Riemann Hypothesis on Ramanujan's Function

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Abstract

Srinivasa Ramanujan studied the function $S_1(x) = \sum_{\rho} \frac{x^{\rho-1}}{\rho \cdot (1-\rho)}$ where ρ runs over the nontrivial zeros of the Riemann ζ function. Under the Riemann hypothesis, we know that $|S_1(x)| \leq \frac{\tau}{\sqrt{x}}$ for $\tau = 2 + \gamma - \log(4 \cdot \pi) \approx 0.04619$. The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. There are several statements equivalent to the famous Riemann hypothesis. In 2011, Solé and Planat stated that, the Riemann hypothesis is true if and only if the inequality $\zeta(2) \cdot \prod_{p \leq x} (1 + \frac{1}{p}) > e^{\gamma} \cdot \log \theta(x)$ holds for all $x \geq 5$, where $\theta(x)$ is the Chebyshev function, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, $\zeta(x)$ is the Riemann zeta function and $\log$ is the natural logarithm. In this note, using Solé and Planat criterion, we prove that, when the Riemann hypothesis is false, then there are infinitely many natural numbers x for which $\frac{\log x}{\sqrt{x}} - \frac{10}{\sqrt{x}} + 2 \cdot \log x + S_1(x) \cdot \sqrt{x} \cdot \log x \leq 2.062$ could be satisfied. In addition, we show that the Riemann hypothesis is true when $S_1(x) \geq \frac{\varepsilon}{\sqrt{x}}$ for $\varepsilon \geq -1.9999999$ and large enough x.

Keywords: Riemann hypothesis, Riemann zeta function, Chebyshev function

MSC Classification: 11M26 , 11A25

1 Introduction

The hypothesis was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems. Leonhard Euler discovered a particular value of the Riemann zeta function (1734).

Proposition 1 *It is known that[1, (1) p. 1070]:*

$$\zeta(2) = \prod_{i=1}^{\infty} \frac{p_i^2}{p_i^2 - 1} = \frac{\pi^2}{6}.$$

Proposition 2 *[2]. For $x \geq 10^8$:*

$$\begin{aligned} \sum_{p>x} \log\left(\frac{p^2}{p^2-1}\right) &\geq \sum_{p\geq y}\left(\frac{1}{p^2}\right) \\ &\geq \frac{1}{y\cdot\log y} - \frac{1}{y\cdot\log^2 y} + \frac{2}{y\cdot\log^3 y} - \frac{9}{y\cdot\log^4 y} \\ &\geq \frac{1}{x\cdot\log x} - \frac{10}{x\cdot\log^2 x} \end{aligned}$$

where $y = x + 1$ and $\log$ is the natural logarithm.

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p\leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x. For $x \geq 2$, we say that $\mathsf{Dedekind}(x)$ holds provided that

$$\zeta(2)\cdot\prod_{p\leq x}\left(1+\frac{1}{p}\right) > e^{\gamma}\cdot\log\theta(x)$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. Next, we have Solé and Planat Theorem:

Proposition 3 $\mathsf{Dedekind}(x)$ *holds for all natural numbers $x \geq 5$ if and only if the Riemann hypothesis is true [3, Theorem 4.2 p. 5].*

This is the main insight.

Lemma 1 *If the Riemann hypothesis is false, then there are infinitely many natural numbers $x \geq 5$ for which* $\mathsf{Dedekind}(x)$ *fails (i.e.* $\mathsf{Dedekind}(x)$ *does not hold).*

Srinivasa Ramanujan studied the function $S_1(x) = \sum_{\rho} \frac{x^{\rho-1}}{\rho\cdot(1-\rho)}$ where ρ runs over the nontrivial zeros of the Riemann ζ function [4, Section 65]. Thus, we have a Theorem about this function:

Proposition 4 *[5, (2.16)]. For $x \geq 10^9$:*

$$\sum_{p\leq x} \log\left(1-\frac{1}{p}\right) + \gamma + \log\log\theta(x) + \frac{2}{\sqrt{x}\cdot\log x} + \frac{S_1(x)}{\log x} \leq \frac{2.062}{\sqrt{x}\cdot\log^2 x}$$

This is the main theorem.

Theorem 1 *If the Riemann hypothesis is false, then there are infinitely many natural numbers x for which*

$$\frac{\log x}{\sqrt{x}} - \frac{10}{\sqrt{x}} + 2\cdot\log x + S_1(x)\cdot\sqrt{x}\cdot\log x \leq 2.062$$

could be satisfied.

The following is a key Corollary.

Corollary 1 *The Riemann hypothesis is true, when $S_1(x) \geq \frac{\varepsilon}{\sqrt{x}}$ for $\varepsilon \geq -1.9999999$ and large enough x.*

2 Proof of the Lemma 1

Proof According to Proposition 3, the Riemann hypothesis is false, if there exists some natural number $x_0 \geq 5$ such that $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$:

$$g(x) = \frac{e^{\gamma}}{\zeta(2)}\cdot\log\theta(x)\cdot\prod_{p\leq x}\left(1+\frac{1}{p}\right)^{-1}.$$

We know the bound [3, Theorem 4.2 p. 5]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}$$

where f was introduced in the Nicolas paper [6, Theorem 3 p. 376]:

$$f(x) = e^{\gamma}\cdot\log\theta(x)\cdot\prod_{p\leq x}\left(1-\frac{1}{p}\right).$$

When the Riemann hypothesis is false, then there exists a real number $b < \frac{1}{2}$ for which there are infinitely many natural numbers x such that $\log f(x) = \Omega_{+}(x^{-b})$ [6, Theorem 3 (c) p. 376]. According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0):\ \log f(y) \geq k\cdot y^{-b}.$$

That inequality is equivalent to $\log f(y) \geq \left(k\cdot y^{-b}\cdot\sqrt{y}\right)\cdot\frac{1}{\sqrt{y}}$, but we note that

$$\lim_{y\to\infty}\left(k\cdot y^{-b}\cdot\sqrt{y}\right) = \infty$$

for every possible positive value of k when $b < \frac{1}{2}$. In this way, this implies that

$$\forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0) \colon \ \log f(y) \geq \frac{1}{\sqrt{y}}.$$

Hence, if the Riemann hypothesis is false, then there are infinitely many natural numbers x such that $\log f(x) \geq \frac{1}{\sqrt{x}}$. Since $\frac{2}{x} = o(\frac{1}{\sqrt{x}})$, then it would be infinitely many natural numbers x_0 such that $\log g(x_0) > 0$. □

3 Proof of the Theorem 1

By Lemma 1, there are infinitely many natural numbers $x \geq 5$ for which

$$\gamma + \log\log\theta(x) \geq \sum_{p \leq x} \log\left(1 + \frac{1}{p}\right) + \log(\zeta(2))$$

could be satisfied, when the Riemann hypothesis is false. This implies that there are infinitely many natural numbers $x \geq 10^9$ for which

$$\sum_{p \leq x} \log\left(1 - \frac{1}{p}\right) + \sum_{p \leq x} \log\left(1 + \frac{1}{p}\right) + \log(\zeta(2)) + \frac{2}{\sqrt{x} \cdot \log x} + \frac{S_1(x)}{\log x} \leq \frac{2.062}{\sqrt{x} \cdot \log^2 x}$$

could be also satisfied by Proposition 4. We can write

$$\left(1 - \frac{1}{p}\right) \cdot \left(1 + \frac{1}{p}\right) = \left(1 - \frac{1}{p^2}\right).$$

for every prime [3, p. 3]. Consequently, we obtain that

$$\sum_{p \leq x} \log\left(1 - \frac{1}{p^2}\right) + \log(\zeta(2)) + \frac{2}{\sqrt{x} \cdot \log x} + \frac{S_1(x)}{\log x} \leq \frac{2.062}{\sqrt{x} \cdot \log^2 x}$$

by properties of logarithms. By Proposition 1, we know that

$$\sum_{p > x} \log\left(\frac{p^2}{p^2 - 1}\right) + \frac{2}{\sqrt{x} \cdot \log x} + \frac{S_1(x)}{\log x} \leq \frac{2.062}{\sqrt{x} \cdot \log^2 x}.$$

Using the Proposition 2, we can see that

$$\frac{1}{x \cdot \log x} - \frac{10}{x \cdot \log^2 x} + \frac{2}{\sqrt{x} \cdot \log x} + \frac{S_1(x)}{\log x} \leq \frac{2.062}{\sqrt{x} \cdot \log^2 x}.$$

Let's multiply both sides by $\sqrt{x} \cdot \log^2 x$ to show that

$$\frac{\log x}{\sqrt{x}} - \frac{10}{\sqrt{x}} + 2 \cdot \log x + S_1(x) \cdot \sqrt{x} \cdot \log x \leq 2.062$$

could be satisfied, when the Riemann hypothesis is false. Note that, there are not anomalies of signs after of multiplying by $\sqrt{x} \cdot \log^2 x$ since for $x \geq 10^9$ we obtain that $\log x > 20.7$.

4 Proof of Corollary 1

Proof By Theorem 1, there are infinitely many natural numbers x for which

$$\frac{\log x}{\sqrt{x}} - \frac{10}{\sqrt{x}} + 2 \cdot \log x + \varepsilon \cdot \log x \leq 2.062$$

could be satisfied, when the Riemann hypothesis is false and $S_1(x) \geq \frac{\varepsilon}{\sqrt{x}}$ for $\varepsilon \geq -1.9999999$ and large enough x. However, we can always assure that

$$\frac{\log x}{\sqrt{x}} - \frac{10}{\sqrt{x}} + 2 \cdot \log x + \varepsilon \cdot \log x \leq 2.062$$

never holds for every $\varepsilon \geq -1.9999999$ and large enough x. Note that, the Theorem 1 was proved over the domain of the natural numbers, so that is the same to say that there exists some $y > 0$ such that for all natural numbers $x > y$, then we can always confirm that the inequality never holds for every $\varepsilon \geq -1.9999999$. In conclusion, the Riemann hypothesis is true by principle of non-contradiction.

□

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