On the nontrivial zeros of the Riemann zeta function

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Abstract

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. There are several statements equivalent to the famous Riemann hypothesis. In 2011, Solé and Planat stated that, the Riemann hypothesis is true if and only if the inequality $\zeta(2) \cdot \prod_{p \leq x} (1 + \frac{1}{p}) > e^{\gamma} \cdot \log \theta(x)$ holds for all $x \geq 5$, where $\theta(x)$ is the Chebyshev function, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, $\zeta(x)$ is the Riemann zeta function and $\log$ is the natural logarithm. In this note, using Solé and Planat criterion, we prove that, when the Riemann hypothesis is false, then there are infinitely many natural numbers x for which $\frac{\log x}{\sqrt{x}} + O(\frac{1}{\sqrt{x}}) + 2 \cdot \log x + c \cdot \log x \leq 2.062$ could be satisfied for some $c > 0$. Since the inequality $\frac{\log x}{\sqrt{x}} + O(\frac{1}{\sqrt{x}}) + 2 \cdot \log x + c \cdot \log x \leq 2.062$ does not hold for some $c > 0$ and large enough x, then the Riemann hypothesis is true by principle of non-contradiction.

Keywords: Riemann hypothesis, Riemann zeta function, Chebyshev function

MSC Classification: 11M26 , 11A25

1 Introduction

The Riemann hypothesis is considered by many to be the most important unsolved problem in pure mathematics. It was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems. Leonhard Euler discovered a particular value of the Riemann zeta function (1734).

Proposition 1 *It is known that[1, (1) p. 1070]:*

$$\zeta(2) = \prod_{i=1}^{\infty} \frac{p_i^2}{p_i^2 - 1} = \frac{\pi^2}{6}.$$

Proposition 2 *For* $x > 24317$*:*

$$\sum_{p>x} \log\left(\frac{p^2}{p^2-1}\right) = \frac{1}{x \cdot \log x} + O\left(\frac{1}{x \cdot \log^2 x}\right)$$

where log *is the natural logarithm [2].*

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x. For $x \geq 2$, we say that $\mathsf{Dedekind}(x)$ holds provided that

$$\zeta(2) \cdot \prod_{p \leq x} \left(1 + \frac{1}{p}\right) > e^{\gamma} \cdot \log \theta(x)$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. Next, we have Solé and Planat Theorem:

Proposition 3 $\mathsf{Dedekind}(x)$ *holds for all natural numbers* $x \geq 5$ *if and only if the Riemann hypothesis is true [3, Theorem 4.2 p. 5].*

This is the main insight.

Lemma 1 *If the Riemann hypothesis is false, then there are infinitely many natural numbers* $x \geq 5$ *for which* $\mathsf{Dedekind}(x)$ *fails (i.e.* $\mathsf{Dedekind}(x)$ *does not hold).*

Srinivasa Ramanujan studied the function $S_1(x) = \sum_{\rho} \frac{x^{\rho-1}}{\rho \cdot (1-\rho)}$ where ρ runs over the nontrivial zeros of the Riemann ζ function [4, Section 65].

Proposition 4 *[4, Section 65]. Ramanujan also proved that*

$$S_1(x) = \frac{c}{\sqrt{x}}$$

which appears in the Equation (332) of the Ramanujan's old notes, where a constant $c > 0$ *is deduced because of all nontrivial zeros of the Riemann* ζ *function remains in the Critical Strip. Certainly, we have that*

$$\mid S_1(x) \mid \leq \frac{1}{\sqrt{x}} \cdot \sum_{\rho} \frac{1}{\rho \cdot (1-\rho)} = \frac{1}{\sqrt{x}} \cdot \sum_{\rho} \left(\frac{1}{\rho} + \frac{1}{1-\rho} \right) = \frac{\tau}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \cdot \sum_{\rho} \frac{1}{1-\rho}$$

where $\tau = 2 \cdot \sum_{\rho} \frac{1}{\rho} = 2 + \gamma - \log(4 \cdot \pi) \approx 0.046$ *is a well-known value [5, p. 272].*

Thus, we have Nicolas Theorem:

Proposition 5 *[6, (2.16)]. For all* $x \geq 10^9$*:*

$$\sum_{p \leq x} \log \left(1 - \frac{1}{p}\right) + \gamma + \log \log \theta(x) + \frac{2}{\sqrt{x} \cdot \log x} + \frac{S_1(x)}{\log x} \leq \frac{2.062}{\sqrt{x} \cdot \log^2 x}$$

This is our main theorem.

Theorem 1 *If the Riemann hypothesis is false, then there are infinitely many natural numbers* x *for which*

$$\frac{\log x}{\sqrt{x}} + O\left(\frac{1}{\sqrt{x}}\right) + 2 \cdot \log x + c \cdot \log x \leq 2.062$$

could be satisfied for some $c > 0$.

The following is a key Corollary.

Corollary 1 *The Riemann hypothesis is true.*

2 Proof of the Lemma 1

Proof According to Proposition 3, the Riemann hypothesis is false, if there exists some natural number $x_0 \geq 5$ such that $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$:

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(x) \cdot \prod_{p \leq x} \left(1 + \frac{1}{p}\right)^{-1}.$$

We know the bound [3, Theorem 4.2 p. 5]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}$$

where f was introduced in the Nicolas paper [7, Theorem 3 p. 376]:

$$f(x) = e^{\gamma} \cdot \log \theta(x) \cdot \prod_{p \leq x} \left(1 - \frac{1}{p}\right).$$

When the Riemann hypothesis is false, then there exists a real number $b < \frac{1}{2}$ for which there are infinitely many natural numbers x such that $\log f(x) = \Omega_{+}(x^{-b})$ [7, Theorem 3 (c) p. 376]. According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0)\colon\ \log f(y) \geq k \cdot y^{-b}.$$

That inequality is equivalent to $\log f(y) \geq \left(k \cdot y^{-b} \cdot \sqrt{y}\right) \cdot \frac{1}{\sqrt{y}}$, but we note that

$$\lim_{y \to \infty} \left(k \cdot y^{-b} \cdot \sqrt{y}\right) = \infty$$

for every possible positive value of k when $b < \frac{1}{2}$. In this way, this implies that

$$\forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0)\colon\ \log f(y) \geq \frac{1}{\sqrt{y}}.$$

Hence, if the Riemann hypothesis is false, then there are infinitely many natural numbers x such that $\log f(x) \geq \frac{1}{\sqrt{x}}$. Since $\frac{2}{x} = o(\frac{1}{\sqrt{x}})$, then it would be infinitely many natural numbers x_0 such that $\log g(x_0) > 0$. □

3 Proof of the Theorem 1

By Lemma 1, there are infinitely many natural numbers $x \geq 5$ for which

$$\gamma + \log\log\theta(x) \geq \sum_{p \leq x} \log\left(1 + \frac{1}{p}\right) + \log(\zeta(2))$$

could be satisfied. This implies that there are infinitely many natural numbers $x \geq 10^9$ for which

$$\sum_{p \leq x} \log\left(1 - \frac{1}{p}\right) + \sum_{p \leq x} \log\left(1 + \frac{1}{p}\right) + \log(\zeta(2)) + \frac{2}{\sqrt{x} \cdot \log x} + \frac{S_1(x)}{\log x} \leq \frac{2.062}{\sqrt{x} \cdot \log^2 x}$$

could be also satisfied by Proposition 5. We can write

$$\left(1 - \frac{1}{p}\right) \cdot \left(1 + \frac{1}{p}\right) = \left(1 - \frac{1}{p^2}\right).$$

for every prime [3, p. 3]. Consequently, we obtain that

$$\sum_{p \leq x} \log\left(1 - \frac{1}{p^2}\right) + \log(\zeta(2)) + \frac{2}{\sqrt{x} \cdot \log x} + \frac{S_1(x)}{\log x} \leq \frac{2.062}{\sqrt{x} \cdot \log^2 x}$$

by properties of logarithms. By Proposition 1, we know that

$$\sum_{p > x} \log\left(\frac{p^2}{p^2 - 1}\right) + \frac{2}{\sqrt{x} \cdot \log x} + \frac{S_1(x)}{\log x} \leq \frac{2.062}{\sqrt{x} \cdot \log^2 x}.$$

Using the Propositions 2 and 4, we can see that

$$\frac{1}{x \cdot \log x} + O\left(\frac{1}{x \cdot \log^2 x}\right) + \frac{2}{\sqrt{x} \cdot \log x} + \frac{c}{\sqrt{x} \cdot \log x} \leq \frac{2.062}{\sqrt{x} \cdot \log^2 x}.$$

Let's multiply both sides by $\sqrt{x} \cdot \log^2 x$ to assume that

$$\frac{\log x}{\sqrt{x}} + O\left(\frac{1}{\sqrt{x}}\right) + 2 \cdot \log x + c \cdot \log x \leq 2.062$$

could be satisfied for some $c > 0$, when the Riemann hypothesis is false.

4 Proof of Corollary 1

Proof By Theorem 1, we proved that there exists some c' such that there are infinitely many natural numbers x for which

$$\frac{\log x}{\sqrt{x}} + \frac{c'}{\sqrt{x}} + 2 \cdot \log x + c \cdot \log x \leq 2.062$$

could be satisfied for some $c > 0$, when the Riemann hypothesis is false. However, no matter how small could be the value of c', we can always assure that

$$\frac{\log x}{\sqrt{x}} + \frac{c'}{\sqrt{x}} + 2 \cdot \log x + c \cdot \log x \leq 2.062$$

does not hold for some $c > 0$ and large enough x. In conclusion, the Riemann hypothesis is true by principle of non-contradiction.

□

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