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$m-$ SUBHARMONIC AND STRONGLY $m-$ SUBHARMONIC FUNCTIONS

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Abstract. *In this paper is considered whether the function* $u=\ln\left(x_1^2+x_2^2+...+x_n^2\right)=\ln[(z_1+\overline{z}_1)^2+(z_2+\overline{z}_2)^2+...+(z_n+\overline{z}_n)^2]-2\ln 2$ *belongs to the class of m-subharmonic and strongly m-subharmonic functions.*

Keywords: *m-subharmonic function, strongly m-subharmonic function, vector hessian, eigenvalues of the Hermit matrix.*

$m-$ СУБГАРМОНИЧЕСКИЕ И СИЛЬНО $m-$ СУБГАРМОНИЧЕСКИЕ ФУНКЦИИ

Аннотация. *В работе рассматривается принадлежность функции* $u=\ln\left(x_1^2+x_2^2+...+x_n^2\right)=\ln[(z_1+\overline{z}_1)^2+(z_2+\overline{z}_2)^2+...+(z_n+\overline{z}_n)^2]-2\ln 2$ *к классу* $m-$ *субгармонических и сильно* $m-$*субгармонических функций.*

Ключевые слова. $m-$*субгармоническая функция, сильно* $m-$*субгармоническая функция, вектор гессиан, собственные значения матрицы Эрмита.*

INTRODUCTION

The pluripotential theory which connected with nonlinear Monge-Ampere equation and plurisubharmonic functions was built in 80s of last century. This theory was constructed due to fundamental results of mathematicians from USA, Poland, Sweden, France and Uzbekistan (E.Bedford, B.A.Taylor, J. Sisiak, H-J.Bremerman, U. Cegrell, A.Zeriahi, A.Sadullaev) and etc. In the 1990s there were many attempts to develop and expand the pluripotensial theory to broader classes of functions. One such class is the $m-$ subharmonic $(m-sh)$ functions $(1\le m\le n)$.

The theory of potentials in the class of $m-sh$ functions was developed by A. Sadullaev and B. Abdullaev (see [1]) and is now successfully used in a number of areas of mathematics.

By B.I.Abdullayev in the article[5], this $u=\ln\left(x_1^2+x_2^2+x_3^2\right)=\ln[(z_1+\overline{z}_1)^2+(z_2+\overline{z}_2)^2+(z_3+\overline{z}_3)^2]-2\ln 2$, (1) (where z_i is equal to the following $z_j=x_j+y_j,\ j=1,2,3$) function is proved to belong to the classes $1-sh$ and $2-sh$.

Also, the question of whether this function belongs to the class of strongly 2-subharmonic was raised. In the article, we show that the function $u=\ln\left(x_1^2+x_2^2+...+x_n^2\right)=\ln[(z_1+\overline{z}_1)^2+(z_2+\overline{z}_2)^2+...+(z_n+\overline{z}_n)^2]-2\ln 2$ (2) is sh_m when $n-\left(\left[\frac{n}{2}\right]-1\right)\le m\le n,\ \ n\ge 2$ and $m-sh$ is when $2\le m\le n$.

MATERIALS AND METHODS

$m-$subharmonic and strongly $m-$subharmonic functions.

For a twice differentiable function $u\ \text{О}C^2(D)$ the second-order differential form

$$dd^c u = \frac{i}{2}\sum_{j,k} u_{j,\bar{k}} dz_j \wedge d\overline{z_k} \qquad (3)$$

where $u_{j\bar{k}} = \frac{\partial^2 u}{\partial z_j \partial \overline{z_k}}$, is Hermitian quadratic form at a fixed point $z^0 \in D$.

Here, as usual

$$du = \partial u + \bar{\partial} u = \sum_{k=1}^{n} \frac{\partial u}{\partial z_k} dz_k + \sum_{k=1}^{n} \frac{\partial u}{\partial \overline{z_k}} d\overline{z_k}$$

$$d^c u = \frac{1}{4i}\left(\sum_{k=1}^{n} \frac{\partial u}{\partial z_k} dz_k - \sum_{k=1}^{n} \frac{\partial u}{\partial \overline{z_k}} d\overline{z_k} \right).$$

After a suitable unitary transformation [3] at a fixed point point $z^0 \in D$, the Hermitian differential form reduced to the diagonal view:

$$dd^c u = \frac{i}{2}\left(\lambda_1 dz_1 \wedge d\overline{z_1} + \lambda_2 dz_2 \wedge d\overline{z_2} + \ldots + \lambda_n dz_n \wedge d\overline{z_n} \right). \qquad (4)$$

In this case, $l_1, \ldots, l_n$ are the eigenvalues of the Hermitian $\left(u_{j,\bar{k}}\right)$ matrix, and $\lambda = \left(\lambda_1, \ldots, \lambda_n\right) \in \mathbb{R}^n$ are real values. The operator $\left(dd^c u\right)^m \wedge \beta^{n-m}$ can be written as $\left(dd^c u\right)^m \wedge \beta^{n-m} = m!(n-m)!\mathrm{H}_m(u)\beta^n$, where $H_m(u) = \sum_{1 \le j_1 < j_2 < \ldots < j_m \le n} \lambda_{j_1}\lambda_{j_2}\ldots\lambda_{j_m} -$ is $mth-$ order Hessian of the vector $\lambda\left(u, z^0\right) = \lambda = \left(\lambda_1, \ldots, \lambda_n\right) \in \mathbb{R}^n$.

Definiton 1. An upper semicontinuous in a domain $D \subset \mathbb{C}^n$ function $u(z)$ is said to be $m-$subharmonic in D, $u \in m-sh(D)$, if $dd^c u \wedge \beta^{m-1} \ge 0$, in the generalized sense, as current, i.e.

$$dd^c u \wedge \beta^{m-1}(\omega) = \int u\beta^{m-1} \wedge dd^c \omega \ge 0, \quad \forall \omega \in F^{n-m,n-m}, \ \omega \ge 0.$$

Here $\beta = dd^c |z|^2 -$ is the standard volume form of the space $\mathbb{C}^n$ and $F^{n-m,n-m}$ is the space of compactly supported in D smooth differential forms of bi-degree $(n-m, n-m)$. Note that

$$psh(D) = 1-sh(D) \subset m-sh(D) \subset n-sh(D) = sh(D).$$

This definition can be expressed following for twice smooth functions:

Definition 2. If for a twice smooth $u \in C^2(D)$ $D \subset \mathbb{C}^n$ function (at a fixed point $z^0 \in D$) the inequality

$$dd^c u \wedge \beta^{m-1} \ge 0 \qquad (5)$$

Holds, then this function is called $m-$ subharmonic $(1 \le m \le n)$ function in a domain D.

We express the definition of strongly $m-$subharmonic function as follows:

Definition3. If for a twice smooth $u \in C^2(D)$ $D \subset \mathbb{C}^n$ function (at a fixed point $z^0 \in D$) the inequalities

$$\left(dd^c u\right)^k \wedge \beta^{n-k} \ge 0, \quad \forall k = 1, 2, \ldots, n-m+1 \qquad (6)$$

Holds, then this function is called strongly $m-$ subharmonic $(1 \le m \le n)$ function in a domain D.

Eigenvalues of the Hermit matrix. We find the $\lambda_1, \lambda_2, ..., \lambda_n$ eigenvalues of the function (2) Hermite matrix $\left(u_{j\bar{k}}\right)$. First, we perform the following calculations:

$$\frac{\partial u}{\partial z_1} = \frac{1}{|z+\bar{z}|^2} 2\left(z_1+\bar{z}_1\right) = 2\frac{z_1+\bar{z}_1}{|z+\bar{z}|^2};$$

$$\frac{\partial u}{\partial z_2} = \frac{1}{|z+\bar{z}|^2} \cdot 2(z_2+\bar{z}_2) = 2\frac{z_2+\bar{z}_2}{|z+\bar{z}|^2};$$

$$\frac{\partial u}{\partial z_3} = \frac{1}{|z+\bar{z}|^2} \cdot 2(z_3+\bar{z}_3) = 2\frac{z_3+\bar{z}_3}{|z+\bar{z}|^2};$$

....

$$\frac{\partial u}{\partial z_n} = \frac{1}{|z+\bar{z}|^2} \cdot 2(z_n+\bar{z}_n) = 2\frac{z_n+\bar{z}_n}{|z+\bar{z}|^2};$$

1) $$u_{1\bar{1}} = \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_1} = \frac{2|z+\bar{z}|^2}{|z+\bar{z}|^4} - \frac{2(z_1+\bar{z}_1)\cdot 2(z_1+\bar{z}_1)}{|z+\bar{z}|^4} = \frac{2}{|z+\bar{z}|^2} - \frac{4(z_1+\bar{z}_1)^2}{|z+\bar{z}|^4};$$

2) $$u_{1\bar{2}} = \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_2} = -\frac{2(z_1+\bar{z}_1)\cdot 2(z_2+\bar{z}_2)}{|z+\bar{z}|^4} = -4\frac{(z_1+\bar{z}_1)(z_2+\bar{z}_2)}{|z+\bar{z}|^4};$$

3) $$u_{1\bar{3}} = \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_3} = -4\frac{(z_1+\bar{z}_1)(z_3+\bar{z}_3)}{|z+\bar{z}|^4};$$

4) $$u_{2\bar{1}} = \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_1} = -4\frac{(z_1+\bar{z}_1)(z_2+\bar{z}_2)}{|z+\bar{z}|^4};$$

5) $$u_{2\bar{2}} = \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_2} = \frac{2}{|z+\bar{z}|^2} - \frac{4(z_2+\bar{z}_2)^2}{|z+\bar{z}|^4};$$

6) $$u_{2\bar{3}} = \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_3} = -4\frac{(z_2+\bar{z}_2)(z_3+\bar{z}_3)}{|z+\bar{z}|^4};$$

7) $$u_{3\bar{1}} = \frac{\partial^2 u}{\partial z_3 \partial \bar{z}_1} = -4\frac{(z_1+\bar{z}_1)(z_3+\bar{z}_3)}{|z+\bar{z}|^4};$$

8) $$u_{3\bar{2}} = \frac{\partial^2 u}{\partial z_3 \partial \bar{z}_2} = -4\frac{(z_2+\bar{z}_2)(z_3+\bar{z}_3)}{|z+\bar{z}|^4}$$

$$u_{3\bar{3}} = \frac{\partial^2 u}{\partial z_3 \partial \bar{z}_3} = \frac{2}{|z+\bar{z}|^2} - \frac{4(z_3+\bar{z}_3)^2}{|z+\bar{z}|^4};$$

$$\Rightarrow u_{s\bar{t}} = \frac{\partial^2 u}{\partial z_s \partial \bar{z}_t} = -4\frac{(z_s+\bar{z}_s)(z_t+\bar{z}_t)}{|z+\bar{z}|^4}, \quad s,t=\overline{1,n};\ s\neq t$$

$$u_{s\bar{s}} = \frac{\partial^2 u}{\partial z_s \partial \bar{z}_s} = \frac{2}{|z+\bar{z}|^2} - \frac{4(z_s+\bar{z}_s)^2}{|z+\bar{z}|^4}, \quad s=\overline{1,n}.$$

Now let's look at the following determinant:

$$\begin{vmatrix} u_{1\bar{1}}-\lambda & \cdots & u_{1\bar{n}} \\ \vdots & \ddots & \vdots \\ u_{n\bar{1}} & \cdots & u_{n\bar{n}}-\lambda \end{vmatrix} = \begin{vmatrix} \frac{2}{|z+\bar{z}|^2} - \frac{4(z_1+\bar{z}_1)^2}{|z+\bar{z}|^4} - \lambda & \cdots & -4\frac{(z_1+\bar{z}_1)(z_n+\bar{z}_n)}{|z+\bar{z}|^4} \\ \vdots & \ddots & \vdots \\ -4\frac{(z_1+\bar{z}_1)(z_n+\bar{z}_n)}{|z+\bar{z}|^4} & \cdots & \frac{2}{|z+\bar{z}|^2} - \frac{4(z_n+\bar{z}_n)^2}{|z+\bar{z}|^4} - \lambda \end{vmatrix} =$$

$$\left(-4\frac{(z_1+\bar{z}_1)}{|z+\bar{z}|^4}\right)\cdots\left(-4\frac{(z_n+\bar{z}_n)}{|z+\bar{z}|^4}\right) \begin{vmatrix} \frac{|z+\bar{z}|^4}{-4(z_1+\bar{z}_1)}\left(\frac{2}{|z+\bar{z}|^2} - \frac{4(z_1+\bar{z}_1)^2}{|z+\bar{z}|^4} - \lambda\right) & \cdots & (z_n+\bar{z}_n) \\ \vdots & \ddots & \vdots \\ (z_1+\bar{z}_1) & \cdots & \frac{|z+\bar{z}|^4}{-4(z_n+\bar{z}_n)}\left(\frac{2}{|z+\bar{z}|^2} - \frac{4(z_n+\bar{z}_n)^2}{|z+\bar{z}|^4} - \lambda\right) \end{vmatrix} = 0$$

To make it easier to calculate the above determinant, we will make the following transformation:

$$x_k = \frac{2}{|z+\bar{z}|^2} - \frac{4(z_k+\bar{z}_k)^2}{|z+\bar{z}|^4} - \lambda, \quad k=\overline{1,n}$$

$$a_k = z_k + \bar{z}_k,\ k=\overline{1,n}$$

Thus, we come to calculate the following (see [4]) determinant:

$$\begin{vmatrix} x_1 & a_2 \dots & a_n \\ a_1 & \ddots & \vdots \\ \vdots & & \\ a_1 & a_2 \cdots & x_n \end{vmatrix} = (x_1-a_1)(x_2-a_2)\dots(x_n-a_n)\left(1+\frac{a_1}{x_1-a_1}+\frac{a_2}{x_2-a_2}+\dots+\frac{a_n}{x_n-a_n}\right);$$

$$x_1 - a_1 = \frac{|z+\bar{z}|^4}{-4(z_1+\bar{z}_1)}\left(\frac{2}{|z+\bar{z}|^2} - \frac{4(z_1+\bar{z}_1)^2}{|z+\bar{z}|^4} - \lambda\right) - (z_1+\bar{z}_1) = \frac{2|z+\bar{z}|^2 - |z+\bar{z}|^4\lambda}{-4(z_1+\bar{z}_1)}$$

$$\frac{a_1}{x_1-a_1} = \frac{-4(z_1+\bar{z}_1)^2}{2|z+\bar{z}|^2 - |z+\bar{z}|^4\lambda}, \dots, \frac{a_n}{x_n-a_n} = \frac{-4(z_n+\bar{z}_n)^2}{2|z+\bar{z}|^2 - |z+\bar{z}|^4\lambda}$$

$$1+\frac{a_1}{x_1-a_1}+\frac{a_2}{x_2-a_2}+\dots+\frac{a_n}{x_n-a_n} = 1 - \frac{4}{2-|z+\bar{z}|^2\lambda} = \frac{-2-|z+\bar{z}|^2\lambda}{2-|z+\bar{z}|^2\lambda} \quad \Rightarrow$$

$$\begin{vmatrix} u_{1\bar{1}}-\lambda & \cdots & u_{1\bar{n}} \\ \vdots & \ddots & \vdots \\ u_{n\bar{1}} & \cdots & u_{n\bar{n}}-\lambda \end{vmatrix} = \frac{\left(2-|z+\bar{z}|^2\lambda\right)^{n-1}\left(-2-|z+\bar{z}|^2\lambda\right)}{|z+\bar{z}|^{2n}} =$$

$$= -\left(\frac{2}{|z+\bar{z}|^2}-\lambda\right)^{n-1}\left(\frac{2}{|z+\bar{z}|^2}-\lambda\right)=0 \qquad \Rightarrow$$

$$\lambda_{1,2,3,\dots,n-1}=\frac{2}{|z+\bar{z}|^2}, \quad \lambda_n=-\frac{2}{|z+\bar{z}|^2}$$

RESULTS

$m-$**subharmonicity of the function** $u=\ln\left(x_1^2+x_2^2+\dots+x_n^2\right)$. Let's check the given function for $m-$ subharmonicity. We get the equation

$$\left(dd^c|z|^2\right)^n = n!\left(\frac{i}{2}\right)^n dz_1\wedge d\bar{z}_1\wedge\dots\wedge dz_n\wedge d\bar{z}_n = n!\,dx_1\wedge dx_2\wedge\dots\wedge dx_{2n}=dV,$$

where dV is the volume element in $\mathbf{C}^n\approx\mathbf{R}^{2n}$. From the fact that

$$\left(dd^c|z|^2\right)^{m-1}=(m-1)!\left(\frac{i}{2}\right)^{m-1}\sum_{1\le j_1<\dots<j_{m-1}\le n} dz_{j_1}\wedge d\bar{z}_{j_1}\dots\wedge dz_{j_{m-1}}\wedge d\bar{z}_{j_{m-1}}$$

and condition $dd^cu\wedge\left(dd^c|z|^2\right)^{m-1}\ge 0$ in the definition of $m-sh$ function, we get the following form:

$$dd^cu\wedge\left(dd^c|z|^2\right)^{m-1}=$$

$$=(m-1)!\left(\frac{i}{2}\right)^m\sum_{\substack{1\le j_1<\dots<j_{m-1}\le n,\\ 1\le j\le n}}\lambda_j dz_j\wedge d\bar{z}_j\wedge dz_{j_1}\wedge d\bar{z}_{j_1}\dots\wedge dz_{j_{m-1}}\wedge d\bar{z}_{j_{m-1}}\ge 0.$$

Now multiply both sides of the above inequality by the positive form $\left(\frac{i}{2}\right)^{n-m} dz_{k_1}\wedge d\bar{z}_{k_1}\dots\wedge dz_{k_{n-m}}\wedge d\bar{z}_{k_{n-m}}$, $1\le k_1<\dots<k_{n-m}\le n$, and get the system of inequalities

$$\lambda_{j_1}+\dots+\lambda_{j_m}\ge 0, \quad 1\le j_1<\dots<j_m\le n. \qquad (7)$$

If we take into account that $\lambda_{1,2,3,\dots,n-1}=\frac{2}{|z+\bar{z}|^2}$, $\lambda_n=-\frac{2}{|z+\bar{z}|^2}$, then the system of inequalities (7) is valid for $2\le m\le n.$

Theorem1. $u=\ln\left(x_1^2+x_2^2+\dots+x_n^2\right)=\ln[(z_1+\bar{z}_1)^2+(z_2+\bar{z}_2)^2+\dots+(z_n+\bar{z}_n)^2]-2\ln 2$ function is $m-sh$ function when $2\le m\le n.$

Strongly $m-$**subharmonicity of the function** $u=\ln\left(x_1^2+x_2^2+\dots+x_n^2\right)$. Now we check the function (2) for a strongly $m-$ subharmonicity**:**

since equality $\left(dd^c u\right)^m \wedge \beta^{n-m} = m!(n-m)!H_m(u)\beta^n$ holds, if the Hessian vector $H_m(u) = \sum_{1\le j_1<\ldots<j_m\le n} \lambda_{j_1}\lambda_{j_2}\ldots\lambda_{j_m}$ – is positively defined, then $\left(dd^c u\right)^m \wedge \left(dd^c |z|^2\right)^{n-m} \ge 0$ is satisfied. So, for the given function to be sh_m, $1\le m\le n$ according to the definition, the following inequalities must be fulfilled:

$$H_1(\lambda(u)) \ge 0, H_2(\lambda(u)) \ge 0, \ldots, H_{n-m+1}(\lambda(u)) \ge 0.$$

In the case where $n=2$, the eigenvalues of the Hermite $\left(u_{j\bar{k}}\right)$ matrix of the function (2) are equal to $\lambda_{1,2} = \pm\frac{2}{|z+\bar{z}|^2}$. In the case where $n=2$, $m=1$, the following 2 inequalities must be fulfilled at the same time: $\begin{cases} H_1(\lambda(u)) \ge 0 \\ H_{n-m+1}(\lambda(u)) = H_{2-1+1}(\lambda(u)) = H_2(\lambda(u)) \ge 0 \end{cases}$

It is not difficult to see that,

$$\begin{cases} H_1(\lambda(u)) = \lambda_1 + \lambda_2 = \frac{2}{|z+\bar{z}|^2} - \frac{2}{|z+\bar{z}|^2} = 0 \\ H_2(\lambda(u)) = \lambda_1\lambda_2 = -\frac{4}{|z+\bar{z}|^4} < 0 \end{cases}$$

And $H_{n-m+1}(\lambda(u)) = H_{2-2+1}(\lambda(u)) = H_1(\lambda(u)) = 0$ when $m=n=2$.

So, the function $u = \ln\left(x_1^2 + x_2^2\right) = \ln[(z_1+\bar{z}_1)^2 + (z_2+\bar{z}_2)^2] - 2\ln 2$ can be sh_2, but not sh_1.

when $n=3$:

$$\lambda_{1,2} = \frac{2}{|z+\bar{z}|^2}, \quad \lambda_3 = -\frac{2}{|z+\bar{z}|^2}.$$

If we take $m=n=3$, then $H_{n-m+1}(\lambda(u)) = H_1(\lambda(u)) = \lambda_1+\lambda_2+\lambda_3 \ge 0$.

If $n=3$, $m=2$, then

$$\begin{cases} H_1(\lambda(u)) = \lambda_1+\lambda_2+\lambda_3 = \frac{2}{|z+\bar{z}|^2} > 0 \\ H_{n-m+1}(\lambda(u)) = H_2(\lambda(u)) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = -\frac{4}{|z+\bar{z}|^4} < 0 \end{cases}$$

So, the function $u = \ln\left(x_1^2 + x_2^2 + x_3^{\,2}\right) = \ln[(z_1+\bar{z}_1)^2 + (z_2+\bar{z}_2)^2 + (z_3+\bar{z}_3)^2] - 2\ln 2$ can be sh_3, but it cannot be sh_2.

And for $n=6,$ function (2) can be $sh_4,$ but cannot be sh_3:

$$\lambda_{1,2,3,4,5}=\frac{2}{\left|z+\bar{z}\right|^2},\ \lambda_6=-\frac{2}{\left|z+\bar{z}\right|^2}$$

$$H_1\left(\lambda(u)\right)=\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6=\frac{8}{\left|z+\bar{z}\right|^2}>0$$

$$H_2\left(\lambda(u)\right)=\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_1\lambda_5+\lambda_1\lambda_6+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_2\lambda_5+\lambda_2\lambda_6+$$
$$\lambda_3\lambda_4+\lambda_3\lambda_5+\lambda_3\lambda_6+\lambda_4\lambda_5+\lambda_4\lambda_6+\lambda_5\lambda_6=\frac{20}{\left|z+\bar{z}\right|^4}-\frac{10}{\left|z+\bar{z}\right|^4}>0$$

$$H_3\left(\lambda(u)\right)=\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_2\lambda_5+\lambda_1\lambda_2\lambda_6+\lambda_1\lambda_3\lambda_4+\lambda_1\lambda_3\lambda_5+\lambda_1\lambda_3\lambda_6+\lambda_1\lambda_4\lambda_5+$$
$$+\lambda_1\lambda_4\lambda_6+\lambda_1\lambda_5\lambda_6+\lambda_2\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_5+\lambda_2\lambda_3\lambda_6+\lambda_2\lambda_4\lambda_5+\lambda_2\lambda_4\lambda_6+\lambda_2\lambda_5\lambda_6+\lambda_3\lambda_4\lambda_5+$$
$$\lambda_3\lambda_4\lambda_6+\lambda_3\lambda_5\lambda_6+\lambda_4\lambda_5\lambda_6=\frac{20}{\left|z+\bar{z}\right|^6}-\frac{20}{\left|z+\bar{z}\right|^6}=0\Rightarrow\left(dd^cu\right)^3\wedge\left(dd^c|z|^2\right)^3=0$$

$H_{6-3+1}\left(\lambda(u)\right)=H_4\left(\lambda(u)\right)<0.$ Thus, in the special case where $n=2$ and $n=3,$ the function (2) is sh_m in $m=n,$ i.e. subharmonic function, In the case of $n=4$ *and* $n=5$, when $n-1\le m\le n,$ it is the sh_m function, and in the case of $n=6$ *and* $n=7,$ it is the sh_m function when $n-2\le m\le n.$ Also, for $n=8$ *and* $n=9,$ when $n-3\le m\le n,$ sh_m function is. In general, when $n-\left(\left[\frac{n}{2}-1\right]\right)\le m\le n,\ n\ge 2$, the given function is sh_m function. We prove that: From the above calculations, we can conclude as follows:

$$H_1\left(\lambda(u)\right)\ge 0,\ H_2\left(\lambda(u)\right)<0 \text{ at } n=2 \text{ and } n=3;$$

$$H_1\left(\lambda(u)\right)\ge 0, H_2\left(\lambda(u)\right)\ge 0, H_3\left(\lambda(u)\right)<0 \text{ at } n=4 \textit{ and } n=5;$$

$$H_1\left(\lambda(u)\right)\ge 0, H_2\left(\lambda(u)\right)\ge 0, H_3\left(\lambda(u)\right)\ge 0, H_4\left(\lambda(u)\right)<0 \text{ at } n=6 \textit{ and } n=7.$$

Indeed, $H_1(\lambda), H_2(\lambda),..., H_k(\lambda)\ge 0,\quad H_{k+1}(\lambda)<0\ n=2k$ *and* $n=2k+1.$

Let's check this for $n=2k+1$:

As we know, $\lambda_{1,2,...,2k-1,2k}=\dfrac{2}{\left|z+\bar{z}\right|^2},\quad \lambda_{2k+1}=-\dfrac{2}{\left|z+\bar{z}\right|^2}$.

There are C_{2k+1}^m terms in the sum of $H_m(\lambda),$ and C_{2k}^m of them are positive, because λ_{2k+1} is not involved in these terms. Those involved in λ_{2k+1} are equal to $C_{2k+1}^m-C_{2k}^m=C_{2k}^{m-1}$, that is, in the sum of $H_m(\lambda),$ $\dfrac{(2k)!}{(m-1)!(2k-m+1)!}$ terms are negative, $\dfrac{(2k)!}{m!(2k-m)!}$ terms are positive, and all terms have the same value.

It is not difficult to see that the inequality

$$\frac{(2k)!}{(m-1)!(2k-m+1)!} \le \frac{(2k)!}{m!(2k-m)!}$$

holds when $m \le k+\frac{1}{2}$. So, $H_m(\lambda) \ge 0$ for $m=\overline{1,k}$ and $H_m(\lambda)<0$ for $m=\overline{(k+1),n}$. For $n=2k$, it can be seen that $H_1(\lambda), H_2(\lambda),\dots, H_k(\lambda) \ge 0$, $H_{k+1}(\lambda)<0$, following the same considerations as above. This means that the function given in $n=2k$ and $n=2k+1$ is the sh_m function when $n-(k-1) \le m \le n$. So we came to the following conclusion:

Theorem2. $u=\ln\left(x_1^2+x_2^2+\dots+x_n^2\right)=\ln[(z_1+\overline{z}_1)^2+(z_2+\overline{z}_2)^2+\dots+(z_n+\overline{z}_n)^2]-2\ln 2$ function is a sh_m function when $n-\left(\left[\frac{n}{2}-1\right]\right) \le m \le n,\ n\ge 2.$

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