Compound Option Model

A vanilla compound option is defined as a European vanilla option upon another European vanilla option, which may be called underlying vanilla option. There are four types of compound options: call-on-call, call-on-put, put-on-call and put-on-put. Due to the call-put parity, basically, we only need to consider call-on-call and call-on-put. In this report, under the assumption that the asset price, which is the underlyer of the underlying option, follows geometrical Brownian motion and that risk-free short rate, dividend yield and volatility are deterministic, we present Black-Scholes/Merton's analytical close form pricing formula for vanilla compound options.

Let $\{St\}$ be the price of a given asset which follows the following SDE

$$\mathrm{d}S_t = \mu_t S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t \ , \qquad t > 0 \ ,$$

where $\{Wt\}$ is a standard R-valued Wiener process, *¹t* and *¾t* are deterministic drift term and volatility, respectively. Let rt be the deterministic risk-free short interest rate. For $t < s$, let us define

$$\tilde{\mu}(t,s) = \int_t^s \mu \mathrm{d}\tau \ , \quad \tilde{\sigma}(t,s) = \sqrt{\int_t^s \sigma^2 \mathrm{d}\tau} \ , \quad \tilde{r}(t,s) = \int_t^s r \mathrm{d}\tau \ .$$

The discounting factor from s back to t, denoted by df($t; s$), can be written as

$$\mathrm{df}(t,s) = \exp[-\tilde{r}(t,s)] \ ,$$

and a forward price seen at *t* matured at *s*, denoted by *F*(*t; s*), can be written as

$$F(t,s) = S_t \cdot \exp(\tilde{\mu}(t,s)) \ .$$

Let *t* < *T* and *ST* = *eZT* . Then we have

$$Z_T \ \sim_t \ \mathrm{N}\left(m_T = \ln S_t + \tilde{\mu}(t,T) - \frac{\tilde{\sigma}^2(t,T)}{2} \ , \ v_T = \tilde{\sigma}^2(t,T) \right) \ , \qquad t \leq T \ .$$

Further, Let *t* < *T*1 < *T*2, *ST*1 = *eZ*1 and *ST*2 = *eZ*2 . Then, *Z*1 »*t* N(*m*1*; v*1), *Z*2 »*t* N(*m*2*; v*2), and relative to the time of *t*, (*Z*1*;Z*2) is jointly normal distributed with the following correlation coefficient

$$\rho = \mathrm{Corr}_t(Z_1, Z_2) = \frac{\tilde{\sigma}(t,T_1)}{\tilde{\sigma}(t,T_2)} \ .$$

Let *f*(¢*;* ¢) be the joint density function of (*Z*1*;Z*2) relative to time *t*, *f*1(¢) is the density function of *Z*1 relative to time *t* and *f*2*j*1(¢ ; ¢) is the density function of *Z*2 conditional on *Z*1 relative to time *t*. Clearly, we have

$$f(z_1, z_2) = f_1(z_1) \cdot f_{2|1}(z_2; z_1) \ ,$$

where

$$
\begin{aligned}
f(z_1) &= \frac{1}{\sqrt{2\pi v_1}} \cdot \exp\left[-\frac{1}{2}\frac{(z_1 - m_1)^2}{v_1}\right] , \\
f(z_1, z_2) &= \frac{1}{2\pi\sqrt{v_1 v_2 (1-\rho^2)}} \cdot \\
&\quad \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(z_1 - m_1)^2}{v_1} - 2\rho\frac{(z_1 - m_1)(z_2 - m_2)}{\sqrt{v_1 v_2}} + \frac{(z_2 - m_2)^2}{v_2}\right]\right\} .
\end{aligned}
$$

Let T be a maturity of the compound option with a strike $K > 0$, $T1 > T$ be the maturity of the underlying option with a strike $K1 > 0$ and a call-put index °1. Then the compound option payoff at the maturity of T becomes

$$
[p(T, S_T; K_1, T_1, \gamma_1) - K]^+ \ .
$$

We have

$$
p_c(t, S_t; K, \Gamma) = \mathrm{df}(t, T) \cdot \mathrm{E}_t\left[\{p(T, S_T; K_1, T_1, \gamma_1) - K\}^+\right] \ .
$$

After substituting

$$
p(T, S_T; K_1, T_1, \gamma_1) = \mathrm{df}(T, T_1) \cdot \mathrm{E}_T\left[\gamma_1 \cdot (S_{T_1} - K_1)^+\right]
$$

In the following section, we will obtain analytical close form pricing formulae for the call-on-call and call-on-put compound options.

The price dynamic follows:

$$\frac{\partial}{\partial S} p(T, S; K_1, T_1, 1) = \mathrm{df}(T, T_1) \exp[\tilde{\mu}(T, T_1)] \Phi_1(D) > 0 \ ,$$

$$D = \frac{\ln \dfrac{F(T, T_1)}{K_1} + \dfrac{\tilde{\sigma}^2(T, T_1)}{2}}{\tilde{\sigma}(T, T_1)} \ .$$

References:

https://finpricing.com/lib/FiBond.html