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RESEARCH MANUAL 2

ECMWF FORECAST MODEL

ADIABATIC PART

ECMWF Research Department

This manual describes the adiabatic part of the operational forecast model. It supercedes previously published documentation. When changes are made to the model, update sheets will be issued.

This Meteorological Bulletin is bound separately.

Update sheets should be retained at the back of the manual.

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CHAPTER 1

Adiabatic Formulation

1.1 INTRODUCTION

In this volume the technical details of the adiabatic part of the ECMWF operational model are described. The first two chapters describe the governing equations, the coordinates and the discretization schemes used in the operational ECMWF model. Attention is concentrated on the representation of the explicitly resolved adiabatic processes, but a derivation of the equations including terms requiring parametrization is included in Appendix A1. Detailed descriptions of the parametrizations themselves are given in the Physical Parametrization Manual (RM-3).

The ECMWF model is formulated in spherical harmonics. After the inter-model comparisons by Jarraud et al. (1981) and Girard and Jarraud (1982) truncated expansions in terms of spherical harmonics were adopted for the representation of dynamical fields. The transform technique developed by Eliasen et al. (1970), Orszag (1970) and Machenhauer and Rasmussen (1972) is used such that non-linear terms, including parameterizations, are evaluated at a set of almost regularly distributed grid points.

In the vertical, a flexible coordinate is used, enabling the model to use either the usual terrain-following sigma coordinate (Phillips, 1957), or a hybrid coordinate for which upper-level model surfaces "flatten" over steep terrain, becoming surfaces of constant pressure in the stratosphere (Simmons and Burridge, 1981, Simmons and Strüfing, 1981). Moist processes are treated in a consistent way in both the dynamical equations and parameterization schemes.

The second section chapter 1presents the continuous form of the governing equations. Chapter 2 gives details of the spectral discretization and of the vertical coordinate and its associated vertical finite difference scheme. The temporal finite-difference scheme, which includes not only a conventional semi-implicit treatment of gravity-wave terms (Robert et al, 1972) but also a semi-implicit treatment of the advection of vorticity and moisture (Jarraud et al, 1982), is also described, as is the formulation chosen for horizontal diffusion.

The third chapter deals with the non-linear normal mode initialization procedure which is used to balance the fields of mass and winds in the initial data, in order to suppress fast moving gravity waves in the forecast.

Chapter 4 contains a description of the structure of the model's computer code. Flow diagrams are given to show the organisation of the model. The subroutines dealing with the run control, the dynamics and the initialization are described.

In chapter 5 the various diagnostics that can be extracted during a model run are described. Finally the structure of the files used by the model, the handling of input and output, and the memory manager designed to supervise the use of the computer's central memory, are explained in chapter 6.

1.2 THE CONTINUOUS EQUATIONS FOR A GENERAL PRESSURE-BASED VERTICAL COORDINATE

Although the model has been programmed for one particular form of vertical coordinate, which is introduced in Sect.2.2, it is convenient to introduce the equations and their spectral discretization for a general pressure-based terrain-following vertical coordinate, $\eta(p,p_s)$. This must be a monotonic function of pressure, p, and depend also on surface pressure, p_s, in such a way that

$$\eta(0,p_s) = 0 \text{ and } \eta(p_s, p_s) = 1$$

For such a coordinate, the continuous formulation of the primitive equations for a dry atmosphere may be directly derived from their basic height-coordinate forms following Kasahara (1974).

During the design of the model, a detailed derivation of the corresponding equations for a moist atmosphere, including a separation into terms to be represented explicitly in the subsequent discretized form of the equations and terms to be parametrized, was carried out. It is shown in Appendix A1 that under certain approximations, the momentum, thermodynamic and moisture equations may be written:

$$\frac{\partial U}{\partial t} - (f + \xi)V + \dot{\eta}\frac{\partial U}{\partial \eta} + \frac{R_d T_v}{a}\frac{\partial \ln p}{\partial \lambda} + \frac{1}{a}\frac{\partial}{\partial \lambda}(\phi + E) = P_U + K_U \qquad (1.2.1)$$

$$\frac{\partial V}{\partial t} + (f + \xi)U + \dot{\eta}\frac{\partial V}{\partial \eta} + \frac{R_d T_v}{a}(1-\mu^2)\frac{\partial \ln p}{\partial \mu} + \frac{(1-\mu^2)}{a}\frac{\partial}{\partial \mu}(\phi + E) = P_V + K_V \qquad (1.2.2)$$

$$\frac{\partial T}{\partial t} + \frac{U}{a(1-\mu^2)}\frac{\partial T}{\partial \lambda} + \frac{V}{a}\frac{\partial T}{\partial \mu} + \dot{\eta}\frac{\partial T}{\partial \eta} - \frac{\kappa T_v \omega}{(1+(\delta-1)q)p} = P_T + K_T \qquad (1.2.3)$$

and

$$\frac{\partial q}{\partial t} + \frac{U}{a(1-\mu^2)}\frac{\partial q}{\partial \lambda} + \frac{V}{a}\frac{\partial q}{\partial \mu} + \dot{\eta}\frac{\partial q}{\partial \eta} = P_q + K_q \qquad (1.2.4)$$

The continuity equation is

$$\frac{\partial}{\partial \eta}\left(\frac{\partial p}{\partial t}\right) + \nabla \cdot \left(\underset{\sim}{v}_h \frac{\partial p}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right) = 0 \qquad (1.2.5)$$

and the hydrostatic equation takes the form

$$\frac{\partial \phi}{\partial \eta} = -\frac{R_d T_v}{p}\frac{\partial p}{\partial \eta} \qquad (1.2.6)$$

The pressure coordinate vertical velocity is given by

$$\omega = - \int_0^{\eta} \nabla \cdot (\underset{\sim}{v}_h \frac{\partial p}{\partial \eta}) d\eta + \underset{\sim}{v}_h \cdot \nabla p \, , \qquad (1.2.7)$$

and explicit expressions for the rate of change of surface pressure, and for $\dot{\eta}$, are obtained by integrating (1.2.5), using the boundary conditions $\dot{\eta}=0$ at $\eta=0$ and $\eta=1$:

$$\frac{\partial p_s}{\partial t} = - \int_0^1 \nabla \cdot (\underset{\sim}{v}_h \frac{\partial p}{\partial \eta}) d\eta \qquad (1.2.8)$$

and

$$\dot{\eta} \frac{\partial p}{\partial \eta} = - \frac{\partial p}{\partial t} - \int_0^{\eta} \nabla \cdot (\underset{\sim}{v}_h \frac{\partial p}{\partial \eta}) d\eta \qquad (1.2.9)$$

(1.2.8) may also be written

$$\frac{\partial \ln p_s}{\partial t} = - \frac{1}{p_s} \int_0^1 \nabla \cdot (\underset{\sim}{v}_h \frac{\partial p}{\partial \eta}) d\eta \qquad (1.2.10)$$

Variables and constants are defined in Table 1.

In the special case of sigma coordinates ($\eta = \sigma = p/p_s$), the above equations are the same as those used in the first operational ECMWF model, apart from the factor $(1 + (\delta-1)q)$ in (1.2.3), which differs from unity by an amount of the same order as the difference between temperature and virtual temperature, and apart also from differences in the terms written symbolically on the right-hand sides of (1.2.1) - (1.2.4), which are those requiring parametrization. Following the derivation given in Appendix A1, the terms P_U, P_V, P_T and P_q are written:

$$P_U = -g\cos\theta \left(\frac{\partial p}{\partial \eta}\right)^{-1} \frac{\partial}{\partial \eta} J_U \qquad (1.2.11)$$

$$P_V = -g\cos\theta \left(\frac{\partial p}{\partial \eta}\right)^{-1} \frac{\partial}{\partial \eta} J_V \qquad (1.2.12)$$

$$P_T = \frac{1}{c_p} \left\{Q_R + Q_L + Q_D - g \left(\frac{\partial p}{\partial \eta}\right)^{-1} \left[\frac{\partial}{\partial \eta} J_s - c_{pd} T \, (\delta-1) \frac{\partial}{\partial \eta} J_q\right]\right\} \qquad (1.2.13)$$

$$P_q = S_q - g \left(\frac{\partial p}{\partial \eta}\right)^{-1} \frac{\partial}{\partial \eta} J_q \quad , \qquad (1.2.14)$$

where $c_p = c_{pd} \, (1+(\delta-1)q)$

In (1.2.11) - (1.2.14), J_U, J_V, J_s and J_q represent net parametrized vertical fluxes of momentum, dry static energy ($c_pT+\phi$), and moisture. They include fluxes due to convection and boundary-layer turbulence. Q_R, Q_L and Q_D represent heatings due respectively to radiation, to internal phase changes (including the evaporation of precipitation) and to the internal dissipation of kinetic energy associated with the P_U and P_V terms. S_q denotes the rate of change of q due to rainfall and snowfall. Details of the calculation of these terms are given in the ECMWF Physical Parameterization Manual (RM-3).

The terms K_U, K_V, K_T, and K_q in (1.2.1) - (1.2.4) represent the influence of unresolved horizontal scales. Their treatment differs from that of the P_U, P_V, P_T and P_q terms in that it does not involve a physical model of sub grid-scale processes, but rather a numerically convenient form of scale-selective diffusion of a magnitude determined empirically to ensure a realistic behaviour of resolved scales. These terms are specified in Sect.2.4.

In order to apply the spectral method, Eqs. (1.2.1) and (1.2.2) are written in vorticity and divergence form (Bourke, 1972). They become

$$\frac{\partial \xi}{\partial t} = \frac{1}{a(1-\mu^2)} \frac{\partial}{\partial \lambda} (F_V + P_V) - \frac{1}{a} \frac{\partial}{\partial \mu} (F_U + P_U) + K_\xi \tag{1.2.15}$$

$$\frac{\partial D}{\partial t} = \frac{1}{a(1-\mu^2)} \frac{\partial}{\partial \lambda} (F_U + P_U) + \frac{1}{a} \frac{\partial}{\partial \mu} (F_V + P_V) - \nabla^2 G + K_D \tag{1.2.16}$$

where

$$F_U = (f+\xi)V - \dot{\eta}\frac{\partial U}{\partial \eta} - \frac{R_d T_v}{a} \frac{\partial \ln p}{\partial \lambda} \tag{1.2.17}$$

$$F_V = -(f+\xi)U - \dot{\eta}\frac{\partial V}{\partial \eta} - \frac{R_d T_v}{a} (1-\mu^2) \frac{\partial \ln p}{\partial \mu} \tag{1.2.18}$$

and

$$G = \phi + E \tag{1.2.19}$$

We also note that a streamfunction ψ and velocity potential χ may be introduced such that

$$\left.\begin{aligned} U &= \frac{1}{a} \left\{-(1-\mu^2) \frac{\partial \psi}{\partial \mu} + \frac{\partial \chi}{\partial \lambda}\right\} , \\ V &= \frac{1}{a} \left\{\frac{\partial \psi}{\partial \lambda} + (1-\mu^2) \frac{\partial \chi}{\partial \mu}\right\} , \\ \xi &= \nabla^2 \psi , \\ \text{and} \quad D &= \nabla^2 \chi . \end{aligned}\right\} \tag{1.2.20}$$

Table 1 List of constants and symbols

Symbol	Meaning	Value	Unit	Section
a	radius of the earth	6.371×10^{6}	m	1.2
A_k	constant defining the vertical coordinate		Pa	2.2
B_k	constant defining the vertical coordinate		-	2.2
c_ℓ	normal mode coefficient		-	3.2
C_p	$= C_{pd}[1+(\delta-1)q]$: specific heat of moist air at constant pressure		$J\ kg^{-1}\ K^{-1}$	1.2
C_{pd}	specific heat of dry air at constant pressure	1005.46	$J\ kg^{-1}\ K^{-1}$	1.2
C_{pv}	specific heat of water vapour at constant pressure	1869.46	$J\ kg^{-1}\ K^{-1}$	A1.7
D	$= \frac{1}{a}(\frac{1}{1-\mu^2}\frac{\partial U}{\partial\lambda}+\frac{\partial V}{\partial\mu})$ divergence		s^{-1}	1.2
E	$= \frac{1}{2}(U^2+V^2)/(1-\mu^2)$: kinetic energy/unit mass		m^2s^{-2}	1.2
f	$= 2\Omega\sin\theta$: Coriolis parameter		s^{-1}	1.2
g	acceleration of gravity	9.80665	$m\ s^{-2}$	1.2
G	$= \phi+E$: total dry energy per unit mass		m^2s^{-2}	1.2
K	horizontal diffusion coefficient		m^4s^{-2}	2.4
K_X	tendency of variable X due to horizontal diffusion		$[X]\ s^{-1}$	1.2
m	zonal wave number		-	2.1
n	meridional index		-	2.1
P	auxiliary potential for initialisation		m^2s^{-2}	3.2
p	pressure		Pa	1.2
$P_n^m(\mu)$	associated Legendre function of the first kind		-	2.1
p_o	reference pressure for vertical coordinate		Pa	2.2
p_s	surface pressure		Pa	1.2
P_X	parametrized tendency of variable X		$[X]s^{-1}$	1.2
R_d	gas constant for dry air	287.05	$J\ kg^{-1}\ K^{-1}$	1.2

Table 1 continued

Symbol	Meaning	Value	Unit	Section
R_v	gas constant for water vapour	461.51	$J\ kg^{-1}\ K^{-1}$	A1.6
q	specific humidity		kg/kg	1.2
q_i	specific ice content		kg/kg	A1.7
q_ℓ	specific liquid water content		kg/kg	A1.7
Q_Y	heating rate due to physical process Y		$K\ s^{-1}$	1.2
S_q	rate of change of humidity due to precipitation		s^{-1}	1.2
t	time		s	1.2
T	temperature		K	1.2
T_v	$= T[1+(\frac{1}{\varepsilon} - 1)q]$: virtual temperature		K	1.2
u	zonal wind		$m\ s^{-1}$	1.2
U	$= u\cos\theta$: scaled zonal wind		$m\ s^{-1}$	1.2
v	meridional wind		$m\ s^{-1}$	1.2
$\underset{\sim}{v}_h$	(u,v): horizontal wind vector		$m\ s^{-1}$	
V	$= v\cos\theta$: scaled meridional wind		$m\ s^{-1}$	1.2
$w(\mu)$	quadrature (or Gaussian) weight		-	4.3
z	height		m	1.2
δ	$= C_{pv}/C_{pd}$		-	1.2
ε	$= R_d/R_v$		-	A1.2
η	$\eta_k = A_k/p_o + B_k$: generalized vertical coordinate		-	1.2
$\dot{\eta}$	$= \frac{d\eta}{dt}$: η-coordinate vertical velocity		s^{-1}	1.2
θ	latitude		-	1.2
κ	$= R_d/C_{pd}$		-	1.2

Table 1 continued

Symbol	Meaning	Value	Unit	Section
λ	longitude		-	1.2
μ	$= \sin\theta$		-	1.2
ξ	$= \frac{1}{a}(\frac{1}{1-\mu^2}\frac{\partial V}{\partial\lambda} - \frac{\partial U}{\partial\mu})$: relative vorticity		s^{-1}	1.2
ξ_ℓ	horizontal normal mode		-	3.2
ρ	density		$kg\ m^{-3}$	A1.1
σ	$= p/p_s$		-	1.2
ϕ	$= gz$: geopotential height		$m^2\ s^{-2}$	1.2
ϕ_s	surface geopotential height		$m^2\ s^{-2}$	2.1
χ	velocity potential		$m^2\ s^{-1}$	1.2
ψ	streamfunction		$m^2 s^{-1}$	1.2
ψ_m	vertical normal mode		-	3.2
ω	$= \frac{dp}{dt}$: p-coordinate vertical velocity		$Pa\ s^{-1}$	1.2
Ω	angular velocity of earth	7.292×10^{-5}	s^{-1}	1.2

CHAPTER 2

The Discrete Equations

2.1 HORIZONTAL DISCRETIZATION

2.1.1 Spectral representation

The basic prognostic variables of the model are ξ, D, T, q and $\ln p_s$. They, and the surface geopotential ϕ_s, are represented in the horizontal by truncated series of spherical harmonics:

$$X(\lambda,\mu,\eta,t) = \sum_{m=-M}^{M} \sum_{n=m}^{N(m)} X_n^m(\eta,t)\, P_n^m(\mu) e^{im\lambda} \tag{2.1.1}$$

where X is any variable. The $P_n^m(\mu)$ are the Associated Legendre Functions of the first kind, defined here by

$$P_n^m(\mu) = \sqrt{(2n+1)\frac{(n-m)!}{(n+m)!}}\ \frac{1}{2^n n!}\,(1-\mu^2)^{m/2}\,\frac{d^{n+m}}{d\mu^{n+m}}(\mu^2-1)^n\ ,\ m\geq 0\ ,$$

and (2.1.2)

$$P_n^{-m}(\mu) = P_n^m(\mu)$$

This definition is such that

$$\frac{1}{2}\int_{-1}^{1} P_n^m(\mu)\, P_s^m(\mu)\,d\mu = \delta_{ns} \tag{2.1.3}$$

where δ_{ns} is the Kronecker delta function.

The X_n^m are the complex-valued spectral coefficients of the field X, and they are given by

$$X_n^m(\eta,t) = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} X(\lambda,\mu,\eta,t)\, P_n^m(\mu) e^{-im\lambda}\, d\lambda\, d\mu \tag{2.1.4}$$

Since X is real

$$X_n^{-m} = (X_n^m)^*\ , \tag{2.1.5}$$

where ( )* denotes the complex conjugate. The model thus deals explicitly only with the X_n^m for $m\geq 0$.

The Fourier coefficients of X, $X_m(\mu,\eta,t)$ are defined by

$$X_m(\mu,\eta,t) = \frac{1}{2\pi}\int_0^{2\pi} X(\lambda,\mu,\eta,t)e^{-im\lambda}\,d\lambda \quad , \tag{2.1.6}$$

or using (2.1.1), by

$$X_m(\mu,\eta,t) = \sum_{n=m}^{N(m)} X_n^m(\eta,t)\,P_n^m(\mu) \tag{2.1.7}$$

with

$$X(\lambda,\mu,\eta,t) = \sum_{m=-M}^{M} X_m(\mu,\eta,t)e^{im\lambda} \tag{2.1.8}$$

Horizontal derivatives are given analytically by

$$\left(\frac{\partial X}{\partial\lambda}\right)_m = im\,X_m \tag{2.1.9}$$

and
$$\left(\frac{\partial X}{\partial\mu}\right)_m = \sum_{n=m}^{N(m)} X_n^m\,\frac{dP_n^m}{d\mu} \tag{2.1.10}$$

where the derivative of the Legendre Function is given by the recurrence relation:

$$(1-\mu^2)\,\frac{dP_n^m}{d\mu} = -\,n\,\epsilon_{n+1}^m\,P_{n+1}^m + (n+1)\,\epsilon_n^m\,P_{n-1}^m \tag{2.1.11}$$

with

$$\epsilon_n^m = \left(\frac{n^2-m^2}{4n^2-1}\right)^{1/2} \tag{2.1.12}$$

An important property of the spherical harmonics is:

$$\nabla^2\left\{P_n^m(\mu)\,e^{im\lambda}\right\} = -\,\frac{n(n+1)}{a^2}\,P_n^m(\mu)\,e^{im\lambda} \tag{2.1.13}$$

Relationships (1.2.20) may thus be used to derive expressions for the Fourier velocity coefficients, U_m and V_m, in terms of the spectral coefficients ξ_n^m and D_n^m. It is convenient for later reference to write these expressions in the form:

$$U_m = U_{\xi m} + U_{Dm} \tag{2.1.14}$$

$$V_m = V_{\xi m} + V_{Dm} \tag{2.1.15}$$

where

$$U_{\xi m} = -a \sum_{n=m}^{N(m)} \frac{1}{n(n+1)} \xi_n^m H_n^m(\mu) \qquad (2.1.16)$$

$$U_{Dm} = -a \sum_{n=m}^{N(m)} \frac{im}{n(n+1)} D_n^m P_n^m(\mu) \qquad (2.1.17)$$

$$V_{\xi m} = -a \sum_{n=m}^{N(m)} \frac{im}{n(n+1)} \xi_n^m P_n^m(\mu) \qquad (2.1.18)$$

$$V_{Dm} = a \sum_{n=m}^{N(m)} \frac{1}{n(n+1)} D_n^m H_n^m(\mu) \qquad (2.1.19)$$

and

$$H_n^m(\mu) = -(1-\mu^2) \frac{dP_n^m}{d\mu} \qquad (2.1.20)$$

The H_n^m can be computed from the recurrence relation (2.1.11).

The model is programmed to allow for a flexible pentagonal truncation, depicted in Fig. 2.1 (Baede et al, 1979). This truncation is completely defined by the three parameters J, K and M illustrated in the Figure. The common truncations are special cases of the pentagonal one:

Triangular	$M = J = K$
Rhomboidal	$K = J + M$
Trapezoidal	$K = J$, $K > M$

The summation limit, N(m) is given by

$N = J + |m|$ if $J + |m| < K$,

and

$N = K$ if $J + |m| > K$.

Operationally, the truncation is triangular, with $J = K = M = N = 106$.

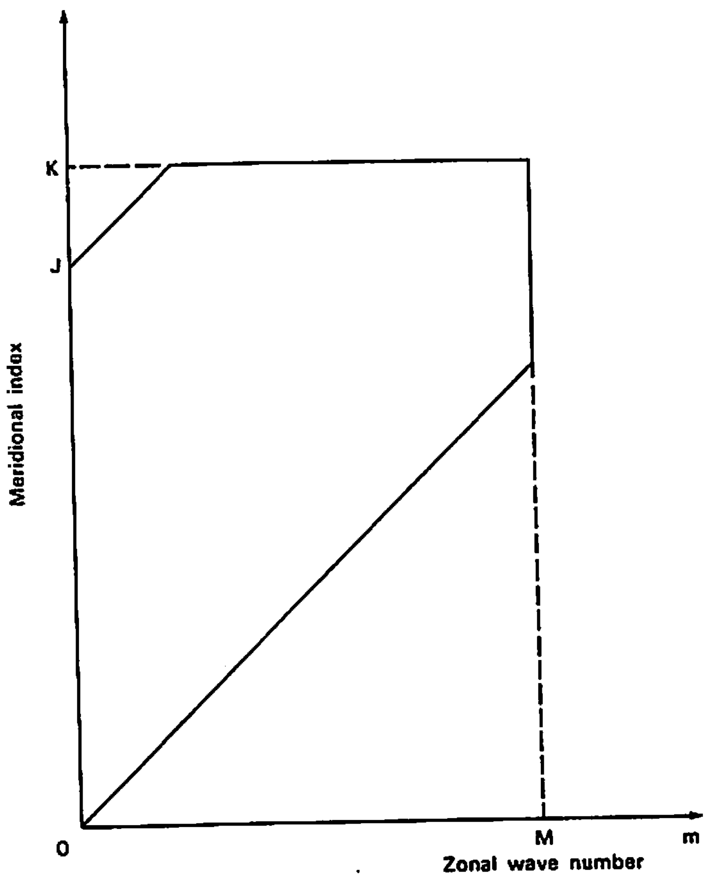


Fig. 2.1 Pentagonal truncation

2.1.2 Spectral/grid-point transforms, and the evaluation of spectral tendencies

The general form of the calculations follows that of the early multi-level spectral models described by Bourke (1974) and Hoskins and Simmons (1975), although the present model differs in its use of an advective form for the temperature and moisture equations (1.2.3) and (1.2.4). The prognostic equations for ξ, D, T, q and $\ell n p_s$ are (1.2.15), (1.2.16), (1.2.3), (1.2.4) and (1.2.10). Equations for the corresponding spectral coefficients are obtained by multiplying each side of these equations by $P_n^m e^{-im\lambda}$, and integrating over the sphere. This yields, from (2.1.4),

$$\frac{\partial \xi_n^m}{\partial t} = \frac{1}{4\pi a} \int_{-1}^{1} \int_{0}^{2\pi} \left\{ \frac{1}{1-\mu^2} \frac{\partial}{\partial \lambda} (F_V + P_V) - \frac{\partial}{\partial \mu} (F_U + P_U) \right\} P_n^m(\mu) e^{-im\lambda} d\lambda d\mu + (K_\xi)_n^m \qquad (2.1.21)$$

$$\frac{\partial D_n^m}{\partial t} = \frac{1}{4\pi a} \int_{-1}^{1} \int_{0}^{2\pi} \left\{ \frac{1}{1-\mu^2} \frac{\partial}{\partial \lambda} (F_U + P_U) + \frac{\partial}{\partial \mu} (F_V + P_V) \right\} P_n^m(\mu) e^{-im\lambda} d\lambda d\mu - \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} (\nabla^2 G) P_n^m(\mu) e^{-im\lambda} d\lambda d\mu + (K_D)_n^m \qquad (2.1.22)$$

$$\frac{\partial T_n^m}{\partial t} = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} (F_T + P_T) P_n^m(\mu) e^{-im\lambda} d\lambda d\mu + (K_T)_n^m \qquad (2.1.23)$$

$$\frac{\partial q_n^m}{\partial t} = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} (F_q + P_q) P_n^m(\mu) e^{-im\lambda} d\lambda d\mu + (K_q)_n^m \qquad (2.1.24)$$

and

$$\frac{\partial (\ell n ps)_n^m}{\partial t} = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} F_p P_n^m(\mu) e^{-im\lambda} d\lambda d\mu \qquad (2.1.25)$$

where F_U, F_V and G are given by (1.2.17) - (1.2.19), and

$$F_T = -\frac{U}{a(1-\mu^2)} \frac{\partial T}{\partial \lambda} - \frac{V}{a} \frac{\partial T}{\partial \mu} - \dot{\eta} \frac{\partial T}{\partial \eta} + \frac{\kappa T_v \omega}{(1+(\delta-1)q)p} \qquad (2.1.26)$$

$$F_q = -\frac{U}{a(1-\mu^2)} \frac{\partial q}{\partial \lambda} - \frac{V}{a} \frac{\partial q}{\partial \mu} - \dot{\eta} \frac{\partial q}{\partial \eta} \qquad (2.1.27)$$

$$F_p = -\frac{1}{p_s} \int_{0}^{1} \nabla \cdot \left(\underset{\sim}{v}_h \frac{\partial p}{\partial \eta} \right) d\eta \qquad (2.1.28)$$

Equations (2.1.23) - (2.1.25) are in the form used in the model. The corresponding forms for the vorticity and divergence equations are obtained from (2.1.21) and (2.1.22) by integration by parts and use of (2.1.13):

$$\frac{\partial \xi_n^m}{\partial t} = \frac{1}{4\pi a} \int_{-1}^{1} \int_{0}^{2\pi} (1-\mu^2)^{-1} \{ im(F_V + P_V) P_n^m(\mu) - (F_U + P_U) H_n^m(\mu) \} e^{-im\lambda} d\lambda d\mu + (K_\xi)_n^m \quad (2.1.29)$$

$$\frac{\partial D_n^m}{\partial t} = \frac{1}{4\pi a} \int_{-1}^{1} \int_{0}^{2\pi} (1-\mu^2)^{-1} \{ im(F_U + P_U) P_n^m(\mu) + (F_V + P_V) H_n^m(\mu) \} e^{-im\lambda} d\lambda$$
$$+ \frac{n(n+1)}{4\pi a^2} \int_{-1}^{1} \int_{0}^{2\pi} G \, P_n^m(\mu) e^{-im\lambda} \, d\lambda d\mu + (K_D)_n^m \quad , \quad (2.1.30)$$

where $H_n^m(\mu)$ is given by (2.1.20).

An outline of the model's computation of spectral tendencies may now be given. First, a grid of points covering the sphere is defined. Using the basic definition of the spectral expansions (2.1.1) and equations (2.1.14) - (2.1.19), values of ξ, D, U, V, T, q and $\ell n p_s$ are calculated at the gridpoints, as also are the derivatives $\frac{\partial T}{\partial \lambda}$, $\frac{\partial T}{\partial \mu}$, $\frac{\partial q}{\partial \lambda}$, $\frac{\partial q}{\partial \mu}$, $\frac{\partial \ell n p_s}{\partial \lambda}$ and $\frac{\partial \ell n p_s}{\partial \mu}$ using (2.1.9) and (2.1.10). The resulting gridpoint values are sufficient to calculate gridpoint values of F_U, F_V, F_T, F_q, F_p and G, together with the parametrized tendencies P_U, P_V, P_T and P_q, since prognostic surface fields associated with the parametrization are defined and updated on the same grid. The integrands of the prognostic equations (2.1.29), (2.1.30), (2.1.23), (2.1.24) and (2.1.25) are thus known at each gridpoint, and spectral tendencies are calculated by numerical quadrature.

The grid on which the calculations are performed is chosen to give an exact (given the spectral truncation of the fields, and within round-off error) contribution to spectral tendencies from quadratic non-linear terms. The integrals with respect to λ involve the product of three trigonometric functions, and as shown by Machenhauer and Rasmussen (1972) they may be evaluated exactly using a regularly-spaced grid of at least 3M+1 points. For the latitudinal integrals, Eliasen et al. (1970) showed that quadratic non-linear terms lead to integrands which are polynomials in μ of a certain order.

They may thus be computed exactly using Gaussian quadrature (e.g. Krylov, 1962), with points located at the (approximately equally-spaced) latitudes which satisfy $P^0_{N_G}(\mu) = 0$, for a sufficiently large integer N_G. These latitudes form what are referred to as the "Gaussian latitudes".

In order to find the necessary number of Gaussian latitudes for the pentagonal truncation, the product truncation for quadratic terms must be constructed. The general form of this product truncation together with the original truncation is shown in Fig.2.2 in which the quantity L is defined as:

$$L = K-J$$

It should be noted that the triangular indentation in the upper boundary disappears if $M-L \geq 2L$, i.e. if $M \geq 3L$. From this figure and the exactness condition for the Gaussian integration it may be shown that the number of Gaussian latitudes N_G must fulfil one of the following conditions:

$$\text{if } M \leq 2(K-J) \text{ , } N_G \geq \frac{2J+K+M+1}{2} \text{ ,}$$

$$\text{if } M \geq 2(K-J) \text{ , } N_G \geq \frac{3K+1}{2} \text{ .}$$

These conditions reduce to the following for the common truncations:

triangular or trapezoidal: $N_G \geq \frac{3K+1}{2}$ (since $M \geq 2(K-J)=0$),

rhomboidal: $N_G \geq \frac{3J+2M+1}{2}$ (since $2(K-J)=2M>M$).

An asymptotic property of the Legendre functions which may be derived directly from the definition (2.1.2) is

$$P^m_n(\mu) \sim (1-\mu^2)^{m/2} \quad \text{as } \mu \to \pm 1.$$

Thus for large m the functions become vanishingly small as the poles are approached, and the contributions to the integrals (2.1.21) - (2.1.25) from polar regions become less than unavoidable round-off error for sufficiently large zonal wavenumbers. This means that in practice 3M+1 longitudinal points may not be needed at all latitudes, and a decreasing number of points may be used as the poles are approached without significant loss of accuracy in the calculation of quadratic terms (Machenhauer, 1979). An option for the model

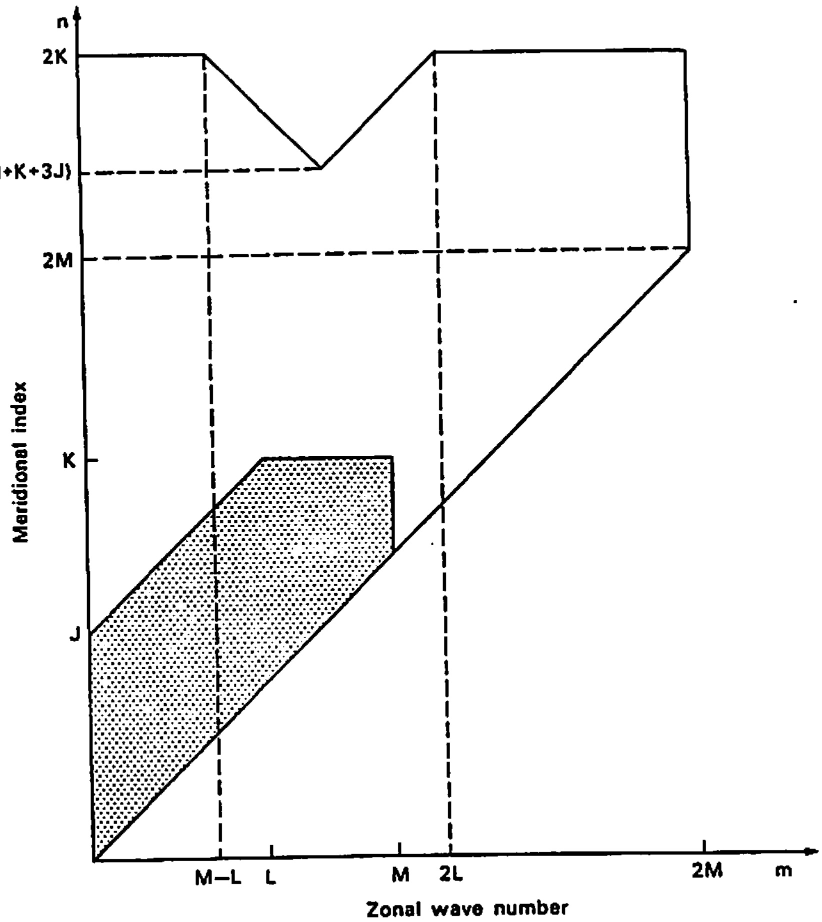


Fig. 2.2 Product truncation

to use a different number of longitudinal points at different latitudes has yet to be fully developed.

Operationally, the number of longitudinal points is 320 for each line of latitude. The number of latitudes, N_G, is 160. These latitudes are specified in Table 2.1, and are approximated (in a least-squares fit) by a regular latitudinal spacing of 1.121° starting from 89.16°N.

Table 2.1. The "Gaussian" latitudes of the computational grid of the operational model.

No. from pole	Lat(°)	No. from pole	Lat(°)
1	89.14	41	44.30
2	88.03	42	43.18
3	86.91	43	42.06
4	85.79	44	40.93
5	84.67	45	39.81
6	83.55	46	38.69
7	82.43	47	37.57
8	81.31	48	36.45
9	80.19	49	35.33
10	79.06	50	34.21
11	77.94	51	33.08
12	76.82	52	31.96
13	75.70	53	30.84
14	74.58	54	29.72
15	73.46	55	28.60
16	72.34	56	27.48
17	71.21	57	26.36
18	70.09	58	25.23
19	68.97	59	24.11
20	67.85	60	22.99
21	66.73	61	21.87
22	65.61	62	20.75
23	64.49	63	19.63
24	63.36	64	18.50
25	62.24	65	17.38
26	61.12	66	16.26
27	60.00	67	15.14
28	58.88	68	14.02
29	57.76	69	12.90
30	56.64	70	11.78
31	55.51	71	10.65
32	54.39	72	9.53
33	53.27	73	8.41
34	52.15	74	7.29
35	51.03	75	6.17
36	49.91	76	5.05
37	48.78	77	3.93
38	47.66	78	2.80
39	46.54	79	1.68
40	45.42	80	0.56

2.2 VERTICAL DISCRETIZATION

2.2.1 The hybrid vertical representation

To represent the vertical variation of the dependent variables ξ, D, T and q the atmosphere is divided into NLEV layers as illustrated in Fig.2.3. These layers are defined by the pressures of the interfaces between them (the "half levels"), and these pressures are given by

$$p_{k+\frac{1}{2}} = A_{k+\frac{1}{2}} + B_{k+\frac{1}{2}} \, p_s \tag{2.2.1}$$

for k=0, 1, 2, ...NLEV. The $A_{k+\frac{1}{2}}$ and $B_{k+\frac{1}{2}}$ are constants whose values effectively define the vertical coordinate. Necessary values are

$$A_{\frac{1}{2}} = B_{\frac{1}{2}} = A_{NLEV+\frac{1}{2}} = 0 \quad , \; B_{NLEV+\frac{1}{2}} = 1 \tag{2.2.2}$$

The usual sigma coordinate is obtained as the special case

$$A_{k+\frac{1}{2}} = 0 \quad , \; k = 0, 1, 2....NLEV \tag{2.2.3}$$

This form of hybrid coordinate has been chosen because it is particularly efficient from a computational viewpoint. It also allows a simple direct control over the "flattening" of coordinate surfaces as pressure decreases, since the A's and B's may be determined by specifying the distribution of half-level pressures for a typical sea-level surface pressure and for a surface pressure typical of the lowest expected to be attained in the model. Coordinate surfaces are surfaces of constant pressure at levels where $B_{k+\frac{1}{2}}$ = 0.

The prognostic variables ξ, D, T and q are represented by their values at intermediate ("full-level") pressures, p_k. Values for p_k are not explicitly required by the model's vertical finite-difference scheme, which is described in the following section, but they are required by parametrization schemes, in the creation of initial data, and in the interpolation to pressure levels that forms part of the post-processing. Alternative forms for p_k have been discussed by Simmons and Burridge (1981) and Simmons and Strüfing (1981). Little sensitivity has been found, and the simple form

$$p_k = \frac{1}{2} \left(p_{k+\frac{1}{2}} + p_{k-\frac{1}{2}} \right) \tag{2.2.4}$$

has been adopted, where half-level values are as given by (2.2.1).

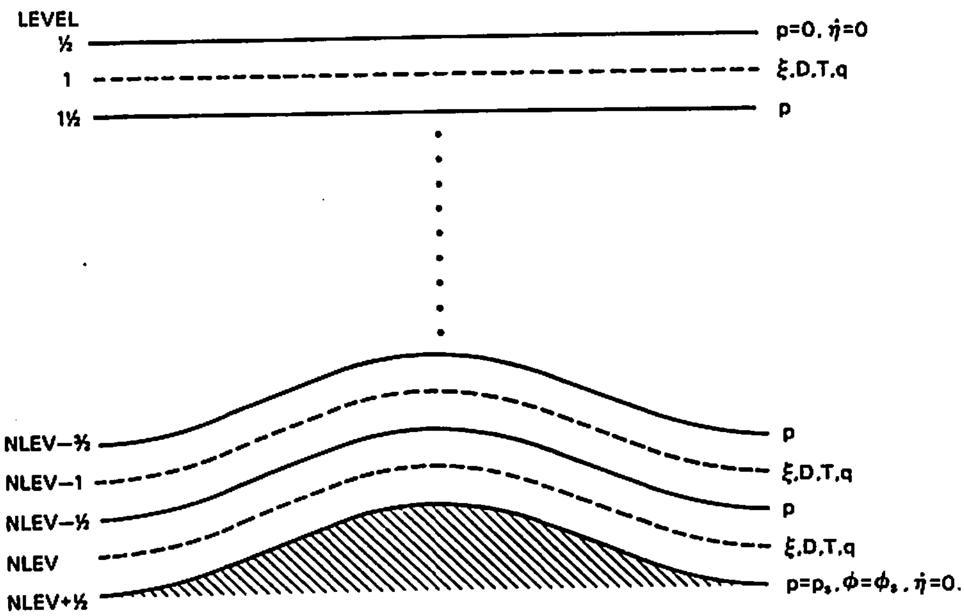


Fig. 2.3 Vertical distribution of variables

The explicit relationship between p and p_s defined for model half levels implicitly determines a vertical coordinate η. The model formulation is in fact such that this coordinate need not be known explicitly, as demonstrated in the following section. However, it is computationally convenient to define η for the radiative parametrization and for the vertical interpolation used in the post-processing. The half-level values are given by

$$\eta_{k+\frac{1}{2}} = A_{k+\frac{1}{2}}/p_o + B_{k+\frac{1}{2}} \tag{2.2.5}$$

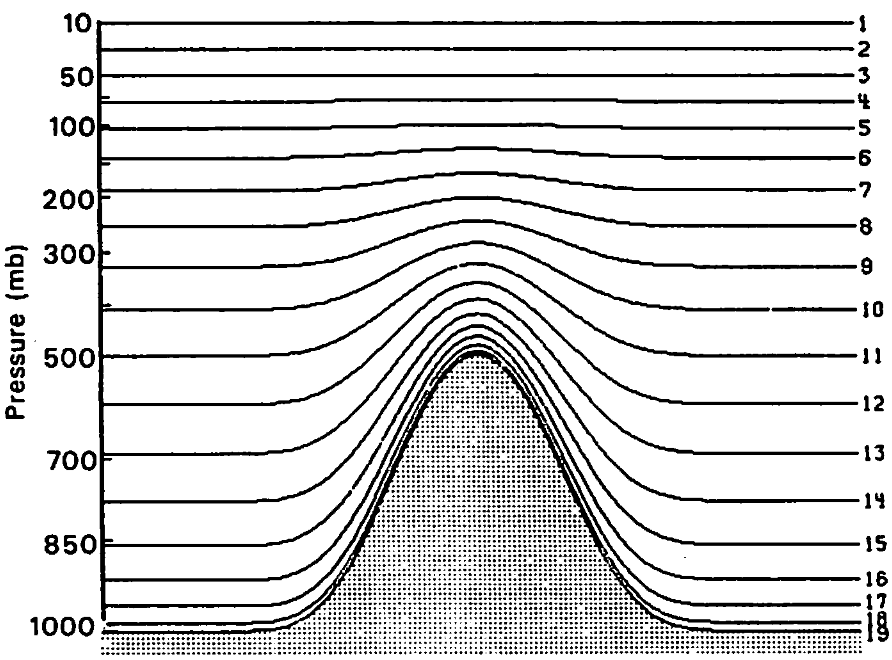
where p_o is a constant pressure. From (1.5.1) it is seen that this coordinate is identical to the usual σ when $A_{k+\frac{1}{2}} = 0$, and in general equals σ when $p_s = p_o$. $\eta = p/p_o$ at levels where coordinate surfaces are surfaces of constant pressure. Values of η in between half-levels are given by linear interpolation:

$$\eta = \eta_{k+\frac{1}{2}} + (p - p_{k+\frac{1}{2}})(\eta_{k+\frac{1}{2}} - \eta_{k-\frac{1}{2}})/(p_{k+\frac{1}{2}} - p_{k-\frac{1}{2}}) \text{ for } p_{k-\frac{1}{2}} \leq p \leq p_{k+\frac{1}{2}} \tag{2.2.6}$$

A 19-layer version is used operationally, and the corresponding values of the $A_{k+\frac{1}{2}}$ and $B_{k+\frac{1}{2}}$ are given in Table 2.2. The distribution of full-level pressures is shown in Fig.2.4. The top two layers are at constant pressures, and the lowest two layers are pure sigma layers. The value of p_o used for the definition of η is the reference sea-level pressure of 101325 Pa.

Table 2.2 Parameters specifying the vertical coordinate of the 19-layer operational model

k	$A_{k+\frac{1}{2}}$ (P_a)	$B_{k+\frac{1}{2}}$
0	0.000000	0.0000000000
1	2000.000000	0.0000000000
2	4000.000000	0.0000000000
3	6046.110595	0.0003389933
4	8267.92756	0.0033571866
5	10609.513232	0.0130700434
6	12851.100169	0.0340771467
7	14698.498086	0.0706498323
8	15861.125180	0.1259166826
9	16116.236610	0.2011954093
10	15356.924115	0.2955196487
11	13621.460403	0.4054091989
12	11101.561987	0.5249322235
13	8127.144155	0.6461079479
14	5125.141747	0.7596983769
15	2549.969411	0.8564375573
16	783.195032	0.9287469142
17	0.000000	0.9729851852
18	0.000000	0.9922814815
19	0.000000	1.0000000000



Full-level pressures in the operational model

Fig. 2.4 Distribution of full-level pressures

2.2.2 The vertical finite-difference scheme

The vertical finite-difference scheme is a generalization to the hybrid coordinate with form (2.2.1) of the scheme adopted in the first operational ECMWF model (Burridge and Haseler, 1977), apart from a small modification concerned with the conservation of angular momentum. The generalized scheme has been discussed by Simmons and Burridge (1981) and by Simmons and Strüfing (1981), and the presentation here is restricted to a prescription of the finite-diffference forms of the various terms of the continuous equations that involve η.

a) The surface-pressure tendency

The finite-difference analogue of (1.2.10) is

$$\frac{\partial \ln p_s}{\partial t} = -\frac{1}{p_s}\sum_{k=1}^{NLEV} \nabla \cdot (\underset{\sim}{v}_k \Delta p_k) \qquad (2.2.7)$$

where the subscript "k" denotes a value for the k-th layer, and

$$\Delta p_k = p_{k+\frac{1}{2}} - p_{k-\frac{1}{2}} \qquad (2.2.8)$$

From (2.2.1) we obtain

$$\frac{\partial \ln p_s}{\partial t} = -\sum_{k=1}^{NLEV} \left\{\frac{1}{p_s} D_k \Delta p_k + (\underset{\sim}{v}_k \cdot \nabla \ln p_s)\Delta B_k\right\} \qquad (2.2.9)$$

where $\Delta B_k = B_{k+\frac{1}{2}} - B_{k-\frac{1}{2}}$ (2.2.10)

b) The continuity equation

(1.2.9) gives

$$\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k+\frac{1}{2}} = -\frac{\partial p_{k+\frac{1}{2}}}{\partial t} - \sum_{j=1}^{k} \nabla \cdot (\underset{\sim}{v}_j \Delta p_j) \qquad (2.2.11)$$

and from (2.2.1)

$$\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k+\frac{1}{2}} = -p_s\left[B_{k+\frac{1}{2}}\frac{\partial \ln p_s}{\partial t} + \sum_{j=1}^{k}\left\{\frac{1}{p_s} D_j \Delta p_j + (\underset{\sim}{v}_j \cdot \nabla \ln p_s)\Delta B_j\right\}\right] \qquad (2.2.12)$$

where $\frac{\partial \ln p_s}{\partial t}$ is given by (2.2.9).

c) Vertical advection

Given $\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k+\frac{1}{2}}$ computed from (2.2.12), vertical advection of a variable X is given by

$$\left(\dot{\eta}\frac{\partial X}{\partial \eta}\right)_k = \frac{1}{2\Delta p_k}\left\{\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k+\frac{1}{2}}(X_{k+1} - X_k) + \left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k-\frac{1}{2}}(X_k - X_{k-1})\right\} \tag{2.2.13}$$

This form ensures that there is no spurious source or sink of kinetic energy, potential energy or moisture due to the finite-difference representation of vertical advection.

d) The hydrostatic equation

The form chosen for the finite-difference analogue of (1.2.6) is

$$\phi_{k+\frac{1}{2}} - \phi_{k-\frac{1}{2}} = - R_d(T_v)_k \ln \frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}} \tag{2.2.14}$$

which gives

$$\phi_{k+\frac{1}{2}} = \phi_s + \sum_{j=k+1}^{NLEV} R_d(T_v)_j \ln \frac{p_{j+\frac{1}{2}}}{p_{j-\frac{1}{2}}} \tag{2.2.15}$$

Full level values of geopotential are given by

$$\phi_k = \phi_{k+\frac{1}{2}} + \alpha_k R_d(T_v)_k \tag{2.2.16}$$

where $\alpha_1 = \ln 2$ (2.2.17)

and, for k>1,

$$\alpha_k = 1 - \frac{p_{k-\frac{1}{2}}}{\Delta p_k} \ln \left(\frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) \tag{2.2.18}$$

Reasons for this particular choice of the α_k are given below.

e) The pressure gradient term

It is shown by Simmons and Strüfing (1981) that if the geopotential is given by (2.2.16), the form

$$R_d (T_v \nabla \ln p)_k = \frac{R_d(T_v)_k}{\Delta p_k}\left\{\left(\ln \frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right)\nabla p_{k-\frac{1}{2}} + \alpha_k \nabla(\Delta p_k)\right\}. \tag{2.2.19}$$

for the pressure-gradient term ensures no spurious source or sink of angular momentum due to the vertical differencing. This expression is adopted in the model, but with the α_k given by (2.2.18) for all k. This ensures that the pressure-gradient term reduces to the familiar form $R_d(T_v)_k \nabla \ln p_s$ in the case of sigma coordinates, and the angular momentum conserving property of the scheme still holds in the case in which the first half-level below p=0 is a surface of constant pressure. The choice $\alpha_1=1$ in the hydrostatic equation would have given angular momentum conservation in general, but a geopotential ϕ_1 inappropriate to the pressure-level $p = p_1 = \Delta p/2$. If, alternatively, ϕ_1 were to be interpreted not as a value for a particular level, but rather the mass-weighted layer-mean value, then the choice $\alpha_1 = 1$ would be appropriate.

Using the form (2.2.1) for the half-level pressures (2.2.19) may be written

$$R_d(T_v \nabla \ln p)_k = \frac{R_d(T_v)_k}{\Delta p_k} \left\{\Delta B_k + C_k \frac{1}{\Delta p_k} \ln \frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right\} \nabla p_s \qquad (2.2.20)$$

where

$$C_k = A_{k+\frac{1}{2}} B_{k-\frac{1}{2}} - A_{k-\frac{1}{2}} B_{k+\frac{1}{2}} \qquad (2.2.21)$$

(f) Energy-conversion term

To obtain a form for the term $\kappa T_v \omega/(1+(\delta-1)q)p$ in (1.2.3) we use (1.2.7) to write

$$\left(\frac{\kappa T_v \omega}{(1+(\delta-1)q)p}\right)_k = \frac{\kappa (T_v)_k}{1+(\delta-1)q_k} \left(\frac{\omega}{p}\right)_k \qquad (2.2.22)$$

where

$$\left(\frac{\omega}{p}\right)_k = -\frac{1}{p}\int_0^{\eta_k} \nabla\cdot\left(\underset{\sim}{v}\frac{\partial p}{\partial \eta}\right)d\eta + (\underset{\sim}{v}\cdot\nabla \ln p)_k \qquad (2.2.23)$$

An expression for $\left(\frac{\omega}{p}\right)_k$ is then determined by the requirement that the difference scheme conserves the total energy of the model atmosphere for adiabatic, frictionless motion. This is achieved by

(i) evaluating the first term on the right-hand side of (2.2.23) by

$$-\frac{1}{\Delta p_k}\left\{\left(\ln \frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}}\right) \sum_{j=1}^{k-1} \nabla\cdot(\underset{\sim}{v}_j \Delta p_j) + \alpha_k \nabla\cdot(\underset{\sim}{v}_k \Delta p\quad\right\} \qquad (2.2.24)$$

where the α_k are as given by (2.2.17) and (2.2.18), and as in (2.2.9) and (2.2.11)

$$\nabla\cdot(\underset{\sim}{v}_k \Delta p_k) = D_k \Delta p_k + p_s (\underset{\sim}{v}_k\cdot\nabla \ln p_s)\Delta B_k \qquad (2.2.25)$$

(ii) using the form of (2.2.20) to evaluate the second term on the right-hand side of (2.2.23) by

$$(\underset{\sim}{v} \cdot \nabla \ell n p)_k = \frac{p_s}{\Delta p_k} \left\{ \Delta B_k + C_k \frac{1}{\Delta p_k} \ell n \frac{p_{k+\frac{1}{2}}}{p_{k-\frac{1}{2}}} \right\} \underset{\sim}{v}_k \cdot \nabla \ell n p_s \qquad (2.2.26)$$

2.3 TIME SCHEME

A semi-implicit time scheme is used for equations of divergence, temperature and surface pressure, based on the work of Robert et al. (1972). The growth of spurious computational modes is inhibited by a time filter (Asselin, 1972). In addition, it uses a semi-implicit method for the zonal advection terms in the vorticity and moisture equations, following results obtained by Robert (1981, 1982), who showed that in a semi-implicit shallow water equation model the time-step criterion was determined by the explicit treatment of the vorticity equation. Facilities also exist for selective damping of short model scales to allow use of longer timesteps. These are incorporated within the horizontal diffusion routines of the model, and are described in Sect.2.4.

The semi-implicit schemes are formally given by:

$$\delta_t \xi = ZT \qquad - \frac{1}{2a} \beta_{ZQ} \frac{U_r(\mu)}{(1-\mu^2)} \frac{\partial}{\partial\lambda} \Delta_{tt} \xi \tag{2.3.1}$$

$$\delta_t q = QT \qquad - \frac{1}{2a} \beta_{ZQ} \frac{U_r(\mu)}{(1-\mu^2)} \frac{\partial}{\partial\lambda} \Delta_{tt} q \tag{2.3.2}$$

$$\delta_t D = DT - \nabla^2 G \qquad - \frac{1}{2} \beta_{DT} \nabla^2 \{\gamma \Delta_{tt} T + R_d T_r \Delta_{tt} \ln p_s\} \tag{2.3.3}$$

$$\delta_t T = TT \qquad - \frac{1}{2} \beta_{DT} \tau \Delta_{tt} D \tag{2.3.4}$$

$$\delta_t \ln p_s = PT \qquad - \frac{1}{2} \beta_{DT} \nu \Delta_{tt} D \tag{2.3.5}$$

Here the terms ZT, QT, DT, G, TT, PT represent those on the right-hand sides of equations (1.2.15),(1.2.4), (1.2.16), (1.2.3) and (1.2.10), apart from the diffusion terms, which are neglected here. Adiabatic components are evaluated at the current time, t, and parametrized components are generally evaluated using values of fields at the previous timestep, $t-\Delta t$, full details of the latter being given in the Physical Parametrization Manual. The treatment of diffusion terms is described in the following section.

The remaining terms on the right-hand sides of (2.3.1) - (2.3.5) are corrections associated with the semi-implicit time schemes, and are discussed more fully below. The operators δ_t and Δ_{tt} are given by

$$\delta_t X = (X^+ - X_f^-)/2\Delta t \tag{2.3.6}$$

and $$\Delta_{tt}X = (X^{+} + X_f^{-} - 2X) \qquad (2.3.7)$$

where X represents the value of a variable at time t, X^+ the value at time $t+\Delta t$, and X_f^- a time-filtered value at time $t-\Delta t$. A further operator that will be used is

$$\tilde{\Delta}_{tt}X = X_f^{-} - 2X \qquad (2.3.8)$$

The time filtering is defined by

$$X_f = X + \varepsilon_f(X_f^{-} - 2X + X^{+}) \ , \qquad (2.3.9)$$

and it is computationally convenient to split it into two parts:

$$\tilde{X}_f = X + \varepsilon_f(X_f^{-} - 2X) \qquad (2.3.10)$$

$$X_f = \tilde{X}_f + \varepsilon_f X^{+} \qquad (2.3.11)$$

Operationally, Δt = 15 min, and $\varepsilon_f = 0.1$.

a) <u>The semi-implicit treatment of vorticity and moisture</u>

Referring to equations (2.3.1) and (2.3.2), an explicit treatment of the vorticity and moisture equations is obtained by setting $\beta_{ZQ} = 0$. Otherwise $\beta_{ZQ} = 1$ and $U_r(\mu)$ is a zonally-uniform reference zonal velocity, multiplied by $\cos\theta$. Terms describing advection by this reference velocity are represented implicitly by the arithmetic mean of values at times $t+\Delta t$ and $t-\Delta t$, while the remainder of the tendencies are represented explicitly by values at time t. $U_r(\mu)$ may vary in the vertical.

Because of the use of integration by parts in the derivation of the prognostic equation (2.1.29) for the spectral coefficients of vorticity, it is necessary to treat the vorticity and moisture equations separately. Considering first the moisture equation, we obtain from (2.3.2) and (2.3.6) - (2.3.8):

$$(1 + 2\Delta t\alpha(\mu)\frac{\partial}{\partial\lambda})\, q^{+} = q_f^{-} + 2\Delta t\ QT - 2\Delta t\alpha(\mu)\frac{\partial}{\partial\lambda}\, \tilde{\Delta}_{tt}q \qquad (2.3.12)$$

where

$$\alpha(\mu) = \frac{1}{2a}\, \beta_{ZQ}\, \frac{U_r(\mu)}{1-\mu^2} \qquad (2.3.13)$$

Transforming to Fourier space then gives

$$q_m^+ = b_m(\mu)\{(q_f^- + 2\Delta t\ QT)_m - 2im\Delta t\alpha(\mu)\tilde{\Delta}_{tt}q_m\} \tag{2.3.14}$$

where

$$b_m(\mu) = (1 + 2im\Delta t\alpha(\mu))^{-1} \tag{2.3.15}$$

New values $(q_n^m)^+$ of the spectral coefficients of q are then computed by Gaussian integration as described in Sect.2.12.

For the vorticity equation, (1.2.15) is used to write

$$ZT = \frac{1}{a(1-\mu^2)}\frac{\partial}{\partial\lambda}(F_V + P_V) - \frac{1}{a}\frac{\partial}{\partial\mu}(F_U + P_U) \tag{2.3.16}$$

where the horizontal diffusion term has for convenience been neglected, since as specified in the following section it merely modifies the value of vorticity computed for time t+Δt. Proceeding as for the moisture equation, we obtain

$$\xi_m^+ = b_m(\mu)\{(\xi_f^- + \frac{2im\Delta t}{a(1-\mu^2)}(F_V + P_V))_m - 2im\Delta t\alpha(\mu)\tilde{\Delta}_{tt}\xi_m - \frac{2\Delta t}{a}\frac{\partial}{\partial\mu}(F_U + P_U)_m\} \tag{2.3.17}$$

The factor $b_m(\mu)$ renders the right-hand side of this equation unsuitable for direct integration by parts, but a suitable form is found from the relation

$$b_m(\mu)\frac{\partial}{\partial\mu}(F_U + P_U) = \frac{\partial}{\partial\mu}\{b_m(\mu)(F_U + P_U)\} - c_m(\mu)(F_U + P_U) \tag{2.3.18}$$

where

$$c_m(\mu) = \frac{\partial}{\partial\mu} b_m(\mu) \tag{2.3.19}$$

This gives

$$\xi_m^+ = \tilde{Z}_{\lambda m}(\mu) + \frac{\partial}{\partial\mu}\tilde{Z}_{\mu m}(\mu) \tag{2.3.20}$$

where

$$\tilde{Z}_{\lambda m}(\mu) = b_m(\mu)(\xi_f^-)_m + 2\Delta t\{imb_m(\mu)[\frac{(F_V + P_V)_m}{a(1-\mu^2)} - \alpha(\mu)\tilde{\Delta}_{tt}\xi_m] + \frac{1}{a}c_m(\mu)(F_U + P_U)_m\} \tag{2.3.21}$$

and $$\tilde{Z}_{\mu m}(\mu) = -\frac{2\Delta t}{a}b_m(\mu)\ (F_U + P_U)_m \tag{2.3.22}$$

New values $(\xi_n^m)^+$ are obtained from (2.3.20) by Gaussian quadrature, using integration by parts as illustated by (2.1.21) and (2.1.29) for the continuous form of the equations.

Operationally, $\beta_{ZQ} = 1$. $U_r(\mu)$ is the arithmetic mean of the maximum and minimum velocities multiplied by $\cos\theta$, as computed for each latitude and model level at time step $t-\Delta t$. Different values are thus used for different levels.

b) The semi-implicit treatment of divergence, temperature and surface pressure

Referring to equations (2.3.3) - (2.3.5), an explicit treatment of the divergence, temperature and surface pressure equations is obtained by setting $\beta_{DT}=0$. For $\beta_{DT}=1$, the nature of the semi-implicit correction is such that gravity wave terms for small amplitude motion about a basic state with isothermal temperature T_r and surface pressure p_r are treated implicitly by the arithmetic mean of values at times $t+\Delta t$ and $t-\Delta t$, while the remainder of tendencies are represented explicitly by values at time t. The choice of an isothermal reference temperature is governed by considerations of the stability of the semi-implicit time scheme (Simmons et al, 1978), while the appropriate choice of p_r for the hybrid vertical coordinate is discussed by Simmons and Burridge (1981) and Simmons and Strüfing (1981).

γ, τ and ν in equations (2.3.3) - (2.3.5) are operators obtained from linearizing the finite-difference forms specified in Sect.(2.2.2) about the reference state (T_r, P_r). Their definitions are

$$(\gamma T)_k = \alpha_k^r R_d T_k + \sum_{j=k+1}^{NLEV} R_d T_j \ \ln\left(\frac{p_{j+\frac{1}{2}}^r}{p_{j-\frac{1}{2}}^r}\right) \tag{2.3.23}$$

$$(\tau D)_k = \kappa T_r \left\{\frac{1}{\Delta p_k^r}\left(\ln \frac{p_{k+\frac{1}{2}}^r}{p_{k-\frac{1}{2}}^r}\right) S_{k-\frac{1}{2}}^r + \alpha_k^r D_k\right\} \tag{2.3.24}$$

and

$$\nu D = \frac{1}{p_r} S_{NLEV+\frac{1}{2}}^r \tag{2.3.25}$$

where

$$p^r_{k+\frac{1}{2}} = A_{k+\frac{1}{2}} + p_r\ B_{k+\frac{1}{2}}$$

$$\Delta p^r_k = p^r_{k+\frac{1}{2}} - p^r_{k-\frac{1}{2}} \qquad (2.3.26)$$

$$S^r_{k+\frac{1}{2}} = \sum_{j=1}^{k} D_j \Delta p^r_j$$

and the α^r_k are defined by (2.2.17) and (2.2.18), but with half-level pressures replaced by reference values $p^r_{k+\frac{1}{2}}$.

Expanding (2.3.3) - (2.3.5) using (2.3.6) and (2.3.7), and writing ℓ to denote ℓnp_s, we obtain

$$D^+ = D^-_f + 2\Delta t(DT) - 2\Delta t\nabla^2\{G + \frac{1}{2}\beta_{DT}[\gamma(T^+ + T^-_f - 2T) + R_d T_r(\ell^+ + \ell^-_f - 2\ell)]\} \qquad (2.3.27)$$

$$T^+ = T_1 - \Delta t\ \beta_{DT}\ \tau D^+ \ , \qquad (2.3.28)$$

and

$$\ell^+ = \ell_1 - \Delta t\ \beta_{DT}\ \nu D^+ \ , \qquad (2.3.29)$$

where

$$T_1 = T^-_f + 2\Delta t(TT) - \Delta t\beta_{DT}\tau\tilde{\Delta}_{tt}D \ , \qquad (2.3.30)$$

and

$$\ell_1 = \ell^-_f + 2\Delta t(PT) - \Delta t\beta_{DT}\nu\tilde{\Delta}_{tt}D \ , \qquad (2.3.31)$$

Substituting (2.3.28) and (2.3.29) into (2.3.27) then gives

$$(1 - \Gamma\ \nabla^2)D^+ = DT' \qquad (2.3.32)$$

where

$$\Gamma = (\beta_{DT})^2(\Delta t)^2\ (\gamma\tau + R_d T_r \nu) \ , \qquad (2.3.33)$$

$$DT' = D^-_f + 2\Delta t(DT) + \nabla^2 R = \tilde{D}_\lambda + \tilde{D}_\mu + \nabla^2 R \qquad (2.3.34)$$

with

$$\tilde{D}_\lambda = D_f^- + \frac{2\Delta t}{a(1-\mu^2)} \frac{\partial}{\partial\lambda} (F_U + P_U) \tag{2.3.35}$$

$$\tilde{D}_\mu = \frac{2\Delta t}{a} \frac{\partial}{\partial\mu} (F_V + P_V) \tag{2.3.36}$$

and

$$R = -2\Delta t \left\{ G + \frac{B_{DT}}{2} (\gamma T_2 + R_d T_r \ell_2) \right\} \tag{2.3.37}$$

Here

$$T_2 = T_1 + T_f^- - 2T \tag{2.3.38}$$

and $$\ell_2 = \ell_1 + \ell_f^- - 2\ell \tag{2.3.39}$$

The sequence of these semi-implicit calculations in the model is thus as follows. The expressions (2.3.30), (2.3.31) and (2.3.37) - (2.3.39) are computed on the Gaussian grid to form the gridpoint values of R. The spectral expansion of DT' is then derived by Gaussian quadrature, using integration by parts as illustrated by (2.1.22) and (2.1.30) for the continuous form of the equations. Since

$$\left\{(1 - \Gamma\nabla^2)D^+\right\}_n^m = (1 + \frac{n(n+1)}{a^2}\Gamma)\ (D^+)_n^m ,$$

the spectral coefficients of divergence at time $t+\Delta t$ are given from (2.3.32) by

$$(D^+)_n^m = (1 + \frac{n(n+1)}{a^2}\Gamma)^{-1}(DT')_n^m , \tag{2.3.40}$$

where this operation involves, for each (m,n), multiplication of the vector of NLEV values of $(DT')_n^m$ by a pre-computed NLEVxNLEV matrix whose elements are independent of time and determined by writing the operators γ, τ and ν in matrix and vector form. Finally, (2.3.28) and (2.3.29) are applied in spectral space to compute spectral coefficients of T and $\ell n p_s$ at time $(t+\Delta t)$ in terms of the spectral coefficients of T_1 and ℓ_1 (again determined by Gaussian quadrature) and those of D^+.

Operationally, $\beta_{DT} = 0.75$, $T_r = 300K$ and $p_r = 800$ hPa.

2.4 HORIZONTAL DIFFUSION

2.4.1 Basic scheme

The basic "horizontal" smoothing of vorticity, divergence and specific humidity is represented by a simple linear 4th-order diffusion applied along the hybrid coordinate surfaces:

$$K_X = - K\nabla^4 X \tag{2.4.1}$$

where $X = \xi$, D or q. It is applied in spectral space to $t+\Delta t$ values such that if X_n^m is a spectral coefficient of X computed for timestep $t+\Delta t$ prior to diffusion, then the diffused value $\bar{X}_n^m$ is given by

$$\bar{X}_n^m = X_n^m - 2K\Delta t \left(\frac{n(n+1)}{a^2}\right)^2 \bar{X}_n^m \tag{2.4.2}$$

or

$$\bar{X}_n^m = X_n^m \left\{1+2K\Delta t \left(\frac{n(n+1)}{a^2}\right)^2\right\}^{-1} \tag{2.4.3}$$

A modified diffusion is used for temperature to avoid an unrealistic warming of mountain tops and excessive summer precipitation associated with substantial mixing in the vicinity of steep mountain slopes. A computationally convenient form which approximates diffusion on pressure surfaces is used. Equation (2.4.3) becomes

$$\bar{T}_n^m = (T_c)_n^m + (T_n^m - (T_c)_n^m)\left\{1+2K\Delta t \left(\frac{n(n+1)}{a^2}\right)^2\right\}^{-1} \tag{2.4.4}$$

where

$$T_c = \left(p_s \frac{\partial p}{\partial p_s} \frac{\partial T}{\partial p}\right)_{ref} \ln p_s \tag{2.4.5}$$

and $(\ )_{ref}$ denotes reference values varying only with the model level. These values are based on the standard ICAO atmosphere, and that for level k is given by

$$\left(p_s \frac{\partial p}{\partial p_s} \frac{\partial T}{\partial p}\right)_{ref} = 0.5\,(B_{k+\frac{1}{2}} + B_{k-\frac{1}{2}})\,\alpha\, T_{rc} p_{rs}/p_{rk}\ , \quad T_{rc} > T_{rt}\ ,$$

$$= 0 \qquad\qquad T_{rc} < T_{rt}\ , \tag{2.4.5}$$

$$\text{where } T_{rc} = T_{rs}\,(p_{rk}/p_{rs})^{\alpha} \tag{2.4.6}$$

and

$$p_{rk} = \frac{1}{2}(A_{k+\frac{1}{2}} + A_{k-\frac{1}{2}}) + \frac{1}{2}(B_{k+\frac{1}{2}} + B_{k-\frac{1}{2}})p_{rs} \tag{2.4.7}$$

A facility also exists to increase diffusion at stratospheric levels. The coefficient K is multiplied at level N by a factor $(\text{edif})^{\ell}$, where $\ell=0$ for $N>N_2$ and $\ell=\min\{1+N_2-N,\ 1+N_2-N_1\}$ otherwise.

Operationally, $K = 10^{15}\ m^4\ s^{-1}$, for all fields except divergence, for which the value $2.5 \times 10^{15}\ m^4\ s^{-1}$ is used. The parameters of the ICAO reference atmosphere are:

$$p_{rs} = 1013.20 \text{ hPa} \quad ,$$

$$T_{rs} = 288\text{K} \quad ,$$

$$\alpha = 1/5.256 \quad ,$$

and

$$T_{rt} = 216.5 \text{ K} \quad .$$

The increased stratospheric diffusion is specified by edif=2, $N_1=2$ and $N_2=5$.

2.4.2 Enhanced diffusion to enable use of longer timesteps

Experience with earlier versions of the operational forecasting system has revealed two situations which limit the timestep possible in the model. The first is the occurrence of a strong polar-night jet in the stratosphere during late winter in the Southern Hemisphere. The second is a strong tropospheric jet stream over the Western Pacific during the Northern Hemisphere winter. Facilities now exist for selective enhancement of horizontal diffusion to allow use of longer timesteps than would otherwise be possible in such situations.

The first entails a general substantial increase in horizontal diffusion for the smallest scales at upper model levels. For each level k, a critical total wavenumber n_k is defined. The diffusion coefficient K specified in Sect.2.4.1 is then used for $n < n_k$, while for $n > n_k$ the value $K\,h^c$ is used, where c is the number of levels (below and including k) for which the level number, ℓ, is such that both $\ell > k$ and $n_\ell < n$. Operationally h = 10 and

$$n_k = 82,\ 84,\ 86,\ 88,\ 90,\ 93,\ 96,\ 100,\ 103,\ 105,\ 106,\ldots 106 \text{ for}$$

$$k = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ldots\ldots \text{NLEV}.$$

This effectively acts as a reduction in model resolution at stratospheric levels, without generating the noise found in tests in which the highest-wavenumber components were simply set to zero at these levels.

The second option is to increase damping at model levels where the maximum wind exceeds a critical value. For a particular model level, spectral components whose total wavenumber exceeds a critical value n_{crit} (which depends on the maximum windspeed at that level) are damped at the timestep in question by a factor

$$1 + \alpha \frac{\Delta t}{a} \left[\mathrm{Max}\left\{|\underset{\sim}{u}|\right\}\right](n-n_{crit}) \tag{2.4.8}$$

where

$$n_{crit} = \beta/\mathrm{Max}\left\{|\underset{\sim}{u}|\right\} \tag{2.4.9}$$

Values $\alpha > 2$ and $\beta = a/\Delta t$ are sufficient to avoid exponential computational instability for the linear advection equation for a wave of scale a/n and advecting velocity $\mathrm{Max}\left\{|\underset{\sim}{u}|\right\}$. In the model, β is defined through a critical velocity, V_{crit}:

$$\beta = V_{crit} \frac{\Delta t_o n_o}{\Delta t}$$

where $\Delta t_o = 1200s$ and $n_o = 63$. Here V_{crit} should correspond to a critical velocity for stability (with $\alpha=0$) for a model resolution with $\mathrm{Max}\{n\}=63$ and a 20 minute timestep. Operationally, $\alpha = 2.5$ and $V_{crit} = 85\ m\ s^{-1}$, the latter giving $\beta = 1.009\ a/\Delta t$.

CHAPTER 3

Normal Mode Initialization

3.1 INTRODUCTION

Primitive equation models, unlike quasi-geostrophic models, generally admit high frequency gravity wave solutions, as well as the slower moving Rossby wave solutions. If the results of the analysis scheme are used directly as initial conditions for a forecast, subtle imbalances between the mass and wind fields will cause the forecast to be contaminated by spurious high-frequency gravity-wave oscillations of much larger amplitude than are observed in the real atmosphere. Although these oscillations tend to die away slowly due to various dissipation mechanisms in the model, they make the forecast noisy and they may be quite detrimental to the analysis cycle, in which the six-hour forecast is used as a first-guess field for the next analysis. The synoptic changes over the six-hour period may be swamped by spurious changes due to the oscillations, with the consequence that at the next analysis time, good data may be rejected as being too different from the first-guess field. For this reason, an initialization step is performed between the analysis and the forecast, with the object of eliminating the spurious oscillations.

The principle of the method is to express the analysed fields in terms of the normal modes of free oscillation of the model atmosphere, then to modify the coefficients of the fast moving gravity modes in such a way that their rate of change vanishes.

3.2 COMPUTATION OF THE NORMAL MODES

The first step is to compute the modes of free oscillation of the model atmosphere. For this purpose the model equations are linearized about a basic state at rest, with a temperature profile $\bar{T}(\eta)$ function of height only. The model equations can be written in matrix form:

$$\frac{\partial \underline{D}}{\partial t} - f\underline{\xi} + \beta\underline{u} + \nabla^2 \underline{P} = \underline{R}_D = 0$$

$$\frac{\partial \underline{\xi}}{\partial t} + \beta\underline{V} + f\underline{D} = \underline{R}_\xi = 0 \qquad (3.2.1)$$

$$\frac{\partial \underline{P}}{\partial t} + \underline{\underline{B}}\,\underline{D} = \underline{R}_P = 0$$

The terms on the right hand side contain all the nonlinear tendencies and are here set to zero. The vector notation is used in (3.2.1) to represent the values at all the model levels. The vertical structure matrix $\underline{\underline{B}}$ is given in Simmons and Strüfing (1982, Eq.4.5). It depends on the basic state chosen and on the numerical technique used in the vertical discretization. The auxiliary potential P is defined as $P = \bar{\phi} + R\bar{T}\ \ln p_s$. In the definition of the geopotential of the mean state $\bar{\phi}$ a mean surface pressure $\bar{p}_s$ is assumed.

In order to separate the vertical dependence from the horizontal in (3.2.1) the model variables $\underline{D}$, $\underline{\xi}$ and $\underline{P}$ are expressed in terms of the eigenvectors $\underline{\psi}_m$ of matrix $\underline{\underline{B}}$. For example:

$$\underline{D} = \sum_{m=1}^{M} D_m\, \underline{\psi}_m \qquad (3.2.2)$$

The equations obtained after substitution of 3.2.2 into 3.2.1 have the form of M independent systems of shallow-water equations with equivalent geopotential depth ϕ_m, equal to the eigenvalue corresponding to $\underline{\psi}_m$.

After performing the vertical separation, the M two-dimensional systems may be separated in the zonal direction by Fourier transforming the variables; thus we write e.g.

$$D_m(\lambda, \theta, t) = \sum_{k=0}^{N-1} D_{m,k}(\theta, t) \exp(ik\lambda) \qquad (3.2.3)$$

If we now call $\underline{x}_{m,k}$ the vector which contains $D_{m,k}$, $\xi_{m,k}$ and $P_{m,k}$ (scaled to be non-dimensional), the system of linear equations becomes formally:

$$\frac{d\underline{x}_{m,k}}{dt} = i\, \underline{\underline{A}}_{m,k}\, \underline{x}_{m,k} \qquad (3.2.4)$$

The matrix $\underline{\underline{A}}_{m,k}$ is real and symmetric. Hence its eigenvectors are orthogonal. They form a set of horizontal normal modes which can be used to express $\underline{x}$ (dropping the indices m,k for simplicity of notation):

$$\underline{x} = \sum_{\ell=1}^{3L} c_\ell\, \underline{\xi}_\ell \qquad (3.2.5)$$

In fact these modes naturally divide into two classes: symmetric and antisymmetric with respect to the equator. This property is used to reduce the dimension of the matrix $\underline{\underline{A}}$ when finding its eigenvectors which are the normal modes required.

3.3 THE INITIALIZATION PROCESS

Using (3.2.5), the equation (3.2.4) can now be written

$$\frac{dc_\ell}{dt} = i\ \nu_\ell\ c_\ell \qquad (3.3.1)$$

for each ℓ, with ν_ℓ being the eigenvalues of $\underline{\underline{A}}$.

Hence

$$\underline{x}(t) = \sum_\ell c\ (0)\ \exp\ (i\nu_\ell t)\ \underline{\xi}_\ell \qquad (3.3.2)$$

where the amplitudes $c_\ell(0)$ are determined by the values of D, ξ,P at t = 0. At least for the first few vertical modes (with large equivalent depths ϕ_m) there is a clear distinction between low-frequency Rossby wave solutions (small ν_ℓ) and high frequency gravity wave solutions (large ν_ℓ). Only solutions of the former type are observed in the atmosphere with significant amplitude.

If the real model equations were linear, it would be easy to ensure that high frequency gravity waves do not exist by simply reducing to zero the corresponding normal mode coefficient $c_\ell(0)$ of the analysis. But this method does not work for the full nonlinear model.

The equivalent of (3.3.1) for the nonlinear equations is

$$\frac{dc_\ell}{dt} = i\ \nu_\ell\ c_\ell + r_\ell(t) \qquad (3.3.3)$$

The term r_ℓ is the projection of the nonlinear terms of the model equations (computed by running one time step of the model) onto the normal modes.
If we simply make $c_\ell = 0$ for t=0, very soon this mode will reappear, forced by r_ℓ. This was shown by Williamson (1976).

Machenhauer (1977) has proposed an iterative scheme for removing the gravity-mode oscillations by setting the initial time-derivatives of the gravity-mode coefficients to zero. From (3.3.3),

$$\left(\frac{dc_\ell}{dt}\right)_{t=0} = 0 \quad \text{if } c_\ell(0) = -\frac{r_\ell(0)}{i\ \nu_\ell}\ . \qquad (3.3.4)$$

Since the nonlinear term $r_\ell(0)$ depends partly on the gravity-mode coefficients themselves, it is necessary to iterate the procedure; but for a barotropic model (or for the first few vertical modes of a multi-level model) the scheme converges rapidly, and two iterations are perfectly adequate.

In the current version of the analysis cycle we perform two iterations of Machenhauer's procedure, initializing just the first five vertical modes. (The higher internal modes have very low frequencies and thus do not contribute to the problem of spurious high-frequency oscillations). The nonlinear forcing terms are computed by running the model itself for one timestep at each iteration.

Although in principle the non-linear forcing can include the "physics" package as well as the dynamics, in practice this leads to the immediate divergence of the iteration process. Following Wergen (1987), an estimate d_ℓ of the quasi-stationary part of the physical forcing is used, which is kept constant for all iterations. This estimate is computed by time-averaging the physical tendencies during a 2 hour forecast starting from an uninitialized analysis. Only those components which force inertia-gravity waves with periods longer than a certain cut-off period are retained, thus discarding less reliable small-scale structures. Operationally this cut-off period is 11 hours. In order to obtain only the stationary part of the physical forcing, the diurnal cycle is switched off during this 2 hour forecast.

The filter condition (3.3.4) now reads:

$$c_\ell(0) = - \frac{r_\ell(0) + d_\ell}{i \nu_\ell} \qquad (3.3.5)$$

Since the initialization condition (3.3.4) requires stationarity for the initialization of inertia-gravity waves, it clearly mishandles the tidal component of the atmospheric circulation. It should be allowed to propagate westwards and therefore be excluded from the initialization process. Again, following Wergen (1987), this is achieved by performing a time series analysis of the total dynamical and physical tendencies for the ten days preceeding the actual analysis time. The westward propagating component with a 24 hour period for zonal wavenumber one and a 12 hour period for zonal wavenumber two are excluded from (3.3.4) for all five vertical modes and for the eight gravest meridional modes. With t_ℓ being the tidal component of the tendencies, the initialization condition becomes:

$$C_\ell(0) = - \frac{r_\ell(0) + d_\ell - t_\ell}{i \quad \nu_\ell} \tag{3.3.6}$$

The steps of the initialization procedure can be summarized as follows:

1. Run model for 2 hours from the uninitialized analysis to compute time-averaged physical forcing without diurnal cycle.
2. Filter physical forcing field.
3. Run adiabatic model for one timestep to compute non-linear terms.
4. Compute new gravity mode coefficients according to (3.3.6)
5. Restore analysed surface pressure after first iteration.
6. As 3 but starting from results of first iteration step.
7. As 4 (second iteration).