Numerical Solver for NH2018 S3

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1 Discretization of Equation (S3)

Equation (S3) in NH2018 reads

$$\frac{\partial}{\partial\mu}\left[\frac{1}{2\Omega\mu}\frac{\partial}{\partial\mu}(u_{\mathrm{REF}}\cos\phi)\right]+\frac{2\Omega a^2 H\mu}{R(1-\mu^2)}e^{z/H}\frac{\partial}{\partial z}\left[\frac{e^{(\kappa-1)z/H}}{\frac{\partial\tilde{\theta}}{\partial z}}\frac{\partial}{\partial z}u_{\mathrm{REF}}\cos\phi\right]=-a\frac{\partial}{\partial\mu}\left(\frac{q_{\mathrm{REF}}}{2\Omega\mu}\right) \tag{1}$$

where $\mu\equiv\sin\phi$. With the substitutions $\tilde{u}\equiv u_{\mathrm{REF}}\cos\phi$, $\tilde{q}\equiv\frac{q_{\mathrm{REF}}}{2\Omega\mu}$ and static stability $S\equiv\frac{\partial\tilde{\theta}}{\partial z}$, equation (1) becomes

$$\frac{\partial}{\partial\mu}\left[\frac{1}{2\Omega\mu}\frac{\partial}{\partial\mu}\tilde{u}\right]+\frac{2\Omega a^2 H\mu}{R(1-\mu^2)}e^{z/H}\frac{\partial}{\partial z}\left[\frac{e^{(\kappa-1)z/H}}{S}\frac{\partial\tilde{u}}{\partial z}\right]=-a\frac{\partial\tilde{q}}{\partial\mu} \tag{2}$$

Discretizing each term in equation (2) on the uniform ϕ (index j) and z (index k) grids yields:

$$\begin{aligned}
&\frac{\partial}{\partial\mu}\left[\frac{1}{2\Omega\mu}\frac{\partial}{\partial\mu}\tilde{u}\right]\\
=\;&\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left[\frac{1}{2\Omega\sin\phi\cos\phi}\frac{\partial}{\partial\phi}\tilde{u}\right]\\
\approx\;&\frac{1}{2\Omega(\Delta\phi)^2\cos\phi_j}\left[\frac{\tilde{u}_{j+1,k}}{\sin\phi_{j+1/2}\cos\phi_{j+1/2}}+\tilde{u}_{j,k}\left(\frac{1}{\sin\phi_{j+1/2}\cos\phi_{j+1/2}}+\frac{1}{\sin\phi_{j-1/2}\cos\phi_{j-1/2}}\right)+\frac{\tilde{u}_{j-1,k}}{\sin\phi_{j-1/2}\cos\phi_{j-1/2}}\right]
\end{aligned} \tag{3}$$

$$\begin{aligned}
&\frac{2\Omega a^2 H\mu}{R(1-\mu^2)}e^{z/H}\frac{\partial}{\partial z}\left[\frac{e^{(\kappa-1)z/H}}{\frac{\partial\tilde{\theta}}{\partial z}}\frac{\partial}{\partial z}u_{\mathrm{REF}}\cos\phi\right]\\
=\;&\frac{2\Omega a^2 H\sin\phi}{R\cos^2\phi}e^{z/H}\frac{\partial}{\partial z}\left[\frac{e^{(\kappa-1)z/H}}{\frac{\partial\tilde{\theta}}{\partial z}}\frac{\partial}{\partial z}u_{\mathrm{REF}}\cos\phi\right]\\
\approx\;&\frac{2\Omega^2 a^2 H\sin\phi_j}{R\cos^2\phi_j}\frac{e^{z_k/H}}{(\Delta z)^2}\left[\tilde{u}_{j,k+1}\frac{e^{\frac{(\kappa-1)z_{k+1/2}}{H}}}{S_{k+1/2}}-\tilde{u}_{j,k}\left(\frac{e^{\frac{(\kappa-1)z_{k+1/2}}{H}}}{S_{k+1/2}}+\frac{e^{\frac{(\kappa-1)z_{k-1/2}}{H}}}{S_{k-1/2}}\right)+\tilde{u}_{j,k-1}\frac{e^{\frac{(\kappa-1)z_{k-1/2}}{H}}}{S_{k-1/2}}\right]
\end{aligned} \tag{4}$$

$$\begin{aligned}
&-a\frac{\partial\tilde{q}}{\partial\mu}\\
=\;&-\frac{a}{\cos\phi}\frac{\partial\tilde{q}}{\partial\phi}\\
\approx\;&-\frac{a}{2\Delta\phi\cos\phi_j}\left(\tilde{q}_{j+1}-\tilde{q}_{j-1}\right)
\end{aligned} \tag{5}$$

Substituting (3), (4) and (5) into (2) gives

$$a_{j,k}\tilde{u}_{j+1,k}+b_{j,k}\tilde{u}_{j-1,k}+c_{j,k}\tilde{u}_{j,k+1}+d_{j,k}\tilde{u}_{j,k-1}+e_{j,k}\tilde{u}_{j,k}=f_{j,k} \tag{6}$$

where

$$a_{j,k} = \frac{1}{\sin\phi_{j+1/2}\cos\phi_{j+1/2}} \tag{7}$$

$$b_{j,k} = \frac{1}{\sin\phi_{j-1/2}\cos\phi_{j-1/2}} \tag{8}$$

$$c_{j,k} = \frac{2\Omega^2 a^2 H \sin\phi_j}{R\cos\phi_j} e^{z_k/H} \frac{(\Delta\phi)^2}{(\Delta z)^2} \frac{e^{\frac{(\kappa-1)z_{k+1/2}}{H}}}{S_{k+1/2}} \tag{9}$$

$$d_{j,k} = \frac{2\Omega^2 a^2 H \sin\phi_j}{R\cos\phi_j} e^{z_k/H} \frac{(\Delta\phi)^2}{(\Delta z)^2} \frac{e^{\frac{(\kappa-1)z_{k-1/2}}{H}}}{S_{k-1/2}} \tag{10}$$

$$e_{j,k} = -(a_{j,k} + b_{j,k} + c_{j,k} + d_{j,k}) \tag{11}$$

$$f_{j,k} = -\Omega a(\Delta\phi)(\tilde{q}_{j+1} - \tilde{q}_{j-1}) \tag{12}$$

Equation (6) is then solved using the successive over-relaxation algorithm.

2 The Successive Over-relaxation (SOR) Solver

The iterative procedure to solve equation (6) is

$$\tilde{u}^{new}_{j,k} = \tilde{u}^{old}_{j,k} - \omega\frac{\zeta_{j,k}}{e_{j,k}} \tag{13}$$

where $\zeta_{j,k}$ is the residual

$$\zeta_{j,k} \equiv a_{j,k}\tilde{u}_{j+1,k} + b_{j,k}\tilde{u}_{j-1,k} + c_{j,k}\tilde{u}_{j,k+1} + d_{j,k}\tilde{u}_{j,k-1} + e_{j,k}\tilde{u}_{j,k} - f_{j,k} \tag{14}$$

and the over-relaxation parameter at the n-th step is given by

$$\omega_n = \begin{cases} \frac{1}{1-\rho^2_{\text{jacobi}}/2} & \text{when } n = 1 \\ \frac{1}{1-\rho^2_{\text{jacobi}}\omega_{n-1}/4} & \text{when } n > 1 \end{cases}$$

In NH2018, ρ^2_{jacobi} used is 0.95. The condition to stop the iteration is when

$$\sum_{1}^{j_{\max}-1}\sum_{1}^{k_{\max}-1}\|\zeta_{j,k}\| < \epsilon \sum_{1}^{j_{\max}-1}\sum_{1}^{k_{\max}-1}\|f_{j,k}\| \tag{15}$$

where the tolerance parameter $\epsilon = 10^{-5}$ in NH2018.

3 The boundary conditions

The latitudional boundary conditions for $\tilde{u}$ are:

$$\tilde{u}(\phi = \frac{\pi}{2}) = 0 \Rightarrow \tilde{u}_{j_{\max},k} = 0 \tag{16}$$

$$\tilde{u}(\phi = 0) = (\bar{u} + A)\cos\phi \Rightarrow \tilde{u}_{0,k} = (\bar{u}_{0,k} + A_{0,k})\cos\phi_0 \tag{17}$$

The lower boundary condition for $\tilde{u}$ is given by no-slip condition:

$$\tilde{u}(z = 0) = 0 \Rightarrow \tilde{u}_{j,0} = 0 \tag{18}$$

while the upper boundary condition for $\tilde{u}$ is given by the thermal wind relation:

$$\frac{\partial\tilde{u}}{\partial z} = -\frac{R\cos\phi e^{\kappa z_{\text{top}}/H}}{2\Omega a H\mu}\frac{\partial\bar{T}}{\partial\phi} \tag{19}$$

Discretizing (19) yields

$$\frac{\tilde{u}_{j,k_{\text{top}}} - \tilde{u}_{j,k_{\text{top}-2}}}{2\Delta z} = -\frac{R\cos\phi_j e^{\kappa z_{\text{top}-1}/H}}{2\Omega a H\mu_j}\frac{\bar{T}_{j+1,k} - \bar{T}_{j-1,k}}{2\Delta\phi}$$

$$\tilde{u}_{j,k_{\text{top}}} = \tilde{u}_{j,k_{\text{top}-2}} - \frac{R\cos\phi_j e^{\kappa z_{\text{top}-1}/H}}{2\Omega a H\mu_j}\frac{\Delta z}{\Delta\phi}(\bar{T}_{j+1,k_{\text{top}-1}} - \bar{T}_{j-1,k_{\text{top}-1}}) \tag{20}$$