**Association of X and P in Classical Physics and Information/Invariance Part II**

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In part I we argued that Newton's first law implies that constant momentum (i.e. momentum conservation) is associated with spatial invariance i.e. a lack of information in a certain direction. This is equivalent to stating there exists no force in that direction. We argued that one may apply the generator for translation in the spatial direction i.e. d/dx for x and apply it to a function f(x) such that df(x)/dx = (px). The simplest form is f(x)= (px)x. We also argued that it is precisely this function which appears in Fermat's least time principle because time = distance/speed and one may multiply time by |p| = momentum magnitude.

Given that momentum and displacement are parallel vectors for a light ray, p dot r = p distance = (px)x + (py)y for a two dimensional problem (e.g. reflection or refraction). "Distance d" is the key function to use because dd= xx + yy implies that for fixed y, d d/dx = x/d = sin(theta1) (angle measured from y axis) so p d d/dx = (px) which is a conserved quantity if x is invariant. Thus conservation of momentum for two rays is equivalent to extremization. Extremization, we argue, suggests there may be a built-in physical mechanism which allows a free particle to correct itself if perturbed slightly just like a Maxwell-Boltzmann gas which extremizes entropy subject to constant average energy does.

This built-in mechanism should be linked to motion in space. Writing (px)x as ln(exp(ipx), one not only obtains an eigenvalue equation i.e. -d/dx exp(i (px) x) = (px) exp(i px x), but the notion of a relative change arises if exp(i px x) is interpreted as a kind of probability. A relative change is natural because an absolute one ignores the idea that probability may differ with x even though (px) is constant.

Thus from the idea of invariance in space found in Newton's first law, which is equivalent to conservation of momentum in that direction, the equation d/dx f(x) = (px) leads directly to -id/dx exp(i px x) = (px) exp(ipx x) where exp(i px x) represents a physical probability type of wave which stablilizes the constant (px) motion because this motion is equivalent to extremizing time or |p| time or p dot r / speed of light in Fermat's principle. In other words, the quantum mechanics of a free particle is associated with stabilizing wave behaviour exp(ipx x) which was not seen experimentally at the time of Newton or Fermat, but seems to be part of their formalisms, with Fermat's principle derivable from Newton's first law.

## Conservation of Momentum and Extremization

The idea of conservation of momentum follows from Newtonian mechanics. In fact, we argue it is present in the first law indicating that a particle with uniform speed moving in a certain direction continues in this state unless acted on by a force. In Part I we argued that this indicates invariance in this direction (as no force exists in the direction). As it stands, however, this law is imposed on paper so to speak, but how is it imposed in nature?

This notion may be linked to the idea of extremization which is found in both Newtonian mechanics and the equilibrium of a Maxwell-Botlzmann gas and which further seems to be linked with periodic motion. Thus a particle in a stable equilibrium at x=xo may undergo periodic motion about this point if perturbed and a gas perturbed in density may create periodic sound waves. Also if f(e1) is increased or decreased there is a tendency for equilibrium to be restored. In the case of Newton's first and second laws, one does not generally think in terms of extremization.

Extremization, on the other hand, is the basis of Fermat's least time principle. In general, however, Fermat's least time principle is considered separate from Newton's laws. In Part I, however, we argued that an extremization equation follows from Newton's first law. In particular, consider a two dimensional (i.e. two ray problem e.g. reflection) problem with the flat mirror surface along the x axis at y=0. y=0 is a constraint so there is no invariance in the y direction in the sense that the first ray must hit at y=0 and the second must begin at y=0. X, however, is completely invariant, hence motion is uniform in x or more generally one has constant (px). (One may argue that momentum and velocity are not the same. In Newtonian mechanics they differ by the multiplicative constant mo, rest mass, but for light in a refracting medium E=pc so p$\rightarrow$pn and c$\rightarrow$c/n. Thus in general the two are different and we focus on p because it is linked with energy transfer.)

If (px) is constant we may consider (p1x) = (p2x) where p1 refers to the ray before it hits the mirror and (p2x) after. One may further consider (p1y) = (-p2y). In such a case the magnitude of momentum is the same before and after reflection which makes sense from the point of time reversal symmetry. The (p1x) - (p2x) = p sin(theta1) - p sin(theta2) =0 , where theta is measured from the y axis, is in the form of a extremum problem if sin(theta) may be written as d/dx of a function.

The simplest function is: A = (px) x. Then dA/d = (px). Thus for a two ray reflection problem:

A1 + A2 = (px) x + (px)(L-x) and d/dx (A1+A2) = 0 or (px)=(px) ((1))

Here L is linked to the fixed starting and endpoint, namely (x=0, Y) and (L-x, Y).
Fermat's principle uses the more involved form: time= distance/ speed of light with distance1 = sqrt(xx + YY) and d2 = sqrt( (L-x)(L-x) + YY).

We note, however, that the momentum and displacement vectors of each ray are parallel to each other so in two dimensions:

P dot r = (px) x + (py) y = p distance = p sqrt( xx + YY ) or p sqrt( (L-x)(L-x) + YY) ((2)).

Thus as noted in Part I, Fermat's least time principle follows from Newton's first law and establishes a link between d/dx and (px) which is seen in free particle quantum mechanics

**Concept of Probability**

We have argued so far that both conservation of momentum and spatial invariance, say in the x direction ,are found in Newton's first law. Furthermore, we suggest that the spatial generator in this invariant direction may act on a function A to yield the conserved quantity with:

A = (px) x (This may be generalized ultimately to -Et + p dot r).)

One may write: (px)x = -i ln(exp(ipx)). We do not use exp(px x) or exp(-px x) because they grow to infinite for different x regions.

Thus we have again "periodicity" associated with extremization. What, however, does this periodicity mean? We argue that nature should have a mechanism to ensure constant (px) if a particle moves in the x direction with x being invariant. This may be linked to periodic motion, just a perturbed particle in stable static equilibrium is associated with such motion. Thus we consider exp(ipx) to represent a kind of "probability" in space associated with a periodic probability wave linked to constant (px) motion. One may note that the modulus exp(-ipx) exp(ipx) = 1, so P(x) or spatial density is constant in x even though exp(ipx) is not. Such motion could not be detected in Newton's or Fermat's time, but was seen in experiments in the 1920s.

We note that in information theory, ln(probability) is called information. If exp(ipx) is a kind of probability, then (px) x might also be called formally information.

Thus we argue that the formalism of quantum mechanics pertaining to a free particle seems to follow directly from Newton's first law (which in turn implies Fermat's minimum time principle). One not only has the generator of translations in the direction of an invariant variable x i.e. d/dx linked with (px) i.e. momentum in that direction, but one is led to the notion of a kind of probability exp(i px x). This form has an explicit wavelength proportional to 1/(px) which may be seen in experiment. This suggests that conserved motion which may be expressed in terms of extremization may have a physical mechanism which ensures the motion is stable.

**Conclusion**

In Part II we extend the idea of Part I that Newton's law links invariance in a spatial direction, say x, with conservation of momentum (px) in that direction. The generator of translation in x is d/dx so one may imagine a function f(x) such that df/dx = (px). The simplest is f(x)=A(x) = (px)x or p dot r if the p vector lies in a different direction. One may ask: What is the purpose of such a function? We argue it is useful in an extremization problem. For example, consider a two dimensional ray problem (e.g. reflection). (p1x) = (p2x). One may further assume that p1=p2 based on time reversal symmetry. Then: A = (px) x + (py) y. If x is the invariant variable because there exists a flat mirror at y=0, p dot r = p distance = p sqt(xx+YY) and sqrt( (L-x)(L-x) + YY) for the two rays with initial/final points of (0,Y) and (L-x), Y). Taking d/dx of A1 + A2 and setting it equal to 0 creates an extremization equation equivalent to conservation of (px).

At first one may consider this to be simply mathematical formalism, but the idea of extrema appearing in stable equilibrium scenarios is found throughout physics with perturbations associated with periodic motion. One may ask: Is there a physical mechanism ensuring that (px) is conserved (constant) for an invariant x direction? We argue that (px) x = -i ln (exp(ipx x). exp(ipx) may be considered as a kind of probability which does not grow to infinite. Then d/dx { -i ln(exp(ipx x)) is equivalent to a kind of relative change linked to ensuring px remains constant. This is the description of quantum mechanics for a free particle, which we argue follows directly from Newton's first law, via Fermat's least time principle. Ultimately, however, experiment must confirm the existence of a wavelength proportional to 1/(px), but this already happened in the 1920s.