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Lattices, Order

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Abstract

LATTICES, ORDER and their underlying definitions and theorems are presented in this white paper (knowledge base).

keywords: lattices, order, knowledge base

The most updated version of this white paper is available at
https://osf.io/fzpc4/download
https://zenodo.org/record/7267788

Open Mathematics Knowledge Base
http://omkb.org

Introduction

1. [1, 2]

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Notation

2. $\Box :=$ necessarily

3. $x \nleq y := x \leq y$ is false

4. $\veebar :=$ exclusive or

5. $\exists! :=$ exists exactly one

6. $\exists_{\leq 1} :=$ there is at most one

7. `Def` := Definition

8. `Thm` := Theorem

9. $\cong$:= `order`-isomorphism

10. $\cong_\ell$:= $\ell attice$-isomorphism

11. $\mathbb{N}_0 = \{0, 1, 2, ...\}$

12. $(A \vdash B) := A$ proves B

13. $(A \dashv\vdash B) \equiv (A \vdash B \;\wedge\; B \vdash A)$

Order (or partial order)

14. `Def`

$$(P, \leq) := \texttt{order} \text{ (or } \texttt{partial order}\text{) if } \forall x, y, z \in P$$

(i) $x \leq x$ (reflexive)

(ii) $x \leq y \;\wedge\; y \leq z \;\rightarrow\; x \leq z$ (transitive)

(iii) $x \leq y \;\wedge\; y \leq x \;\rightarrow\; x = y$ (antisymmetric)

15. $P :=$ set; $\leq$:= binary relation on P

Discrete order

16. `Def`

$$= \;:=\; \texttt{discrete order}$$

Quasi-order (pre-order)

17. `Def`
    `quasi-order (pre-order)` := reflexive ∧ transitive ∧ ¬□ antisymmetric

Strict inequality

18. `Def`

$$x < y \text{ in } P \;\leftrightarrow\; x \leq y \;\wedge\; x \neq y$$

19. ≤ := order relation

20. ≤ gives rise to <.

Non-comparability

21. `Def`

$$y \nleq x \nleq y \;\leftrightarrow\; x \parallel y$$

Induced order (inherited)

22. `Def`

$$x, y \in Q : x \leq y \text{ in } Q \;\leftrightarrow\; x \leq y \text{ in } P$$

23. P := ordered set; $Q \subseteq P$

24. Q has the `induced order`.

25. Q has the order `inherited from` P.

Chain (linearly/totally ordered set), antichain

26. `Def`

$$(\forall x, y \in P : x \leq y \;\veebar\; y \leq x) \;\equiv\; (P := \texttt{chain})$$

27. `Def`

$$(x \leq y \text{ in } P \leftrightarrow x = y) \;\equiv\; (P := \texttt{antichain})$$

28. P := ordered set

29. `chain` := `linearly ordered set` := `totally ordered set`

30. Notation: S := chain; $\overline{S}$:= antichain.

---

31. `Thm`

$$\text{Any } \textbf{subset} \text{ of a } \textbf{chain} \text{ (}\textbf{antichain}\text{) is a } \textbf{chain} \text{ (}\textbf{antichain}\text{).}$$

n-element set

32. $\boldsymbol{n} = \{0, 1, 2, ..., n-1\}$ is a chain ($\leq$).

33. $\overline{\mathbf{n}}$:= antichain of $\boldsymbol{n}$

Order-isomorphism

34. `Def`

$$\exists \varphi : x \leq y \text{ in } P \;\leftrightarrow\; \varphi(x) \leq \varphi(y) \text{ in } Q$$
$$\equiv \; P \cong Q$$

35. P, Q := ordered sets

36. $(\varphi : P \to Q)$:= map from P onto Q

37. $(P \cong Q)$:= P and Q are `(order-)isomorphic`

38. φ := `order-isomorphism` (it mirrors the order structure)

---

39. `Thm`

$$\varphi \text{ is } \textbf{bijective}.$$

40. `Thm`

$$(\varphi^{-1} : Q \to P) \equiv \text{ order-isomorphism}$$

41. φ^{-1} := `inverse` of φ

Number systems

42. `Thm`

$$\mathbb{R}, \mathbb{N}, \mathbb{N}_0, \mathbb{Q}, \mathbb{Z} \equiv \texttt{chains} \ \ (\leq)$$

43. $\mathbb{N}_0 := \mathbb{N} \cup \{0\} := \omega$

---

44. `Thm`

$$\omega \cong \mathbb{N}$$

45. $\cong \ := \$ order-isomorphism

46. $(n \mapsto n^+ = n+1) :=$ successor function from $\mathbb{N}_0$ to $\mathbb{N}$ (order-isomorphism)

---

47. `Thm`

$$\langle \mathbb{N}_0, \preceq \rangle \equiv \texttt{order} \text{ (not a chain)}$$

48. `Def`

$$m \preceq n \ \leftrightarrow \ \exists k \in \mathbb{N}_0 : km = n \ \ (m \text{ divides } n)$$

Power set

49. `Thm`

$$\langle \mathcal{P}(X), \leq \rangle \equiv \texttt{ordered set}$$

50. `Def`

$$A \leq B \ \leftrightarrow \ A \subseteq B$$

51. $X :=$ set

52. $\mathcal{P}(X) :=$ power set of X

---

53. `Thm`

$$\forall Y \subseteq \mathcal{P}(X) : Y \text{ is ordered } (\subseteq)$$

Sub

54. `Thm`: *The set of all (subgroups, normal subgroups, subspaces of a vector space, subrings, ideals of a ring) are ordered under set inclusion.*

Topological space

55. $(X, \mathcal{T}) :=$ topological space

56. Families of open, closed, and clopen subsets of X can be ordered under set inclusion.

Predicate, power set, order-isomorphism

57. `Thm`

$$\langle \mathcal{P}(X), \subseteq \rangle, \ \langle \mathbb{P}, \Rightarrow \rangle \equiv \texttt{ordered sets}$$

58. `Def`

$$p, q \in \mathbb{P}(X) : p \Rightarrow q \quad \text{iff} \quad \{x \in X \mid p(x) = T\} \subseteq \{x \in X \mid q(x) = T\}$$

59. $(p : X \to \{T, F\}) :=$ predicate (statement true or false)

60. $T :=$ true, $\ F :=$ false

61. $\mathbb{P}(X) :=$ set of predicates on X

62. $\mathcal{P}(X) :=$ power set of X

63. $\Rightarrow \ := \$ order relation (implication)

---

64. `Thm`

$$(\varphi : \mathbb{P}(X) \to \mathcal{P}(X)) \ \equiv \ \texttt{order-isomorphism}$$

65. $\varphi(p) := \{x \in X \mid p(x) = T\}$

Covering

66. `Def`

$$x < y \ \wedge \ x \leq z < y \ \rightarrow \ z = x$$
$$\equiv \ x \text{ is } \texttt{covered by } y \text{ (or } y \ \texttt{covers } x)$$
$$\equiv \ x \prec y \ \equiv \ y \succ x$$

67. $P :=$ ordered set; $\quad x, y \in P$

68. $\nexists\, z \in P : x < z < y$ when $x \prec y$.

69. $\leq, <$:= order relations

70. $\prec, \succ$:= `covering relations`

---

71. `Thm`

$$(P \text{ is finite}) \ \leftrightarrow \ (x < y \ \leftrightarrow \ \exists s : s = (x = x_0 \prec x_1 \prec \cdots \prec x_n = y))$$

72. For finite P, $<$ determines (and is determined by) $\prec$.

Diagrams

73. Check pp. 11-13 of [1].

74. $x < y$ iff there are connected line segments moving upwards from x to y

75. diagram of $\mathcal{P}(\{1, 2, 3\})$:= the cube

Order-isomorphism, covering

76. `Thm`

$$\varphi \text{ is an } \texttt{order-isomorphism}$$
$$\equiv$$
$$x \prec y \text{ in } P \ \text{ iff } \ \varphi(x) \prec \varphi(y) \text{ in } Q$$

77. P, Q := *finite* ordered sets

78. $(\varphi : P \to Q)$:= bijective map

Order-isomorphism, identical diagrams

79. `Thm`

$$P \cong Q \;\text{ iff }\; P, Q \text{ have identical diagrams}$$

80. $P, Q :=$ *finite* ordered sets

81. $\cong \;:=\;$ order-isomorphism

Dual

82. `Def`

$$x \leq y \text{ in } P^{\partial} \;\leftrightarrow\; y \leq x \text{ in } P$$

83. $P :=$ ordered set

84. $P^{\partial} :=$ dual of P

85. diagram for P^{∂} = diagram for P (upside down)

Duality Principle

86. `Thm`

$$\forall P : \Phi \;\leftrightarrow\; \forall P : \Phi^{\partial}$$

87. $P :=$ ordered set

88. $\Phi :=$ true statement in P

89. $\Phi^{\partial} :=$ true dual statement in P

Bottom, Top

90. `Def`

$$(\exists \bot \in P : \forall x \in P,\; \bot \leq x) \;\equiv\; P \text{ has a } \texttt{bottom} \textit{ element}$$

91. `Def`

$$(\exists \top \in P : \forall x \in P,\; x \leq \top) \;\equiv\; P \text{ has a } \texttt{top} \textit{ element}$$

92. P := ordered set

93. $\bot$:= `bottom`

94. $\top$:= `top`

---

95. `Thm`

$$(\exists! \bot \in P) \text{ is dual to } (\exists! \top \in P).$$

96. (95) is *an instance of the duality theorem.*

---

97. An infinite chain may not have bottom or top elements.

Lifting

98. `lifting` := adding a bottom element to ordered sets

99. `Def`

$$x \leq y \;\text{ iff }\; x = \mathbf{0} \text{ or } x \leq y \text{ in } P$$

100. $\mathbf{0} \notin P, \quad P_\bot := P \cup \{\mathbf{0}\}$

---

101. `Thm`

$$\forall S : \overline{S}_\bot \text{ is an } \texttt{ordered set}$$

102. S := set

103. $\overline{S}_\bot$:= `flat` ordered set (lifted antichain)

Maximal, minimal, greatest (maximum), least (minimum)

104. `Def`

$$(a \leq x \;\wedge\; x \in Q) \to a = x$$

$$\equiv\; a \in Q \text{ is a } \texttt{maximal element of } Q$$

105. $P :=$ ordered set; $Q \subseteq P$; $a, x \in Q$

106. A maximal element is greater than all comparable elements in P, different from itself.

---

107. `Thm`

$$(Q \text{ has the order inherited from } P,\ \exists \top_Q) \;\to\; \operatorname{Max} Q = \{\top_Q\}$$

108. $\top_Q :=$ top element of $Q := \max Q$
$:=$ `greatest` (or `maximum`) element of Q

109. $\operatorname{Max} Q :=$ set of `maximal` elements of Q

---

110. `Thm`

$$(Q \text{ has the order inherited from } P,\ \exists \bot_Q) \;\to\; \operatorname{Min} Q = \{\bot_Q\}$$

111. $\bot_Q :=$ bottom element of $Q := \min Q$
$:=$ `least` (or `minimum`) element of Q

112. $\operatorname{Min} Q :=$ set of `minimal` elements of Q

Disjoint union, linear sum, M_n

113. `Def`

$P \dot\cup Q$ is an `ordered set`:

$$x \le y \text{ in } P \dot\cup Q \;\text{ iff }\; (x, y \in P,\ x \le y \text{ in } P) \;\underline{\vee}\; (x, y \in Q,\ x \le y \text{ in } Q)$$

114. $P \cap Q = \varnothing$

115. $P \dot\cup Q :=$ `disjoint union` of P and Q

---

116. `Def`

$$P \oplus Q \coloneqq \texttt{linear sum}$$
$$\text{(defined by the following order relation on } P \cup Q\text{)}$$
$$x \le y \text{ iff } (x, y \in P,\ x \le y \text{ in } P) \vee (x, y \in Q,\ x \le y \text{ in } Q) \vee (x \in P,\ y \in Q)$$

117. $P, Q \coloneqq$ disjoint ordered sets

---

118. $P_\perp \coloneqq \mathbf{0} \oplus P$ (lifting)

119. $P \oplus \mathbf{1}$ (P with a new top element)

---

120. `Thm` (associativity)

$$P \mathbin{\dot\cup} (Q \mathbin{\dot\cup} R) = (P \mathbin{\dot\cup} Q) \mathbin{\dot\cup} R$$

$$P \oplus (Q \oplus R) = (P \oplus Q) \oplus R$$

---

121. `Def`

$$\mathbf{M_n} = \mathbf{1} \oplus \overline{\mathbf{n}} \oplus \mathbf{1}$$

122. $\boldsymbol{n} = \{0, 1, 2, ..., n-1\}$ is a chain ($\le$).

123. $\overline{\mathbf{n}} \coloneqq$ antichain of $\boldsymbol{n}$

Products

124. `Def`: $P_1 \times \cdots \times P_n$ is an `ordered set` if the `coordinatewise order` holds,

$$(x_1, ..., x_n) \le (y_1, ..., y_n) \ \leftrightarrow\ \forall i : x_i \le y_i \text{ in } P_i.$$

125. $P_1, ..., P_n \coloneqq$ ordered sets

Lexicographic order

126. `Def`

$$(x_1, x_2) \leq (y_1, y_2) \;\text{ if }\; x_1 < y_1 \;\vee\; (x_1 = y_1,\; x_2 \leq y_2)$$

Down-set, up-set

127. `Def`

$$x \in Q,\; y \in P,\; y \leq x \;\rightarrow\; y \in Q$$
$$(Q := \texttt{down-set} := \texttt{decreasing set} := \texttt{order ideal})$$

128. `Def`

$$x \in Q,\; y \in P,\; y \geq x \;\rightarrow\; y \in Q$$
$$(Q := \texttt{up-set} := \texttt{increasing set} := \texttt{order filter})$$

129. $P :=$ ordered set, $\quad Q \subseteq P$

Principal

130. `Def`

$$\downarrow Q := \{y \in P \mid \exists x \in Q : y \leq x\}$$

131. `Def`

$$\uparrow Q := \{y \in P \mid \exists x \in Q : y \geq x\}$$

132. `Def`

$$\downarrow x := \{y \in P \mid y \leq x\}$$

133. `Def`

$$\uparrow x := \{y \in P \mid y \geq x\}$$

134. $P :=$ ordered set, $\quad Q \subseteq P, \quad x \in P$

135. $\downarrow Q,\; \downarrow x :=$ down-sets

136. $\uparrow Q,\; \uparrow x :=$ up-sets

137. $\downarrow Q$ reads *down* Q; $\uparrow Q$ reads *up* Q.

---

138. $\downarrow Q$ is the **smallest down-set** containing Q.

---

139. `Thm`

$$Q \equiv \texttt{down-set} \leftrightarrow Q = \downarrow Q$$

140. `Thm`

$$Q \equiv \texttt{up-set} \leftrightarrow Q = \uparrow Q$$

141. `Thm`

$$\downarrow \{x\} = \downarrow x; \quad \uparrow \{x\} = \uparrow x$$

142. $\downarrow x, \uparrow x :=$ `principals`

Family of down-sets, order-isomorphism

143. `Thm`

$$\langle \mathcal{O}, \subseteq \rangle \text{ is an } \texttt{ordered set}.$$

144. $P :=$ ordered set

145. $\mathcal{O}(P) :=$ set of all `down-sets` of P

---

146. `Thm`

$$P \text{ is an } \texttt{antichain} \rightarrow \mathcal{O}(P) = \mathcal{P}(P)$$

147. $\mathcal{P}(X) :=$ power set of X

---

148. `Thm`

$$Q \text{ is a } \texttt{down-set} \leftrightarrow P \backslash Q \text{ is an } \texttt{up-set}$$

149. $\backslash$:= set complementation ("subtraction" of sets)

---

150. `Thm`

$$\mathcal{O}(P)^\partial \;\cong\; \mathcal{O}(P^\partial)$$

(complementation map)

151. ∂ := dual

152. $\cong$:= order-isomorphism

---

153. `Thm`

$$\mathcal{O}(P \oplus \mathbf{1}) \cong \mathcal{O}(P) \oplus \mathbf{1}$$

$$\mathcal{O}(\mathbf{1} \oplus P) \cong \mathbf{1} \oplus \mathcal{O}(P)$$

$$\mathcal{O}(P_1 \dot\cup P_2) \cong \mathcal{O}(P_1) \times \mathcal{O}(P_2)$$

---

154. `Thm`

$$\mathcal{O}(P_1 \dot\cup P_2) \;\cong_U\; \mathcal{O}(P_1) \times \mathcal{O}(P_2)$$

$$U \mapsto (U \cap P_1, U \cap P_2)$$

Ordered relation, down-sets

155. `Thm`

$$x \le y \;\equiv\; \downarrow x \subseteq \downarrow y \;\equiv\; \forall Q \in \mathcal{O}(P) : y \in Q \to x \in Q$$

156. P := ordered set, $x, y \in P$

Order-preserving/embedding

157. `Def`

$$(x \le_P y \to \varphi(x) \le_Q \varphi(y)) \;\to\; \varphi := \texttt{order-preserving (monotone)}$$

$$(x \le_P y \leftrightarrow \varphi(x) \le_Q \varphi(y)) \;\to\; (\varphi : P \hookrightarrow Q) := \texttt{order-embedding}$$

158. P, Q := ordered-sets, $\varphi : P \to Q$

159. order embedding (one-to-one)

---

160. `Thm`

$$\varphi(P) \cong P$$

161. $(\varphi : P \hookrightarrow Q)$:= `order-embedding` map

162. $\varphi(P) = \{\varphi(x) \mid x \in P\}$

163. $\cong$:= order-isomorphism

164. $\varphi(P)$ is *embedded* in P.

Composite, order-preserving maps

165. `Thm:` *The composite of a finite number of order-preserving maps is order-preserving.*

Pointwise order

166. `Def: pointwise order`

$$f \le g \;\equiv\; \forall x \in \mathbb{R} : f(x) \le g(x)$$

167. $f, g : \mathbb{R} \to \mathbb{R}$

168. $\le$:= order relation of maps (functions)

Upper bound, lower bound

169. `Def`

$$(\forall s \in S : s \le x) \;\to\; (x \in P) := \texttt{upper bound} \text{ of } S$$

170. `Def`

$$(\forall s \in S : s \ge x) \;\to\; (x \in P) := \texttt{lower bound} \text{ of } S$$

171. `Def`

$$S^u := \{x \in P \mid \forall s \in S : s \le x\}$$

172. `Def`

$$S^\ell := \{x \in P \mid \forall s \in S : s \geq x\}$$

173. $P :=$ ordered set; $\quad S \subseteq P$

174. $S^u :=$ set of all upper bounds of S

175. $S^\ell :=$ set of all lower bounds of S

176. S^u reads S *upper* and S^ℓ reads S *lower*.

---

177. `Thm`

$$S^u \text{ is an up-set.}$$

178. `Thm`

$$S^\ell \text{ is a down-set.}$$

---

179. `Thm`

$$\exists_{\leq 1}\, a, b : a = \ell ub,\ b = glb$$

180. $\ell ub :=$ `least upper bound (supremum)` $:= \sup S$

181. $glb :=$ `greatest lower bound (infimum)` $:= \inf S$

Join, meet

182. `Def`

$$(x \vee y) := \sup\{x, y\}$$

183. `Def`

$$(x \wedge y) := \inf\{x, y\}$$

184. `Def`

$$\bigvee S := \sup S$$

185. `Def`

$$\bigwedge S := \inf S$$

186. $(x \vee y) \coloneqq x$ `join` $y \coloneqq x$ `supremum` y

187. $(x \wedge y) \coloneqq x$ `meet` $y \coloneqq x$ `infimum` y

188. $\bigvee S \coloneqq$ `join of` $S \coloneqq$ `supremum` of S

189. $\bigwedge S \coloneqq$ `meet of` $S \coloneqq$ `infimum` of S

190. $P \coloneqq$ ordered set

191. $\bigvee_P S \coloneqq$ `join` of S in $P \coloneqq$ `supremum` of S in P

192. $\bigwedge_P S \coloneqq$ `meet` of S in $P \coloneqq$ `infimum` of S in P

Lattice, complete lattice

193. `Def`

$$\forall x, y \in P\ \exists j, m \in P : j = x \vee y,\ m = x \wedge y$$
$$(P \coloneqq \texttt{lattice})$$

194. `Def`

$$\forall S \subseteq P\ \exists j, m : j = \bigvee S,\ m = \bigwedge S$$
$$(P \coloneqq \texttt{complete lattice})$$

195. $\varnothing \neq P \coloneqq$ ordered set

196. $(x \vee y) \coloneqq x$ `join` y

197. $(x \wedge y) \coloneqq x$ `meet` y

198. $\bigvee S \coloneqq$ `join of` S

199. $\bigwedge S \coloneqq$ `meet of` S

---

200. `Thm`

$$\forall a, b, c, d \in P :$$
$$a \leq b\ \rightarrow\ a \vee c \leq b \vee c,\ a \wedge c \leq b \wedge c,$$
$$a \leq b,\ c \leq d\ \rightarrow\ a \vee c \leq b \vee d,\ a \wedge c \leq b \wedge d$$

201. $P :=$ lattice

---

202. `Thm`

$$x \le y \ \rightarrow \ x \vee y = y, \ x \wedge y = x$$

203. $\varnothing \ne P :=$ ordered-set

---

204. `Thm`

To prove that P is a lattice is equivalent to prove
$\forall x, y \in P$ with $x \parallel y : \exists j, m : j = x \vee y,\, m = x \wedge y.$

205. $\varnothing \ne P :=$ ordered-set

---

206. `Thm`

$$\{x, y\}^u = \uparrow y, \quad \{x, y\}^\ell = \downarrow y$$

207. $x, y \in (P :=$ ordered set$), \quad x \le y$

Power set, complete lattice

208. `Thm`

$\langle \mathcal{P}(X), \subseteq \rangle$ is a `complete lattice`

209. `Def`

$$\bigvee A_i = \bigcup A_i$$
$$\bigwedge A_i = \bigcap A_i$$

210. $\mathcal{P}(X) :=$ power set of X

211. $\bigvee A_i := \bigvee \{A_i \mid i \in I\}$

212. $\bigcup A_i := \bigcup \{A_i \mid i \in I\}$

Lattice of sets

213. `Def`

$$\mathfrak{L} \text{ is } \textit{closed} \text{ under } \texttt{finite unions} \text{ and } \texttt{intersections} \rightarrow$$

$$\rightarrow \mathfrak{L} \text{ is a } \texttt{lattice of sets}$$

214. `Def`

$$\mathfrak{L} \text{ is } \textit{closed} \text{ under } \texttt{arbitrary unions} \text{ and } \texttt{intersections} \rightarrow$$

$$\rightarrow \mathfrak{L} \text{ is a } \texttt{complete lattice of sets}$$

215. $\varnothing \neq \mathfrak{L} \subseteq \mathcal{P}(X)$

216. $\mathcal{P}(X) :=$ power set of X

---

217. `Def`

$$\mathfrak{L} \text{ is a lattice of sets } \rightarrow \langle \mathfrak{L}, \subseteq \rangle \text{ is a lattice,}$$
$$A \vee B = A \cup B,\ A \wedge B = A \cap B$$

---

218. `Thm`

$$\mathcal{O}(P) \text{ is a } \texttt{complete lattice of sets}$$
$$(\textbf{down-set lattice} \text{ of } P).$$

219. $P :=$ ordered set

220. $\mathcal{O}(P) :=$ ordered set of all down-sets of P

M_n, lattice

221. `Thm`

$$\mathbf{M_n} \text{ is a } \texttt{lattice}.$$

222. $\mathbf{M_n} = \mathbf{1} \oplus \overline{\mathbf{n}} \oplus \mathbf{1}$ (ordered set for $n \geq 1$)

223. `Def`

$$P \oplus Q := \texttt{linear sum}$$
$$x \le y \text{ iff } (x, y \in P,\ x \le y \text{ in } P) \vee (x, y \in Q,\ x \le y \text{ in } Q) \vee (x \in P,\ y \in Q)$$

Ordered by division

224. `Thm`

$$\langle \mathbb{N}_0, \preceq \rangle \text{ is a } \texttt{lattice},$$
$$m \vee n = \mathrm{lcm}\{m, n\},\ m \wedge n = \mathrm{gcd}\{m, n\}$$

225. `Def`

$$m \preceq n \ \leftrightarrow\ \exists k \in \mathbb{N}_0 : km = n \ \ (m \text{ divides } n)$$

226. lcm := least common multiple

227. gcd := greatest common divisor

Lattice, algebraic structure

228. `Def`

$$a \vee b := \sup\{a, b\},\quad a \wedge b := \inf\{a, b\}$$

229. $a, b \in (L := \text{lattice})$

230. $\vee, \wedge : L^2 \to L$

231. $\langle L, \vee, \wedge \rangle$:= algebraic structure

232. `Thm`

$$\vee, \wedge \text{ are } \texttt{order preserving}.$$

Connecting Lemma

233. `Thm`

$$a \le b\ \equiv\ a \vee b = b\ \equiv\ a \wedge b = a$$

234. $a, b \in (L := \text{lattice})$

Associativity, commutativity, idempotency, absorption

235. `Thm`: (Associativity) $\forall a, b, c \in L$

$$(a \vee b) \vee c = a \vee (b \vee c)$$
$$(a \wedge b) \wedge c = a \wedge (b \wedge c)$$

236. `Thm`: (Commutativity) $\forall a, b, c \in L$

$$a \vee b = b \vee a$$
$$a \wedge b = b \wedge a$$

237. `Thm`: (Idempotency) $\forall a, b, c \in L$

$$a \vee a = a \vee a$$
$$a \wedge a = a \wedge a$$

238. `Thm`: (Absorption) $\forall a, b, c \in L$

$$a \vee (a \wedge b) = a$$
$$a \wedge (a \vee b) = a$$

239. $L :=$ lattice

Join, meet, order relation, algebraic structure

240. `Thm`

$$\forall a, b \in L : a \vee b = b \leftrightarrow a \wedge b = a$$

241. `Thm`

$$(a \vee b = b \rightarrow a \leq b) \;\rightarrow\; (\leq \text{ is an } \texttt{order relation})$$

242. `Thm`

$$(a \vee b = b \rightarrow a \leq b) \;\rightarrow\; \langle L, \leq \rangle \text{ is a } \texttt{lattice},$$
$$\forall a, b \in L : a \vee b = \sup\{a, b\},\; a \wedge b = \inf\{a, b\}$$

243. $\langle L, \vee, \wedge \rangle$:= algebraic structure

244. $\varnothing \neq L$:= lattice

245. $\vee, \wedge$:= binary operations

Sublattice

246. `Def`

$$\text{if } a, b \in M \to (a \vee b \in M, a \wedge b \in M)$$
$$\text{then } M \text{ is a } \texttt{sublattice} \text{ of } L$$

247. L := lattice

248. $\varnothing \neq M \subseteq L$

249. $\operatorname{Sub} L$:= collection of all sublattices of L

250. $\operatorname{Sub}_0 L = \operatorname{Sub} L \cup \{\varnothing\}$

Product of lattices

251. `Thm`

$$L \times K \text{ is a } \texttt{lattice}.$$

252. `Def`

$$(\ell_1, k_1) \vee (\ell_2, k_2) = (\ell_1 \vee \ell_2, k_1 \vee k_2),$$
$$(\ell_1, k_1) \wedge (\ell_2, k_2) = (\ell_1 \wedge \ell_2, k_1 \wedge k_2)$$

253. L, K := lattices

254. `Thm`

$$(\ell_1, k_1) \vee (\ell_2, k_2) = (\ell_2, k_2) \ \leftrightarrow \ (\ell_1, k_1) \leq (\ell_2, k_2)$$

255. $(\ell_1, k_1) \leq (\ell_2, k_2)$ means $\ell_1 \leq \ell_2$ and $k_1 \leq k_2$.

Lattice Homomorphism

256. `Thm`

$$f \text{ is a } \texttt{lattice homomorphism} \text{ if } \forall a, b \in L:$$
$$f(a \vee b) = f(a) \vee f(b), \ \ \textbf{join-preserving},$$
$$f(a \wedge b) = f(a) \wedge f(b), \ \ \textbf{meet-preserving}$$

257. `Def`

$$\texttt{lattice isomorphism} := \text{bijective homomorphism}$$

258. `Def`

$$f \text{ is a } \texttt{one-to-one homomorphism} \ \rightarrow$$
$$\rightarrow \ (\texttt{sublattice} \ f(L) \text{ of } K) \ \cong \ L$$
$$(f \text{ is an } \textbf{embedding of } L \textbf{ into } K)$$

259. $L, K :=$ lattices

260. $f : L \rightarrow K$

261. $\cong \ := \ $ isomorphism

---

262. `Thm`

$$(M \rightarrowtail L) \ \rightarrow \ (M \hookrightarrow L)$$

263. $(M \rightarrowtail L) :=$ sublattice of $L \cong M$

264. $(M \hookrightarrow L) :=$ order-embedding

---

265. `Def`

$$f := \ \textbf{lattice } \{0, 1\}\textbf{-homomorphism}$$

266. $L, K :=$ bounded lattices

267. $f : L \rightarrow K$

268. $f(0) = 0, \ \ f(1) = 1$

Map, order-preserving, homomorphism

269. `Thm`

$$f \text{ is } \texttt{order-preserving} \equiv$$
$$\equiv \forall a, b \in L : f(a \vee b) \geq f(a) \vee f(b)$$
$$\equiv \forall a, b \in L : f(a \wedge b) \leq f(a) \wedge f(b)$$

270. `Thm`

$$f \text{ is a } \texttt{homomorphism} \rightarrow f \text{ is } \texttt{order-preserving}$$

271. $L, K :=$ lattices

272. $f : L \rightarrow K$

Lattice/order-isomorphism

273. `Thm`

$$A \cong_{\ell att} B \quad \text{iff} \quad A \cong_{\leq} B$$

274. $A, B :=$ lattices

275. $\cong_{\ell att}$:= lattice isomorphism

276. $\cong_{\leq}$:= order-isomorphism

Ideals, filters

277. `Thm`

$$J \text{ is an } \texttt{ideal} \text{ if}$$
$$a, b \in J \rightarrow a \vee b \in J$$
$$a \in L,\ b \in J,\ a \leq b \rightarrow a \in J$$

278. `Thm`

$$G \text{ is a } \texttt{filter} \text{ if}$$
$$a, b \in G \rightarrow a \wedge b \in G$$
$$a \in L,\ b \in G,\ a \geq b \rightarrow a \in G$$

279. $L :=$ lattice

280. $\varnothing \neq J \subseteq L, \quad \varnothing \neq G \subseteq L$

281. `ideal` := non-empty down-set closed under join

282. `filter` := order dual of lattice ideal

283. $\mathcal{I}(L),\ \mathcal{F}(L) :=$ set of all ideals and filters of L, respectively $(\subseteq)$

---

284. `Thm:`

$$\forall J : J \text{ is a } \texttt{sublattice} \text{ of } L, \text{ i.e., } \forall a, b \in L : a \wedge b \leq a$$

285. $J :=$ ideal of L

---

286. `Thm`

$$J \text{ is } \texttt{proper} \text{ iff } 1 \notin J$$

287. `Thm`

$$G \text{ is } \texttt{proper} \text{ iff } 0 \notin G$$

288. $J :=$ `ideal` of L with 1 (top)

289. $G :=$ `filter` of L with 0 (bottom)

---

290. `Thm`

$$\forall a \in L : \downarrow a \text{ is an } \texttt{ideal}$$
$$(\texttt{principal ideal} \text{ generated by } a)$$

291. `Thm`

$$\forall a \in L : \uparrow a \text{ is a } \texttt{filter}$$
$$(\texttt{principal filter} \text{ generated by } a)$$

---

292. `Thm`

$$\forall J, G : J, G \text{ are } \texttt{principal}$$
$$J = \downarrow \bigvee J$$
$$G = \uparrow \bigwedge G$$

293. $J :=$ ideal of the *finite* lattice L

294. $G :=$ filter of the *finite* lattice L

---

295. `Thm`

$$f^{-1}(0) \text{ is an } \texttt{ideal}$$
$$f^{-1}(1) \text{ is a } \texttt{filter}$$

296. $L, K :=$ bounded lattices

297. $(f : L \to K) := \{0,1\}$-homomorphism

---

298. `Thm`

$$G \text{ is a } \texttt{filter} \text{ in } \mathcal{P}(X).$$

299. $(X, \mathcal{T}) :=$ topological space

300. $x \in X$

301. $G = \{V \subseteq X \mid \exists U \in \mathcal{T} : x \in U \subseteq V\}$

302. $\mathcal{P}(X) :=$ power set of X

Join, meet, two subsets, ordered sets

303. `Thm`

$$\begin{aligned}
(i) &\quad \forall s \in S : s \leq \bigvee S,\ s \geq \bigwedge S \\
(ii) &\quad x \leq \bigwedge S \leftrightarrow \forall s \in S : x \leq S \\
(iii) &\quad x \geq \bigvee S \leftrightarrow \forall s \in S : x \geq S \\
(iv) &\quad \bigvee S \leq \bigwedge T \leftrightarrow \forall s \in S\ \forall t \in T : s \leq t \\
(v) &\quad S \subseteq T \to \bigvee S \leq \bigvee T,\ \bigwedge S \geq \bigwedge T
\end{aligned}$$

304. $P :=$ ordered set

305. $S, T \subseteq P$

306. $\bigvee S :=$ join of S, $\bigwedge T :=$ meet of T

307. $\bigvee S, \bigvee T, \bigwedge S, \bigwedge T$ exist.

308. $x \in P$

---

309. Thm

$$\bigvee(S \cup T) = (\bigvee S) \vee (\bigvee T)$$
$$\bigwedge(S \cup T) = (\bigwedge S) \wedge (\bigwedge T)$$

Existence, join, meet, finite subset, lattice

310.

$$\forall F \subseteq P \ \exists s, t : s = \bigvee F, \ t = \bigwedge F$$

311. $P :=$ lattice

312. $F :=$ finite, non-empty subset of P

313. $\bigvee F :=$ join of F, $\bigwedge F :=$ meet of F

Finite lattice complete

314. `Thm:` *Every finite lattice is complete.*

Preservation, existing joins and meets

315. `Def:` φ `preserves existing joins` if

$$\exists j \in P : j = \bigvee S \ \rightarrow \ \exists j' \in Q : j' = \bigvee \varphi(S),$$
$$\varphi(\bigvee S) = \bigvee \varphi(S)$$

316. `Def`: φ `preserves existing meets` if

$$\exists m \in P : m = \bigwedge S \ \rightarrow \ \exists m' \in Q : m' = \bigwedge \varphi(S), \\ \varphi(\bigwedge S) = \bigwedge \varphi(S)$$

317. $P, Q :=$ ordered sets

318. $\varphi : P \to Q$

---

319. `Thm`

$$\exists j \in P : j = \bigvee S,\ \exists j' \in Q : j' = \bigvee \varphi(S) \ \rightarrow \ \varphi(\bigvee S) \geq \bigvee \varphi(S)$$

$$\exists m \in P : m = \bigwedge S,\ \exists m' \in Q : m' = \bigwedge \varphi(S) \ \rightarrow \ \varphi(\bigwedge S) \geq \bigwedge \varphi(S)$$

320. $(\varphi : P \to Q) :=$ order-preserving map

321. $S \subseteq P$

---

322. `Thm`

φ preserves all `joins` and `meets`.

323. $(\varphi : P \to Q) :=$ order-isomorphism

Subset induced order

324. `Thm`

$$\exists j \in Q : j = \bigvee_P S \ \rightarrow \ \bigvee_Q S = \bigvee_P S$$

$$\exists m \in Q : m = \bigwedge_P S \ \rightarrow \ \bigwedge_Q S = \bigwedge_P S$$

325. $P :=$ ordered set

326. $S \subseteq Q \subseteq P$

327. Q has the order inherited from P.

Complete lattice, equivalence

328.

$$\begin{aligned} &P \text{ is a } \texttt{complete lattice} \\ \equiv\ &\forall S \subseteq P\ \exists m \in P : m = \bigwedge S \\ \equiv\ &\top \in P,\ \forall S_{\neq\varnothing} \subseteq P\ \exists m \in P : m = \bigwedge S \end{aligned}$$

329. $\varnothing \neq P :=$ ordered set

330. $\top :=$ top element

Subset power set, complete lattice inclusion

331. Thm

$$\left(\forall \{A_i\} \subseteq \mathcal{L} : \bigcap_i A_i \in \mathcal{L}\right) \ \wedge\ (X \in \mathcal{L}) \ \to$$

$$\to\ \mathcal{L} \text{ is a } \texttt{complete lattice:}$$

$$\bigwedge_i A_i = \bigcap_i A_i,$$

$$\bigvee_i A_i = \bigcap \{B \in \mathcal{L} \mid \bigcup_i A_i \subseteq B\}$$

332. $X :=$ set, $\quad i \in I, \quad \{A_i\} \neq \varnothing$

333. $\mathcal{L} :=$ family of subsets of X

334. $(\mathcal{L}, \subseteq) :=$ ordered set

(Topped) Intersection structure

335. Def

$$\forall \{A_i\} \subseteq \mathcal{L} : \bigcap_i A_i \in \mathcal{L}\ \to$$

$$\to\ \mathcal{L} \text{ is an } \texttt{intersection structure} \ (\textstyle\bigcap\texttt{-structure}) \text{ on } X$$

336. `Def`

$$\left(\forall \{A_i\} \subseteq \mathcal{L} : \bigcap_i A_i \in \mathcal{L}\right) \ \wedge \ (X \in \mathcal{L}) \ \rightarrow$$

$$\rightarrow \ \mathcal{L} \text{ is a } \texttt{topped intersection structure} \text{ on } X$$

$$\texttt{(closure system)}$$

337. $\varnothing \neq \mathcal{L}$:= family of subsets of X

Join/meet-irreducibility

338. `Def`

$$x \in L \text{ is } \texttt{join-irreducible} \text{ if}$$

$$(x \neq 0) \text{ and } (\forall a, b \in L : \ x = a \vee b \ \rightarrow \ x = a \text{ or } x = b)$$

339. `Def`

$$x \in L \text{ is } \texttt{meet-irreducible} \text{ if}$$

$$(x \neq 1) \text{ and } (\forall a, b \in L : \ x = a \wedge b \ \rightarrow \ x = a \text{ or } x = b)$$

340. L := lattice

341. `join-irreducible element` := *cannot* be expressed as the `join` of any other elements in the lattice

342. `meet-irreducible element` := *cannot* be expressed as the `meet` of any other elements in the lattice

---

343. $\mathcal{J}(L)$:= set of join-irreducible elements of L

344. $\mathcal{M}(L)$:= set of meet-irreducible elements of L

345. $\mathcal{J}(L)$ and $\mathcal{M}(L)$ inherit L's order relation.

---

346. `Def`

$$\forall a \in P \ \exists A \subseteq Q : a = \bigvee_P A \ \rightarrow \ Q \text{ is } \texttt{join-dense} \text{ in } P$$

347. Def

$$\forall a \in P \; \exists A \subseteq Q : a = \bigwedge_P A \;\rightarrow\; Q \text{ is } \texttt{meet-dense} \text{ in } P$$

348. $P :=$ ordered set

349. $Q \subseteq P$

---

350. Thm

In a finite lattice every element is a join of join-irreducible elements.

Algebraic theory of lattices

351. Thm

$$(L, \leq) \dashv\vdash (L, \vee, \wedge)$$

352. $(L, \leq) :=$ lattice (order relation)

353. $(L, \vee, \wedge) :=$ lattice (algebraic structure)

Equivalence join meet

354. Thm

$$\begin{aligned}
(x \wedge y) \vee (x \wedge z) &\leq x \wedge (y \vee z) \\
x \vee (y \wedge z) &\leq (x \vee y) \wedge (x \vee z) \\
(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) &\leq (x \vee y) \wedge (y \vee z) \wedge (z \vee x) \\
(x \wedge y) \vee (x \wedge z) &\leq x \wedge (y \vee (x \wedge z))
\end{aligned}$$

355. $x, y, z \in (L :=$ lattice$)$

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Join the **Open Mathematics Collaboration**.
Send your contribution to `mplobo@uft.edu.br`.

Supplementary files

The **latex file** for this *white paper* together with other *supplementary files* are available in [5,6].

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