk, mk, qk) $\in S_g$, where m = (p + q) / 2.

The proof is motivated by the following view.

There are two possibilities for S_g, exactly one of which must occur: Either there is an n $\in \mathbb{N}_4$ in addition to all the numbers m defined in S_g or there is not. The latter is equivalent to SSGB and the former is equivalent to ¬SSGB.

Since, due to (C), every n given by ¬SSGB as well as every multiple nk, k $\in \mathbb{N}$, equals a component of some S_g triple that exists by definition, the covering of $\mathbb{N}_3$ by the S_g triples if n exists (¬SSGB) is equal to that if n does not exist (SSGB). This causes a contradiction because in the case SSGB the numbers m defined in S_g take all integer values x ≥ 4 whereas in the case ¬SSGB they don't.

First of all, we note that each of the two properties (C) and (M) is a condition sine qua non for the proof, for the following reasons.

¬(C) immediately implies ¬SSGB, since an n ≥ 4 different from all S_g triple components pk, mk, qk is in particular different from all m in S_g.

The proof would no longer be possible if, for example, we omitted the factor k in the definition of S_g, because then the corresponding (C) could no longer be guaranteed.

Similarly, the property (M) rules out the possibility that there is an $n \geq 4$ different from all m (i.e. $\neg$SSGB) and n is the arithmetic mean of a pair of primes not used in S_g. Thus (M) excludes the possibility that $\neg$SSGB applies due to a missing prime number pair. This means that the proof would no longer be possible here either if we left out any prime number pair in the formulation of SSGB and S_g.

We will now show that ((C) $\wedge$ (M)) leads to a contradiction.

The following proof is independent of the choice of n if there is more than one in the case of $\neg$SSGB. For example, the minimal such n works.

We split S_g into two complementary subsets: For any $y \in \mathbb{N}_3$, $S_g = S_g+(y) \cup S_g-(y)$, with

$S_g+(y) := \{ (pk, mk, qk) \in S_g \mid \exists\, k' \in \mathbb{N} \quad pk = yk' \vee mk = yk' \vee qk = yk' \}$ and

$S_g-(y) := \{ (pk, mk, qk) \in S_g \mid \forall\, k' \in \mathbb{N} \quad pk \neq yk' \wedge mk \neq yk' \wedge qk \neq yk' \}$.

Let $n \in \mathbb{N}_4$ be given by $\neg$SSGB as described above. Then, we have

(*) $\neg$SSGB => $S_g = S_g+(n) \cup S_g-(n)$.

More precisely, under the assumption $\neg$SSGB with the associated n the set S_g can be written as the disjoint union of the following triples.

(i) S_g triples of the form (pk = nk', mk, qk) with k = k' in case n is prime, due to (C)

(ii) S_g triples of the form (pk = nk', mk, qk) with k ≠ k' in case n is composite and not a power of 2, due to (C)

(iii) S_g triples of the form (3k, 4k = nk', 5k) in case n is a power of 2, due to (C)

(iv) all remaining S_g triples of the form (pk = nk', mk, qk), (pk, mk = nk', qk) or (pk, mk, qk = nk')

and

(v) S_g triples of the form (pk ≠ nk', mk ≠ nk', qk ≠ nk'), i.e. those S_g triples where none of the nk' equals a component.

So, $S_g+(n)$ is the union of the triples of the above types (i) to (iv) and $S_g-(n)$ is the union of the triples of type (v).

Now, we define

S_1 := { (pk, mk, qk) $\in S_g$ | $\neg$SSGB holds }

S_2 := { (pk, mk, qk) $\in S_g$ | SSGB holds }.

So, by (*) we obtain

(1) $\neg$SSGB => $S_1 = S_g = S_g+(n) \cup S_g-(n)$.

Since $S_g+(n) \cup S_g-(n)$ is independent of n, we can write

(1') $\forall\, y \in \mathbb{N}_3$ $\neg$SSGB => $S_1 = S_g = S_g+(y) \cup S_g-(y)$.

Under the assumption SSGB there is no n as above. Therefore, under this assumption, we can choose an arbitrary $y \in \mathbb{N}_3$ such that $S_g = S_g+(y) \cup S_g-(y)$. So, we obtain

(2) $\forall\, y \in \mathbb{N}_3$ SSGB => $S_2 = S_g = S_g+(y) \cup S_g-(y)$.

We will make use of the following trivial principle.

If two sets of (possibly infinitely many) x-tuples are equal, then the sets of their corresponding i-th components are equal; $1 \le i \le x$.

To this end, for each $k \in \mathbb{N}$ we define

M(k) := { mk | (pk, mk, qk) $\in S_g$ }

M_1(k) := { mk | (pk, mk, qk) $\in S_1$ }

M_2(k) := { mk | (pk, mk, qk) $\in S_2$ }.

Then, applying the principle above to the middle component of the triples (pk, mk, qk), (1') and (2) imply

(3) $\forall k \in \mathbb{N} \quad \forall y \in \mathbb{N}_3$

$(\neg SSGB \Rightarrow M_1(k) = M(k) = \{ mk \mid (pk, mk, qk) \in S_g+(y) \cup S_g-(y) \})$

$\wedge$

(4) $\forall k \in \mathbb{N} \quad \forall y \in \mathbb{N}_3$

$(\ SSGB \Rightarrow M_2(k) = M(k) = \{ mk \mid (pk, mk, qk) \in S_g+(y) \cup S_g-(y) \})$.

We set $M := M(1)$, $M_1 := M_1(1)$ and $M_2 := M_2(1)$. So we get

(3') $\forall y \in \mathbb{N}_3 \quad (\neg SSGB \Rightarrow M_1 = M = \{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \})$

$\wedge$

(4') $\forall y \in \mathbb{N}_3 \quad (\ SSGB \Rightarrow M_2 = M = \{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \})$.

Since for every $y \in \mathbb{N}_3$ $S_g+(y) \cup S_g-(y)$ equals S_g, there is a set X such that for every $y \in \mathbb{N}_3$ $\{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \}$ equals X. So, from (3') and (4') we obtain

(5) $(\neg SSGB \Rightarrow M_1 = M = X) \quad \wedge \quad (SSGB \Rightarrow M_2 = M = X)$.

The set X is a free variable in (5) that is either equal to $\mathbb{N}_4$ or to some non-empty proper subset Y of $\mathbb{N}_4$. Therefore, (5) splits as follows.

(5.1) $((\neg SSGB \Rightarrow M_1 = M = \mathbb{N}_4) \quad \wedge \quad (SSGB \Rightarrow M_2 = M = \mathbb{N}_4))$

$\vee$

(5.2) $((\neg SSGB \Rightarrow M_1 = M = Y \neq \mathbb{N}_4) \quad \wedge \quad (SSGB \Rightarrow M_2 = M = Y \neq \mathbb{N}_4))$.

On the other hand, under the assumption SSGB the numbers m defined in S_g take all integer values $x \geq 4$ whereas under ¬SSGB they don't. Therefore, we have

(6.1) SSGB => $M_2 = M = \mathbb{N}_4$

(6.2) ¬SSGB => $M_1 = M \neq \mathbb{N}_4$.

((5.1) ∨ (5.2)) together with (6.1) and (6.2) yields

(**(7.11)** ¬SSGB => $M_1 = M = \mathbb{N}_4$
∧
(7.12) ¬SSGB => $M_1 = M \neq \mathbb{N}_4$)

(7) ∨

(**(7.21)** SSGB => $M_2 = M \neq \mathbb{N}_4$
∧
(7.22) SSGB => $M_2 = M = \mathbb{N}_4$).

Since

¬SSGB => $M_1 = M$ and SSGB => $M_1 = \{\,\} \neq M$

and

SSGB => $M_2 = M$ and ¬SSGB => $M_2 = \{\,\} \neq M$,

from ((7.11) ∧ (7.12)) we get that $M = \mathbb{N}_4$ and $M \neq \mathbb{N}_4$ if ¬SSGB actually is true, and from ((7.21) ∧ (7.22)) we get that $M = \mathbb{N}_4$ and $M \neq \mathbb{N}_4$ if SSGB actually is true.

For two statements P and Q, in contrast to the conditional statement

if P is assumed to be true, then (Q ∧ ¬Q)

which symbolically is (P => (Q ∧ ¬Q)), the statement

if P is actually true, then $(Q \wedge \neg Q)$

is symbolically $(P \wedge (P \Rightarrow (Q \wedge \neg Q)))$.

Therefore, from (7) we obtain

(8.1) $(\neg SSGB \wedge (\neg SSGB \Rightarrow (M = \mathbb{N}_4 \wedge M \neq \mathbb{N}_4)))$

$\vee$

(8.2) $(SSGB \wedge (SSGB \Rightarrow (M = \mathbb{N}_4 \wedge M \neq \mathbb{N}_4)))$.

Then, ((8.1) $\vee$ (8.2)) yields (FALSE $\vee$ FALSE), which is equivalent to FALSE.

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