On the nontrivial zeros of the Riemann zeta function

Frank Vega[1*]

[1*]Software Department, CopSonic, 1471 Route de Saint-Nauphary, Montauban, 82000, Tarn-et-Garonne, France.

Corresponding author(s). E-mail(s): vega.frank@gmail.com;

Abstract

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. There are several statements equivalent to the famous Riemann hypothesis. In 2011, Solé and Planat stated that, the Riemann hypothesis is true if and only if the inequality $\zeta(2) \cdot \prod_{p \leq x} (1 + \frac{1}{p}) > e^{\gamma} \cdot \log \theta(x)$ holds for all $x \geq 5$, where $\theta(x)$ is the first Chebyshev function, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, $\zeta(x)$ is the Riemann zeta function and $\log$ is the natural logarithm. In this note, using Solé and Planat criterion, we prove that, when the Riemann hypothesis is false, then there are infinitely many natural numbers x for which $\frac{\log x}{\sqrt{x}} - \frac{10}{\sqrt{x}} + 2 \cdot \log x + \varepsilon \cdot \log x \leq 2.062$ could be satisfied for some $\varepsilon > 0$. Since the inequality $\frac{\log x}{\sqrt{x}} - \frac{10}{\sqrt{x}} + 2 \cdot \log x + \varepsilon \cdot \log x \leq 2.062$ never holds for every $\varepsilon > 0$ and large enough x, then the Riemann hypothesis is true by principle of non-contradiction.

Keywords: Riemann hypothesis, Riemann zeta function, Chebyshev function

MSC Classification: 11M26 , 11A25

1 Introduction

The Riemann hypothesis is considered by many to be the most important unsolved problem in pure mathematics. It was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems. Leonhard Euler discovered a particular value of the Riemann zeta function (1734).

Proposition 1 *It is known that[1, (1) p. 1070]:*

$$\zeta(2) = \prod_{i=1}^{\infty} \frac{p_i^2}{p_i^2 - 1} = \frac{\pi^2}{6}.$$

Proposition 2 *For all $x \geq 10^9$:*

$$\sum_{p > x} \log\left(\frac{p^2}{p^2 - 1}\right) \geq \frac{1}{x \cdot \log x} - \frac{10}{x \cdot \log^2 x}$$

where log *is the natural logarithm [2].*

In mathematics, the first Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x. For $x \geq 2$, we say that $\mathsf{Dedekind}(x)$ holds provided that

$$\zeta(2) \cdot \prod_{p \leq x} \left(1 + \frac{1}{p}\right) > e^{\gamma} \cdot \log \theta(x)$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. Next, we have Solé and Planat Theorem:

Proposition 3 $\mathsf{Dedekind}(x)$ *holds for all natural numbers $x \geq 5$ if and only if the Riemann hypothesis is true [3, Theorem 4.2 p. 5].*

This is the main insight.

Lemma 1 *If the Riemann hypothesis is false, then there are infinitely many natural numbers $x \geq 5$ for which* $\mathsf{Dedekind}(x)$ *fails (i.e.* $\mathsf{Dedekind}(x)$ *does not hold).*

Srinivasa Ramanujan studied the function $S_1(x) = \sum_{\rho} \frac{x^{\rho - 1}}{\rho \cdot (1 - \rho)}$ where ρ runs over the nontrivial zeros of the Riemann ζ function [4, Section 65]. In number theory, the second Chebyshev function $\psi(x)$ is given by

$$\psi(x) = \sum_{p^k \leq x} \log p$$

with the sum extending over all prime powers p^k that are less than or equal to x. Thus, we have two Nicolas Theorems:

Proposition 4 *[2]. For all $x \geq 10^9$, there exists always some $\varepsilon > 0$ such that*

$$S_1(x) = -\int_1^x \frac{\psi(t)}{t^2}\,dt + \log x - 1 - \gamma + \frac{\log(2\cdot\pi)}{x} - \sum_{k=1}^{\infty} \frac{1}{2\cdot k\cdot(2\cdot k+1)\cdot x^{2\cdot k+1}}$$
$$= \frac{\varepsilon}{\sqrt{x}}.$$

Proposition 5 *[5, (2.16)]. For all $x \geq 10^9$:*

$$\sum_{p\leq x} \log\left(1-\frac{1}{p}\right) + \gamma + \log\log\theta(x) + \frac{2}{\sqrt{x}\cdot\log x} + \frac{S_1(x)}{\log x} \leq \frac{2.062}{\sqrt{x}\cdot\log^2 x}$$

This is the main theorem.

Theorem 1 *If the Riemann hypothesis is false, then there are infinitely many natural numbers x for which*

$$\frac{\log x}{\sqrt{x}} - \frac{10}{\sqrt{x}} + 2\cdot\log x + \varepsilon\cdot\log x \leq 2.062$$

could be satisfied for some $\varepsilon > 0$.

The following is a key Corollary.

Corollary 1 *The Riemann hypothesis is true.*

2 Proof of the Lemma 1

Proof According to Proposition 3, the Riemann hypothesis is false, if there exists some natural number $x_0 \geq 5$ such that $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$:

$$g(x) = \frac{e^\gamma}{\zeta(2)}\cdot\log\theta(x)\cdot\prod_{p\leq x}\left(1+\frac{1}{p}\right)^{-1}.$$

We know the bound [3, Theorem 4.2 p. 5]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}$$

where f was introduced in the Nicolas paper [6, Theorem 3 p. 376]:

$$f(x) = e^\gamma\cdot\log\theta(x)\cdot\prod_{p\leq x}\left(1-\frac{1}{p}\right).$$

When the Riemann hypothesis is false, then there exists a real number $b < \frac{1}{2}$ for which there are infinitely many natural numbers x such that $\log f(x) = \Omega_+(x^{-b})$ [6, Theorem 3 (c) p. 376]. According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0)\colon\ \log f(y) \geq k \cdot y^{-b}.$$

That inequality is equivalent to $\log f(y) \geq \left(k \cdot y^{-b} \cdot \sqrt{y}\right) \cdot \frac{1}{\sqrt{y}}$, but we note that

$$\lim_{y \to \infty} \left(k \cdot y^{-b} \cdot \sqrt{y}\right) = \infty$$

for every possible positive value of k when $b < \frac{1}{2}$. In this way, this implies that

$$\forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0)\colon\ \log f(y) \geq \frac{1}{\sqrt{y}}.$$

Hence, if the Riemann hypothesis is false, then there are infinitely many natural numbers x such that $\log f(x) \geq \frac{1}{\sqrt{x}}$. Since $\frac{2}{x} = o(\frac{1}{\sqrt{x}})$, then it would be infinitely many natural numbers x_0 such that $\log g(x_0) > 0$. □

3 Proof of the Theorem 1

By Lemma 1, there are infinitely many natural numbers $x \geq 5$ for which

$$\gamma + \log\log\theta(x) \geq \sum_{p \leq x} \log\left(1 + \frac{1}{p}\right) + \log(\zeta(2))$$

could be satisfied, when the Riemann hypothesis is false. This implies that there are infinitely many natural numbers $x \geq 10^9$ for which

$$\sum_{p \leq x} \log\left(1 - \frac{1}{p}\right) + \sum_{p \leq x} \log\left(1 + \frac{1}{p}\right) + \log(\zeta(2)) + \frac{2}{\sqrt{x} \cdot \log x} + \frac{S_1(x)}{\log x} \leq \frac{2.062}{\sqrt{x} \cdot \log^2 x}$$

could be also satisfied by Proposition 5. We can write

$$\left(1 - \frac{1}{p}\right) \cdot \left(1 + \frac{1}{p}\right) = \left(1 - \frac{1}{p^2}\right).$$

for every prime [3, p. 3]. Consequently, we obtain that

$$\sum_{p \leq x} \log\left(1 - \frac{1}{p^2}\right) + \log(\zeta(2)) + \frac{2}{\sqrt{x} \cdot \log x} + \frac{S_1(x)}{\log x} \leq \frac{2.062}{\sqrt{x} \cdot \log^2 x}$$

by properties of logarithms. By Proposition 1, we know that

$$\sum_{p > x} \log\left(\frac{p^2}{p^2 - 1}\right) + \frac{2}{\sqrt{x} \cdot \log x} + \frac{S_1(x)}{\log x} \leq \frac{2.062}{\sqrt{x} \cdot \log^2 x}.$$

Using the Propositions 2 and 4, we can see that

$$\frac{1}{x \cdot \log x} - \frac{10}{x \cdot \log^2 x} + \frac{2}{\sqrt{x} \cdot \log x} + \frac{\varepsilon}{\sqrt{x} \cdot \log x} \leq \frac{2.062}{\sqrt{x} \cdot \log^2 x}.$$

Let's multiply both sides by $\sqrt{x} \cdot \log^2 x$ to assume that

$$\frac{\log x}{\sqrt{x}} - \frac{10}{\sqrt{x}} + 2 \cdot \log x + \varepsilon \cdot \log x \leq 2.062$$

could be satisfied for some $\varepsilon > 0$, when the Riemann hypothesis is false. Note that, there are not anomalies of signs after of multiplying by $\sqrt{x} \cdot \log^2 x$ since for $x \geq 10^9$ we obtain that $\log x > 20.7$.

4 Proof of Corollary 1

Proof By Theorem 1, we proved that there are infinitely many natural numbers x for which

$$\frac{\log x}{\sqrt{x}} - \frac{10}{\sqrt{x}} + 2 \cdot \log x + \varepsilon \cdot \log x \leq 2.062$$

could be satisfied for some $\varepsilon > 0$, when the Riemann hypothesis is false. However, we can always assure that

$$\frac{\log x}{\sqrt{x}} - \frac{10}{\sqrt{x}} + 2 \cdot \log x + \varepsilon \cdot \log x \leq 2.062$$

never holds for every $\varepsilon > 0$ and large enough x. Note that, the Theorem 1 was proved over the domain of the natural numbers, so that is the same to say that there exists some $y > 0$ such that for all natural numbers $x > y$, then we can always confirm that the inequality never holds for every $\varepsilon > 0$. In conclusion, the Riemann hypothesis is true by principle of non-contradiction.

□

5 Conclusions

Practical uses of the Riemann hypothesis include many propositions that are known to be true under the Riemann hypothesis and some that can be shown to be equivalent to the Riemann hypothesis. Indeed, the Riemann hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf hypothesis, the Large Prime Gap Conjecture, etc. Certainly, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics in general.

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