Graf's addition theorem

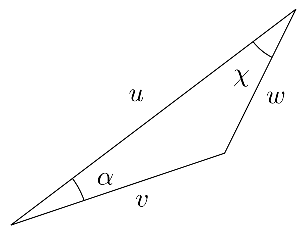


Figure 1: Graf's addition theorem

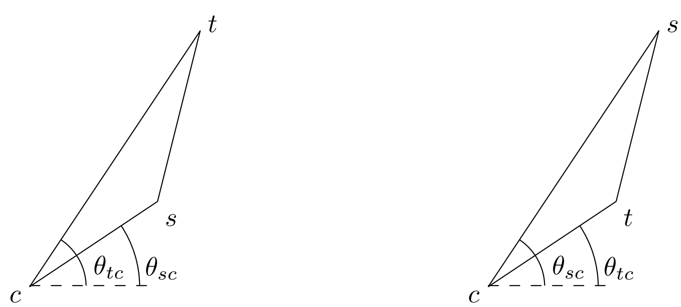
Graf's addition theorem says

$$C_\nu(w)e^{i\nu\chi} = \sum_{k=-\infty}^{\infty} C_{\nu+k}(u)\; J_k(v)e^{ik\alpha}$$

where C_ν can be a Hankel or Bessel function of index ν. This holds when $|u| > |v|$. [1]

Expansions

In all subsequent formulas, θ_{xy} refers to the angle of the vector $x - y$ above the horizontal.



(a) Multipole expansion

(b) Local expansion

Figure 2: Multipole and local expansions

For a multipole expansion, we wish to evaluate $H_0^1(|t-s|)$ where the target is farther from the center than the source. The multipole expansion takes the form

$$H_0^{(1)}(|t-s|) = \sum_{k=-\infty}^{\infty} \underbrace{J_k(|s-c|)e^{-ik\theta_{sc}}}_{\text{coefficients}} H_k^{(1)}(|t-c|)e^{ik\theta_{tc}}.$$

[1]See for instance `http://dlmf.nist.gov/10.23#ii`

In the local expansion, the target is closer to the center than the source. The local expansion takes the form

$$H_0^{(1)}(|t-s|) = \sum_{k=-\infty}^{\infty} \underbrace{H_k^{(1)}(|s-c|)e^{ik\theta_{sc}}}_{\text{coefficients}} J_k(|t-c|)e^{-ik\theta_{tc}}.$$

Multipole-to-multipole and local-to-local translations

We wish to shift the center of the expansion c_1 to a new center c_2 satisfying $|c_1 - c_2| < |s - c_1|$. The goal is to derive a formula for the new coefficients based on the old coefficients and $c_1 - c_2$. This can be done with the help of Graf's addition theorem.

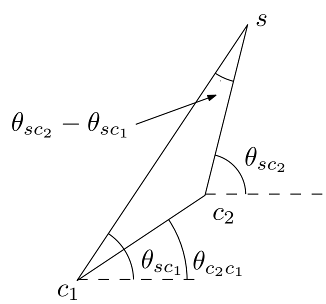


Figure 3: Multipole-to-multipole and local-to-local translation

For shifting the multipole coefficients:

$$J_k(|s-c_2|)e^{-ik\theta_{sc_2}} = \sum_{l=-\infty}^{\infty} \underbrace{J_{k+l}(|s-c_1|)e^{-i(k+l)\theta_{sc_1}}}_{\text{old coefficients}} J_l(|c_2-c_1|)e^{il\theta_{c_2c_1}}.$$

In a similar way, for shifting the local expansion coefficients:

$$H_k^{(1)}(|s-c_2|)e^{ik\theta_{sc_2}} = \sum_{l=-\infty}^{\infty} \underbrace{H_{k+l}^{(1)}(|s-c_1|)e^{i(k+l)\theta_{sc_1}}}_{\text{old coefficients}} J_l(|c_2-c_1|)e^{-il\theta_{c_2c_1}}.$$

Multipole-to-local translation

Given a multipole expansion with center c_1, we wish to shift to center c_2 where c_1 and c_2 satisfy $|c_2 - c_1| > |s - c_1|$. Furthermore, the coefficients at the new center will be coefficients for a local expansion.

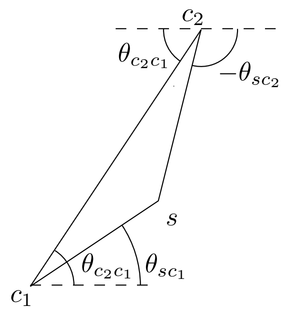


Figure 4: Multipole-to-local translation

The translated coefficients satisfy

$$H_k^{(1)}(|s-c_2|)e^{ik\theta_{sc_2}} = (-1)^k \sum_{l=-\infty}^{\infty} \underbrace{J_l(|s-c_1|)e^{-il\theta_{sc_1}}}_{\text{old coefficients}} H_{k+l}^{(1)}(|c_1-c_2|)e^{i(k+l)\theta_{c_2c_1}}.$$