MATHEMATICAL SCIENCES

ON SOLVING A QUADRATIC DIOPHANTINE EQUATION

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Abstract
Diophantine Equations named after ancient Greek mathematician Diophantus, plays a vital role not only in number theory but also in several branches of science. In this paper, we will solve one of the quadratic Diophantine equations and provide its complete solutions. The method adopted to solve the given equation is using the concept of polar form of a particular complex number. This concept can be generalized for solving similar equations.

Keywords: Quadratic Diophantine Equation, Polar Form, Euler's Formula, Positive Integer Solutions.

1. Introduction

Diophantine Equations were equations whose solutions must be in integers. Since the solutions are integers and most often positive integers, such equations have more practical applications compared to other equations in mathematics. In this paper, we will solve one of the quadratic Diophantine equations in a novel way and present its complete solution in a compact form.

2. Quadratic Diophantine Equation

In this paper, we will try to solve the quadratic Diophantine equation $3x^2 + y^2 = 7^n$ (1), where x, y are positive integers. We will try to obtain a general solution of (1) in closed form. For doing this, we will make use of a particular complex number and a fabulous formula proposed by the greatest mathematician of all times, Leonhard Euler.

3. Solutions to the Equation

We will try to obtain all positive integer solutions (x, y) satisfying (1) for any given natural number n.

Now, we will try to determine the polar form of $\left(2+i\sqrt{3}\right)^n$

$$2+i\sqrt{3} = r\left(\cos\theta + i\sin\theta\right) \Rightarrow r\cos\theta = 2, r\sin\theta = \sqrt{3}$$

From this, we obtain $r^2 = 4+3 = 7 \Rightarrow r = \sqrt{7}, \theta = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (2)

Hence the polar form of $\left(2+i\sqrt{3}\right)^n$ is given by

$$\left(2+i\sqrt{3}\right)^n = 7^{n/2} e^{i n \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)} \quad (3)$$

Now using Euler's Formula in (3), we obtain

$$\left(2+i\sqrt{3}\right)^n = 7^{n/2}\left[\cos\left(n\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) + i\sin\left(n\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right] \quad (4)$$

If we now assume $y + i\sqrt{3}x = \left(2+i\sqrt{3}\right)^n$ (5) then $y - i\sqrt{3}x = \left(2-i\sqrt{3}\right)^n$ (6)

Now multiplying (5) and (6), we get

$$\left(y+i\sqrt{3}x\right)\times\left(y-i\sqrt{3}x\right) = \left(2+i\sqrt{3}\right)^n \times \left(2-i\sqrt{3}\right)^n$$

Simplifying, we obtain $3x^2 + y^2 = 7^n$ which is (1), the original problem which we have considered. Thus the solutions to (1) are given by equating real and imaginary parts of (5). Now using (4) in (5), and for $n \geq 1$ we get

$$\sqrt{3}x = 7^{n/2}\sin\left(n\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) \Rightarrow x = \frac{7^{n/2}}{\sqrt{3}}\sin\left(n\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) \quad (7)$$

$$y = 7^{n/2}\cos\left(n\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) \Rightarrow y = 7^{n/2}\cos\left(n\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) \quad (8)$$

Now from (7) and (8), if we consider $(|x|,|y|)$ then these pairs would provide all positive integer solutions to the given Quadratic Diophantine Equation $3x^2 + y^2 = 7^n$ for any natural number *n*.

4. Conclusion

Considering a quadratic Diophantine equation $3x^2 + y^2 = 7^n$ we have used a novel method to solve it completely in this paper. In particular, equations (7) and (8) provide all required positive integer solutions to the given equation. Further, by considering the polar form of a particular complex number, we have obtained nice closed expressions for the given equations.

In fact, from (7) and (8), we notice that for $n \geq 1$, all positive integer solutions to $3x^2 + y^2 = 7^n$ are given by

$$x = \frac{7^{n/2}}{\sqrt{3}}\left|\sin\left(n\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right|, y = 7^{n/2}\left|\cos\left(n\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right| \quad (9)$$

Thus, for *n* = 1, 2, 3, 4, 5, 6, 7, 8, . . . all positive integer solutions to $3x^2 + y^2 = 7^n$ are given respectively by (1,2); (4,1); (9,10); (8,47); (31,118); (180,143); (503,254); (752,2017); . . .

Thus the values of *x* and *y* from expression (9) provides all possible solutions to the given quadratic Diophantine equation $3x^2 + y^2 = 7^n$. We can adopt similar methods to solve other types of quadratic Diophantine equations using polar forms of suitable complex numbers.

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