The Consequences of Legon's Rectangle
The Rational Giza Design

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Abstract

In 1979, John Legon proposed that Khufu and Menkaure's pyramids were sited at the opposite corners of a $1000\sqrt{2}$ by $1000\sqrt{3}$ cubit rectangle. The problem with this proposal is that it almost works. The $1000\sqrt{3}$ north – south length is fine, but the west – east dimension is 1417.5 cubits rather than the desired 1414. Some researchers argue that this discrepancy invalidates the proposal. I take a different view, and propose that Menkaure as it is now is not what was originally built: we are dealing with a renovated pyramid, which is bedevilling analysis. In this paper I show multiple relationships that follow from adopting Legon's proposal, using an original base size for Menkaure suggested by the mathematics. This produces multiple instances of π, as well as φ, e, and square roots inherent in this arrangement. These relationships strongly argue that Legon's concept design is correct.

Keywords: Egyptology, Giza, History of Mathematics, π, φ.

Contents

1 Introduction

Legon's 1979 paper, The Plan of the Giza Pyramids [1], proposed the conceptual plan linking Khufu and Menkaure, as in Figure 1.

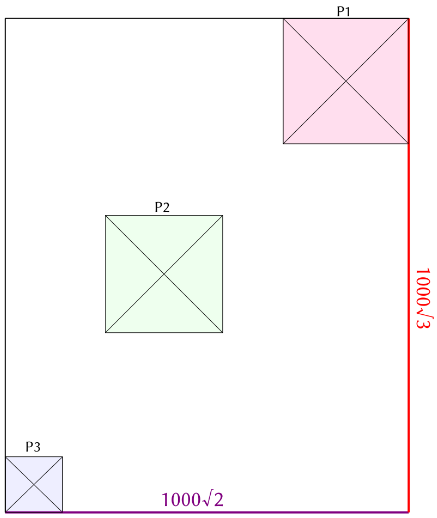

Figure 1: Legon's concept design

The problem with the proposal is that the west – east distance is 1417.5 cubits, rather then the desired 1414. In truth, the problem is even more subtle than that, because Khufu is twisted slightly west of north, while Menkaure is twisted more aggressively east of north (Petrie [2], Nell and Ruggles [3]). So you will get different lengths depending on how you measure. For example, assume we start at Khufu's north east corner. To measure $1000\sqrt{3}$ cubits, we can either measure directly due south, or track along the Khufu's eastern edge at a slight angle east of true south. These two lines will end in different places. Measuring $1000\sqrt{2}$ at 90° west from there, will again land us in two different spots.

A further complication is the values used for $1000\sqrt{3}$ and $1000\sqrt{2}$. Are they 1732.05 and 1414.214 cubits, or do we work to the nearest cubit and use 1732 and 1414 cubits? Each choice we make produces a different end result.

Figure 2 shows where we would end, in Menkuare's current south-west corner, depending on whether we measured with the Cartesian / cardinal directions or Khufu's skewness, and with "full" or rounded values for $1000\sqrt{2}$ and $1000\sqrt{3}$.

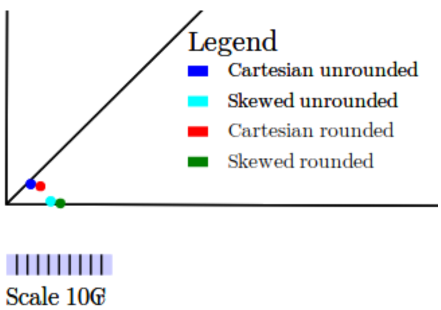

Figure 2: Menkaure's south-west corner

In Zep Tepi Mathematics 101 (ZTM101) [4] I developed a larger double-square site plan, using Legon's rectangle as the starting point. In this paper, I want to make a stronger case for using Legon as base, showing the many results that follow from assuming his paradigm is correct.

"The language of Giza is mathematics."

Robert Bauval

"You will believe."

The architects of Giza

Changelog:
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2 Symbols and values

The usual table of irrationals. Construction is a practical art, so we need to use practical values for certain irrationals. Many results below are ratios between two lengths, usually integers. Irrationals can not be written as fractions, so the result is always an approximation, with varying degrees of accuracy. It appears that while they used 3.1416 as π when needed, they were happy to approximate that to simpler versions like 3.14 or 3.142 when showing intent rather than exactitude was sufficient. Even today, we still use 22/7 or 3.14 in school.

Symbol	Name	"Full" value	Practical value	Practical % Accuracy
π	Archimedes' constant	3.141592653...	3.1416 or 3.142 or 3.14	99.9998, 99.9870 or 99.9493
e	Euler's number	2.718281828...	2.7183 or 2.718 or 2.72	99.9993 or 99.9896 or 99.9368
φ	Golden ratio	1.6180339887...	1.618 $\varphi + 1 = \varphi^2 = 2.618$	99.9979
ρ	Plastic ratio	1.3247179572...	1.3247 or 1.325 $\rho + 1 = \rho^3 = 2.3247$	99.9986 or 99.9458
$\sqrt{2}$	Root 2	1.414213562...	1.4142 or 1.414	99.9990 or 99.9849
$\sqrt{3}$	Root 3	1.732050807...	1.732	99.9971
$\sqrt{5}$	Root 5	2.236067977...	2.236	99.9970
Ṗ	Petrie inch	2.5399977 cm		
₠	Royal cubit	0.5236 m		

Table 1: Symbols, names and values

I use "cubit" for the Royal cubit (₠). Other researchers use other values, but 0.5236 works for me.

Equations shown below use a science/engineering approach, not pure mathematics, so 1.99 = 2.0.

We should remember that we are dealing with a different culture, which may have had a different approach to accuracy and precision, compared to our modern scientific mindset. My Giza analysis suggests they typically worked to 3 or 4 decimal places. This may indicate that they used abacuses or counting tables, or possibly slide rules or logarithms, to calculate. Alternatively, they may just have used four decimals as anything more did not make sense in construction. 0.0001 ₠ is 0.05236 mm, which is about 50 microns, half the smallest distance that can be seen with the naked eye, and approaching the length of a human liver cell.

Instead of the conventional cubit divisions, we use a cubit divided into 100 centicubits, and 1000 millicubits, because this is what works. Petrie (§178) says, "That the cubit was divided decimally in the fourth dynasty we know (see section 139);"

Figure 3 shows how part of such a decimal cubit rod might look, compared to conventional digits, a Petrie-era inch ruler, and a metric ruler. For some reason, it shrinks from true size when printed, but can be checked on screen by viewing the PDF file at 100% zoom factor. Millicubits are quite close in size to 50ths of an inch, the 50ths being 0.50799954 mm and the millicubits 0.5236 mm.

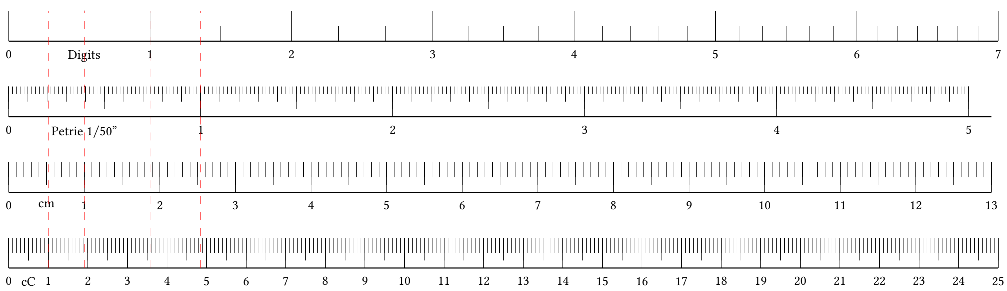

Figure 3: Digits, inches, and centimetres compared to decimal centicubit divisions.

3 How do you solve a problem like Menkaure?

We start with the fact that Menkaure's footprint as it is now is not what it was originally.

There is still considerable debate about Menkaure's size. In "The Pyramids and Temples of Gizeh" [2] (§80), Petrie gives the base as a square with mean side of 4153.6 Ṗ, which translates to 201.5 ₢. Mark Lehner's "Complete Pyramids" [5] gives a base of 102.2 × 104.6 metres, which translates to 195 × about 200 ₢, although I heard that he did not measure it himself, but used the figures from Maragioglio & Rinaldi [6]. M&R in turn cite Goyon for the 102.2 figure, being the west side. So Lehner's figures are west and east sides first. I normally quote the X-axis (south and north sides) first, so that would be 200 × 195 ₢.

In his paper on Menkaure [7], Keith Hamilton cites Lehner's revised version with Zahi Hawass, "Giza and the Pyramids" [8], as giving a base 346 feet or 201 ₢ square, although the conversion is closer to 201.5 ₢.

Legon himself followed Petrie at 201.5 ₢. Hamilton mentions others unnamed as preferring 200 ₢, while goudryan [9] makes the case for a square base of 198 ₢. I have also encountered a 198 ₢ square base in the discussions on Academia.edu. The average of my sides proposed below is also 198 ₢.

Wikipedia's page on Menkaure's pyramid [10] cites Lehner/M&R/Goyon's size, as well as claiming 200 ₢ square as the "original" size. In contrast, Wikipedia's list of Egyptian pyramids [11] gives the base as 103.4 metres, which is 197.5 ₢, again close to 198 ₢.

OpenStreetMap [12] provides the co-ordinates for the four corners. Computing the sides from those gives the north and south edges as 100.7 metres, and the west and east edges as 96.8 metres. That's 192.3 × 184.9 ₢, which seems too small, although it is non-square.

Apart from issues with the base dimensions, the researchers also have trouble deciding on the slope angle. This may be an indication that the base was not originally square, and/or of other alterations.

Petrie (§89) argues that the current size is not the original size:

"From all these details it seems plain that the Third Pyramid was first begun no larger than some of the small Pyramids on the same hill. That it had a passage descending as usual, with a large lintel block over it ; and running horizontally in the rock, into a rock-cut chamber, whose roof was 74.1 above the passage floor. That after this was made, the builders, for some reason, determined on enlarging the Pyramid before it was cased, and on deepening the chamber. They accordingly cut a fresh passage, from the new floor level of the chamber, working this passage from the inside outward. They not only deepened the chamber, but also cut the sloping passage to the lower, granite-lined, coffer chamber ; for the granite lining could not be put in until the second chamber had been deepened to its present extent ; so the granite chamber must be part of the second design, or is perhaps in itself a third design. The old entrance passage was then built over on the outside, and the greater part of its height blocked up."

Ondrash Sabo [13] cites Reisner (not sure which book) as suggesting dividing the construction of the complex into several phases in time:

1. Dynasty IV - time of Menkaure's reign
2. Dynasty IV - time of Shepseskaf's reign
3. Time of Dynasty V
4. Time of Dynasty VI

So clearly at least two experts think this pyramid was not a straight-forward one-time build, even if we disagree on when exactly each phase happened. The issue gets even murkier if we look at the dating exercises. Today we "know" that Menkaure's reign began about 2532 BCE. A hundred years ago, historians were equally adamant that his reign started around 3633 BCE [14]. Even Petrie, in §94, mentions the various dates then current; "Hence, in 4,000 to 6,000 years — the age of the Pyramids by different chronologers —".

Radio-carbon dating of 35 samples collected from the Pyramid of Menkaure revealed the age of the pyramid to be on average 4127 BP [15], which is 2177 BCE. Ian Onvlee's interpretation is "The Lesser Pyramid varied between 3076-2067 BC, or on average 2572 BC." [16].

On the other hand, surface luminescence dating of one sample from an exterior casing stone, which may have been moved during an alteration, returned a date of 3450 ± 950, which is 4400 to 2500 BCE.

Menkaure remains thus a very confusing pyramid, both as to when built and dimensions. Even if we measure it accurately today, that will be the current size, not the original design size.

When I was working on ZTM101, I started with the Wikipedia value at that time, which was 202 ₵. As things developed, and I tried to integrate the 4th and 5th pyramids into the plan, the mathematics simply refused to work. I checked every possible square size from 190 to 210 ₵, in ¼ cubit steps, but nothing would work. Eventually it was suggested to me that Menkuare was not (originally) square… a concept that took time to accept. In truth, there are at least two other non-square pyramids, Djoser and Khui, so the idea is not totally preposterous.

After accepting a non-square base, I let my programs try again and they came up with the dimensions of 201 × 195 ₵. This was while doing the "skeleton blueprint" part of ZTM101. At that point it was suggested to me to apply what I had been doing in the blueprint, which was ratios of the distances between pyramid centres, to the spaces between the pyramids. This is when the $\sqrt{2}$ and $\sqrt{3}$ ratios shown below popped out, which only work with a 201 × 195 size. Then I knew it was correct.

4 Legon's rectangle

During the process of writing ZTM101, it became increasingly clear that the designers were happy to use whole-cubit dimensions for the pyramids, and the spaces between them. This may have been pragmatic, or just considered "good enough."

So we use sides of 1414 and 1732 for Legon's rectangle, and add the diagonal to make a right-angled triangle. This blatantly shows Pythagoras using the first three primes, which is at least one level above the classic Pythagorean 3:4:5 triangle. It also introduces $\sqrt{5}$, which is a recurrent feature at Giza. Due to rounding of the sides, the actual length is 2235.894 ₵ instead of 2236, which is an error of about 5.5 cm on a distance of over 1170 metres.

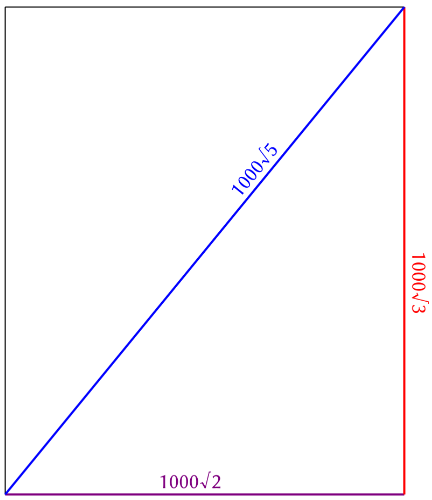

Figure 4: Legon does Pythagoras

5 Siting Khafre

As discussed in ZTM101, Khafre is sited so that it splits the north-south distance between Khufu and Menkaure in the ratio 1:$\sqrt{3}$, and the west-east distance in the ratio 1:$\sqrt{2}$.

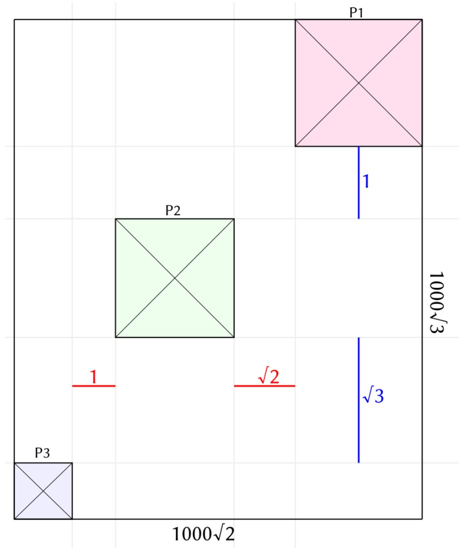

Figure 5: The simple secret behind the Giza layout.

That gives us the dimensions:

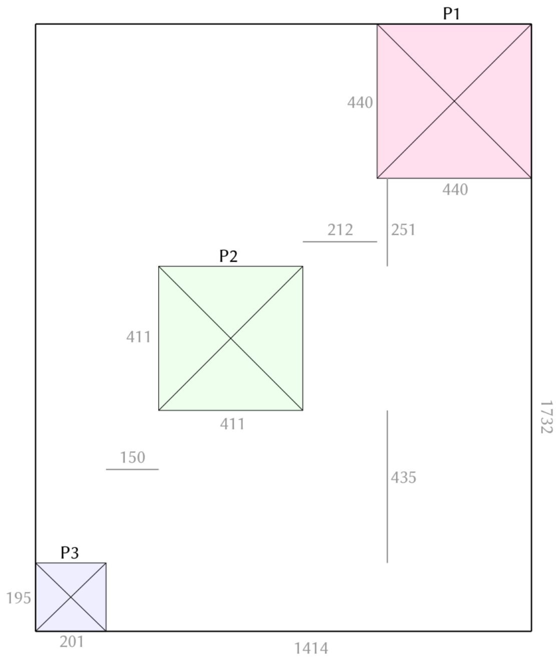

Figure 6: Giza dimensions in ₢.

We can compare these dimensions to those from the Giza Plateau Mapping Project, bearing in mind:

1. The three pyramids are not perfectly aligned with north on the ground, but are in the design.
2. We resized Menkaure from 201.5 square to 201 × 195 ₢.
3. GPMP locations are all ± 5cm.

Since the actual pyramids are twisted and the sides not parallel to the adjacent pyramid, I took the average of the relevant sides, and calculated the distance between them. So for the west-east space between P3 and P2, we calculate the average Eastings for P3 east edge and P2 west edge, and then find the difference in metres and cubits.

Point	Eastings	Average
P2 NW	499558.2	P2: 499558.4
P2 SW	499558.6	
P3 NE	499479.5	P3: 499479.3
P3 SE	499479.1	
Difference m		79.1
Difference ₢		151.07

Table 2: Calculating west-east space between P3 and P2

Similarly, for the west-east space between P1 and P2, we calculate the average Eastings for P2 east edge and P1 west edge, and then find the difference in metres and cubits.

Point	Eastings	Average
P1 NW	499884.6	P1: 499884.75
P1 SW	499884.9	
P2 NE	499773.5	P2: 499773.7
P2 SE	499773.9	
Difference m		111.050
Difference ₢		212.089

Table 3: Calculating west-east space between P1 and P2

For the north-south space between P1 and P2, we calculate the average Northings for P1 south edge and P2 north edge, and then find the difference in metres and cubits.

Point	Northings	Average
P1 SW	99884.7	P1: 99884.8
P1 SE	99884.9	
P2 NW	99753.1	P2: 99753.25
P2 NE	99753.4	
Difference m		131.55
Difference ₢		251.24

Table 4: Calculating north-south space between P1 and P2

For the north-south space between P2 and P3, we calculate the average Northings for P2 south edge and P3 north edge, and then find the difference in metres and cubits.

Point	Northings	Average
P2 SW	99537.8	P2: 99537.95
P2 SE	99538.1	
P3 NW	99312.9	P3: 99912.6
P3 NE	99312.3	
Difference m		225.35
Difference ₢		430.39

Table 5: Calculating north-south space between P2 and P3

We can then compare the current renovated condition against the proposed original design.

Space	GPMP ₲	This design ₲
P1 P2 north-south	251.24	251
P2 P1 west-east	212.09	212
P2 P3 north-south	430.39	435
P3 P2 west-east	151.07	150

Table 6: Comparing current spaces vs design.

As shown in Table 6, the correlation between P1 and P2 is excellent, while the differences between P2 and P3 stem from resizing P3 to a mathematically-determined original size.

In the interests of full disclosure, Table 7 shows a comparison between Giza as it is now, using the GPMP points, versus my proposed design, remembering again that the pyramids are twisted while the design is not, and Menkaure is resized.

Pyramid	Point	GPMP X	GPMP Y	Design X	Design Y	Delta m	Delta ₲
P1	C	500000.0	100000.0	500000.000	100000.000	0.000	0.000
P1	NW	499884.6	100115.1	499884.808	100115.192	0.227	0.434
P1	NE	500115.0	100115.3	500115.192	100115.192	0.220	0.421
P1	SW	499884.9	99884.7	499884.808	99884.808	0.142	0.271
P1	SE	500115.3	99884.9	500115.192	99884.808	0.142	0.271
P2	C	499666.0	99645.6	499666.205	99645.7846	0.276	0.527
P2	NW	499558.2	99753.1	499558.6052	99753.3844	0.495	0.945
P2	NE	499773.5	99753.4	499773.8048	99753.3844	0.305	0.583
P2	SW	499558.6	99537.8	499558.6052	99538.1848	0.385	0.735
P2	SE	499773.9	99538.1	499773.8048	99538.1848	0.127	0.243
P3	C	499426.5	99259.9	499427.4434	99259.3678	1.083	2.069
P3	NW	499373.9	99312.9	499374.8216	99310.4188	2.647	5.055
P3	NE	499479.5	99312.3	499480.0652	99310.4188	1.964	3.751
P3	SW	499373.5	99207.3	499374.8216	99208.3168	1.667	3.185
P3	SE	499479.1	99206.9	499480.0652	99208.3168	1.714	3.274

Table 7: Comparing GPMP points vs design.

Petrie (§93) gives the twists as
Great Pyramid, casing sides -3'43"± 6"
Second Pyramid casing sides -5'26"±16"

The average is -4' 34.5" which is -0.07625°.
To calculate the effect of that twist, we can do length × sin (-0.07625°). The question is, what length? The anchor point for both GPMP and my design is Khufu's centre, but we can't simply twist it around that point, because the whole tectonic plate twisted from some other pivot point.
Regardless, 220 ₲ × sin(-0.07625°) = 0.2928 ₲, and 440 ₲ × sin(-0.07625°) = 0.5856 ₲.

The GPMP X and Y figures are to the nearest 10 cm, so they are all ± 5cm, and thus the deltas could be affected by up to ± 10 cm. This has a knock-on effect on the various accuracies. Furthermore, current dimensions are based off the casing stones, but Petrie (§93), points out that the casing alignment is inferior to the core alignment, which supports my contention that the Dynastics renovated the pyramids rather than building them:

"In considering these results, the difference of the casing and core azimuths of the Great Pyramid shows that probably a re-determination of the N. was made after the core was finished; and it must be remembered that the orientation would be far more difficult to fix after, than during, the construction ; as a high face of masonry, for a plumb-line, would not be available.

"The passages of the Great and Second Pyramids, are the most valuable elements ; as, being so nearly at the polar altitude, a very short plumb-line would transfer the observations to the fixed plane.

"Considering, then, that the Great Pyramid core agrees with the passages far closer than does the casing, the inference seems to be that the casing was fixed by a re-determination of N., by the men who finished the building. These men had not the facilities of the earlier workers ; and are shown, by the inferiority of the later work in the Pyramid, to have been far less careful.

"Hence the casing may probably be left out of consideration, in view of the close agreement of the four other determinations, one of which — the passage — was laid out by the most skilful workmen of the Great Pyramid, with their utmost regularity, the mean variation of the built part being but $\frac{1}{50}$ inch."

The sides lengths are compared in Table 8.

Pyramid	Side	GPMP m	GPMP Cr	Design m	Design Cr
P1	N	230.400	440.031	230.384	440.000
P1	E	230.400	440.031	230.384	440.000
P1	S	230.400	440.031	230.384	440.000
P1	W	230.400	440.031	230.384	440.000
P2	N	215.300	411.192	215.200	411.000
P2	E	215.300	411.192	215.200	411.000
P2	S	215.300	411.192	215.200	411.000
P2	W	215.300	411.192	215.200	411.000
P3	N	105.602	201.684	105.244	201.000
P3	E	105.401	201.300	102.102	195.000
P3	S	105.601	201.682	105.244	201.000
P3	W	105.401	201.300	102.102	195.000

Table 8: Comparing GPMP sides vs design.

6 It all works together: π

With the necessary background matter behind us, we can now look at all the Easter eggs hidden in the design. We start with π.

6.1 π ratios

There are several ratios that round to 3.14.

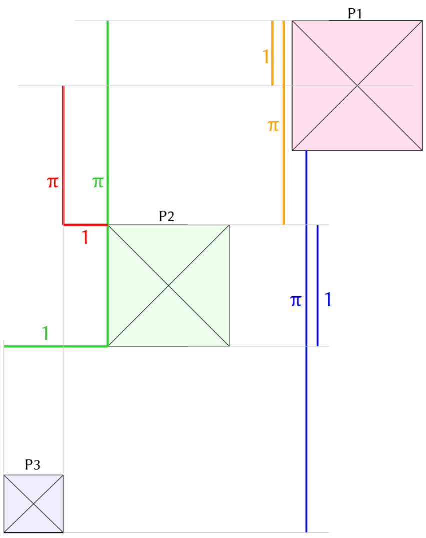

Figure 7: π ratios 1

Red:

$$\frac{220 + 251}{150} = \frac{471}{150} = 3.14 = \pi$$

Green:

$$\frac{440 + 251 + 411}{201 + 150} = \frac{1102}{351} = 3.1396 = 3.14 = \pi$$

Orange:

$$\frac{440 + 251}{220} = \frac{691}{220} = 3.1409 = 3.14 = \pi$$

Blue:

$$\frac{251 + 411 + 435 + 195}{411} = \frac{1292}{411} = 3.1436 = 3.14 = \pi$$

6.2 π Ratio on diagonal

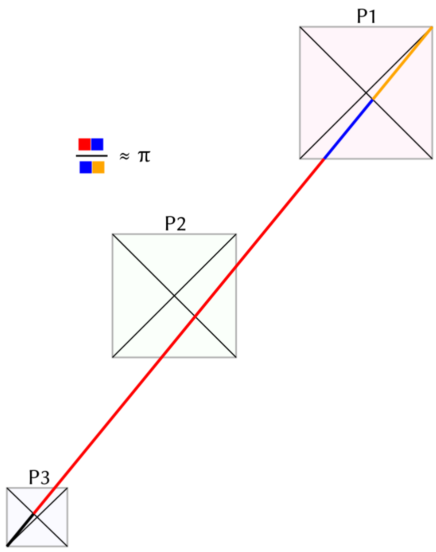

Figure 8: π ratio 2

$$\frac{\text{red} + \text{blue}}{\text{blue} + \text{orange}} = \frac{933.4259}{297.4110} = 3.1385 = 3.14 = \pi$$

6.3 Intersecting π ratios

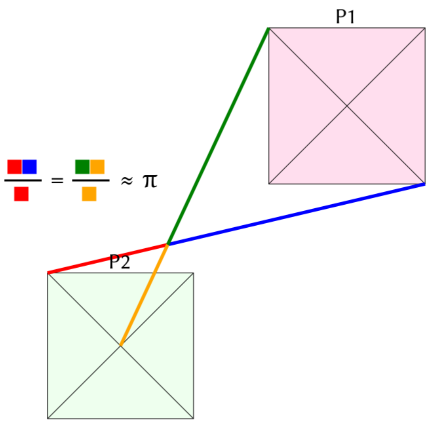

Figure 9: Intersecting π lines

$$\frac{\text{red} + \text{blue}}{\text{red}} = \frac{571.8925}{182.0065} = 3.141 = \pi$$

$$\frac{\text{green} + \text{orange}}{\text{orange}} = \frac{537.8133}{164.8499} = 3.141 = \pi$$

6.4 Pahl's Point

Pahl's Point was discovered by Larry Pahl.

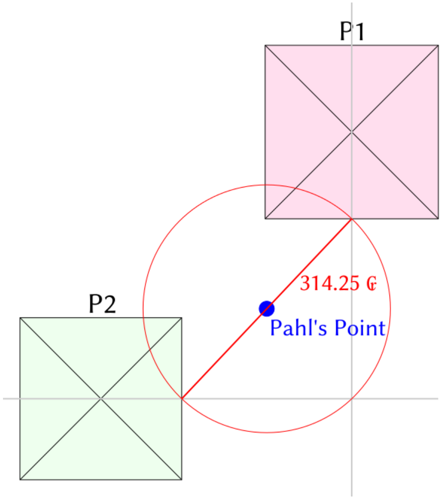

Figure 10: Pahl's Point

$$\frac{\sqrt{(251+205.5)^2+(212+220)^2}}{2} = \frac{628.50}{2} = 314.25 = 100\pi$$

6.5 $\sqrt{2}+\sqrt{3}\approx\pi$

We know that $\sqrt{2}+\sqrt{3}$ gives "an" approximation for π (3.146), so 1414 + 1732 = 3146 is an approximation for 1000π. Also, 1732 - 1414 = 318, from which 1000/318 gives another approximation (3.14465) for π.

6.6 π in metres

By adding the red lengths, and subtracting the blue length in Figure 11, we get an excellent value for π. The blue length is 1618 ₵, discussed in the next section.

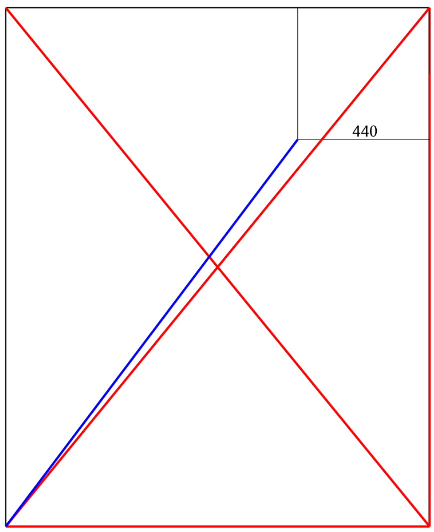

Figure 11: π

Proof: $2236 + 2236 + 1414 + 1732 - 1618 = 6000$ ₵. Convert to metres: $6000 \times 0.5236 = 3141.6$ or 1000π.

It is quite curious how $\sqrt{2}+\sqrt{3}+\sqrt{5}+\sqrt{5}-\varphi = 6.000366336$, or 6.000 rounded. φ is 1.6180339887. I get the impression that this was "considered well-known" by the designers, which is why they used it.

The arrangement can also be interpreted the other way around, albeit with circular logic, similarly to how the King's chamber perimeter is 60 ₵, or 10 π metres.

3141.6 m ÷ 6000 ₵ = 0.5236 m/₵, thus defining the cubit:metre ratio.

6.7 $\sqrt{\pi}$

The right edge diagonal gives us a value for $\sqrt{\pi}$ of 1.772, rounded.

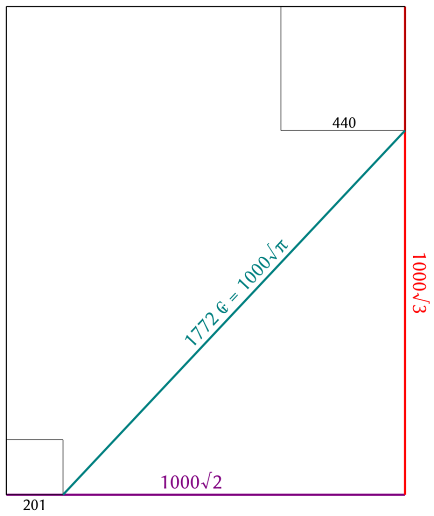

Figure 12: $\sqrt{\pi}$

Proof: $\sqrt{(1414 - 201)^2 + (1732 - 440)^2} = 1772$, rounded to the nearest cubit.
$\sqrt{3.1416} = 1.772$.

6.8 π^2 Between Khufu and Menkaure

Khufu and Menkaure are related via a π approximation.

$$\frac{2 \times \text{area Khufu}}{\text{area Menkaure}} = \frac{2 \times 440^2}{201 \times 195} = 9.8788$$
$$\sqrt{9.8788} = 3.143.$$

6.9 π Ratios in pyramid bases

The pyramid bases are related via π.

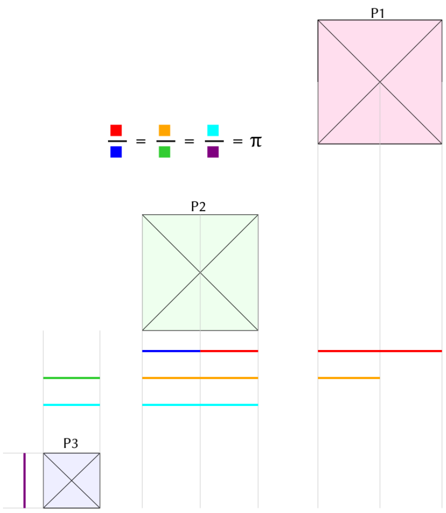

Figure 13: π in the bases

Red and blue:

$$\frac{440 + 205.5}{205.5} = \frac{645.5}{205.5} = 3.1411 = 3.14 = \pi$$

Orange and green:

$$\frac{220 + 411}{201} = \frac{631}{201} = 3.1393 = 3.14 = \pi$$

Cyan and purple:

$$\frac{411 + 201}{195} = \frac{612}{195} = 3.1385 = 3.14 = \pi$$

Another line is P2 plus the space to P3, which is 411 + 150 = 561. This is half a cubit (26 cm on a length of 293 m) out for the square root of 100,000π, as $\sqrt{314160} = 560.5$.

The area of P1 – P2 = $440^2 - 411^2 = 24679 \approx (50\pi)^2$

6.10 π/2 Line ratios

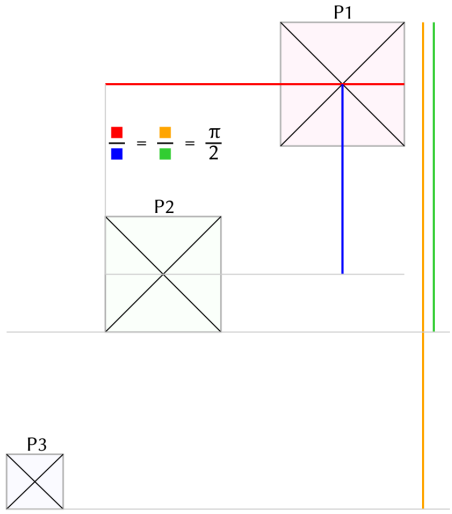

Figure 14: Half π

Red and blue:

$$\frac{411 + 212 + 440}{220 + 251 + 205.5} = \frac{1063}{676.5} = 1.5713 = 1.57 = \frac{\pi}{2}$$

Orange and green:

$$\frac{1732}{440 + 251 + 411} = \frac{1732}{1102} = 1.5717 = 1.57 = \frac{\pi}{2}$$

6.11 $\pi/3$ Line ratios

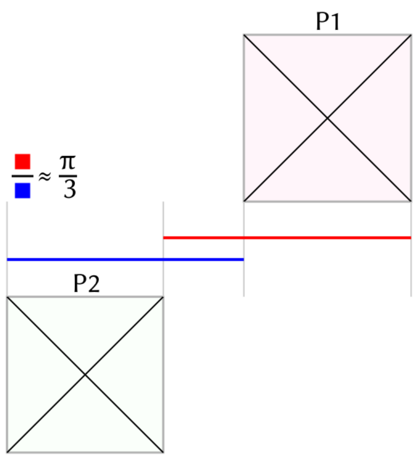

Figure 15: $\pi/3$

Red and blue:

$$\frac{212+440}{411+212} = \frac{652}{623} = 1.047 = \frac{\pi}{3}$$

6.12 $\pi/3$ Areas

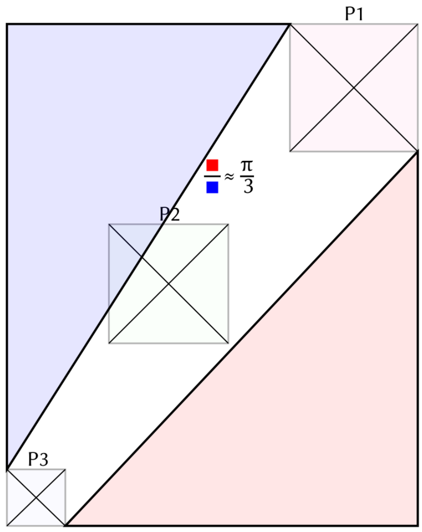

Figure 16: $\pi/3$ areas

Red

$$\frac{(1414-201)\times(1732-440)}{2} = \frac{1567196}{2} = 783598$$

Blue

$$\frac{(1414-440)\times(1732-195)}{2} = \frac{1497038}{2} = 748519$$

Ratio

$$\frac{\text{Red}}{\text{Blue}} = \frac{783598}{748519} = 1.0469 = \frac{\pi}{3}$$

6.13 π Angles

These are reasonably accurate.

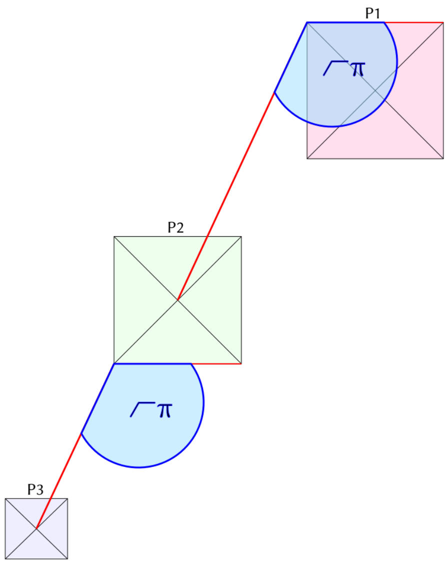

Figure 17: π Angles

$\frac{360}{\pi} = 114.59°$.

From, to	Angle	Error	Accuracy %
P1 to P2	114.97°	0.38°	99.67
P2 to P3	115.19°	0.60°	99.47

Table 9: Analysis of φ angles.

6.14 $\pi - 1$ Angle

$\frac{360}{\pi - 1} = 168.1°$.

$\pi - 1$, like e -1, is one of the numbers that we find at Giza, but nearly nowhere else.

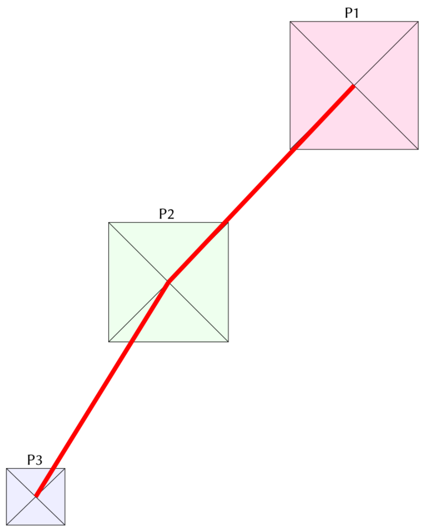

Figure 18: $\pi - 1$ Angle

The angle is 168.411°, out by 0.312°, accuracy is 99.814% .

The angle closely matches the angle in the Bent Pyramid (168.5°)[17].

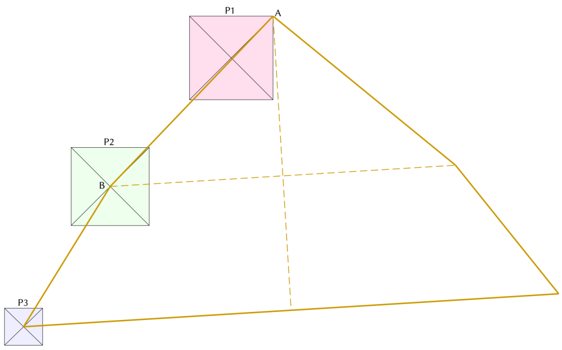

Figure 19: Bent Pyramid in Giza site plan

7 It all works together: φ

7.1 φ Line

The diagonal from P3 SW to P1 SW is 1618 ₲ or 1000φ, which again strongly suggests that the designers knew exactly what they were doing, and that Legon's rectangle is the correct design paradigm.

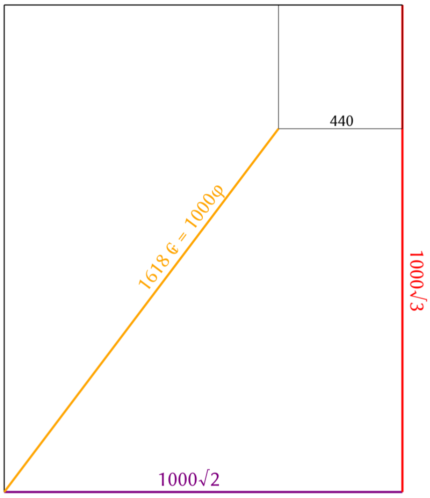

Figure 20: φ.

Proof: $\sqrt{(1414-440)^2+(1732-440)^2}=1618$ or 1000φ, rounded to the nearest cubit.

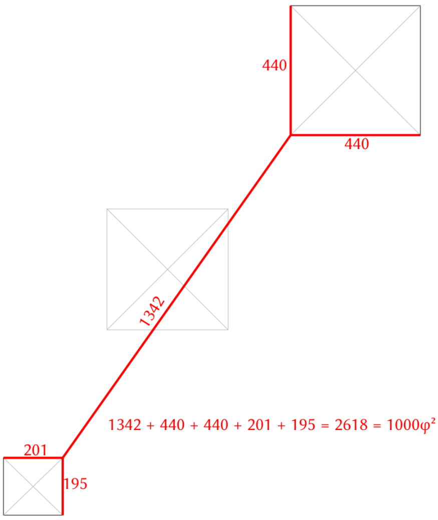

Figure 21: φ^2.

Proof: $\sqrt{(1414-440-201)^2+(1732-440-195)^2}=1342$, rounded to the nearest cubit. 1342 is an anagram of π.

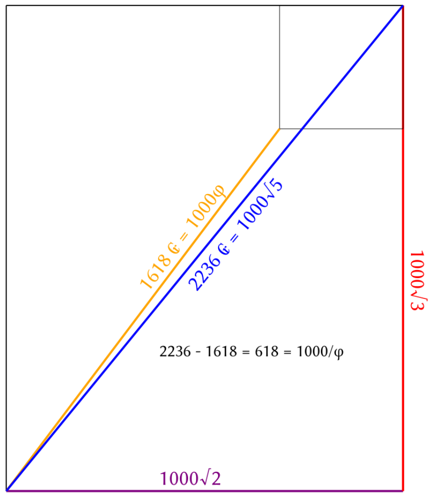

Figure 22: φ^{-1}

The ratio of the φ line to the Legon diagonal is 1618/2236, which is 0.7236 to four places. Squaring that gives 0.5236 to four places, the cubit:metre ratio.

7.2 φ in the bases

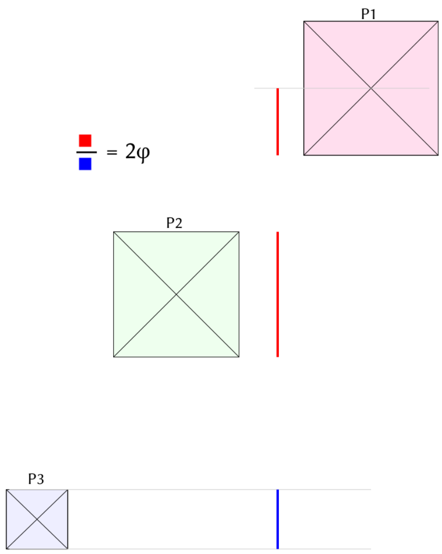

Figure 23: Phi in the bases.

$$\frac{220 + 411}{195} = \frac{631}{195} = 3.236 = 2\varphi$$

7.3 φ ratio

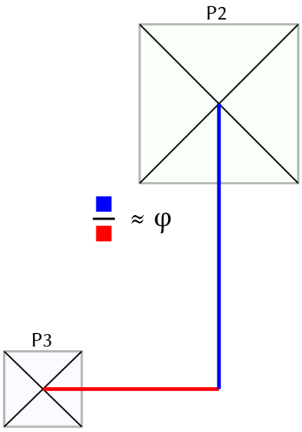

Figure 24: φ ratio.

$$\frac{205.5 + 435 + 97.5}{100.5 + 150 + 205.5} = \frac{738}{456} = 1.618 = \varphi$$

7.4 φ Angles

Six φ angles.

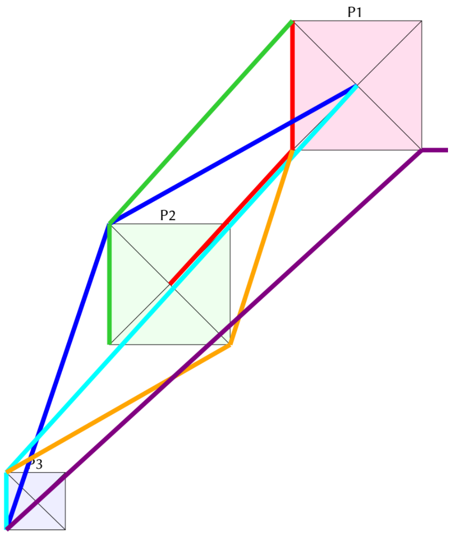

Figure 25: φ angles.

The φ angle is 222.49°. All errors below are less that 0.5°.

Colour	Angle	Error	Accuracy %
Red	222.445°	0.0472°	99.979
Blue	222.174°	0.318°	99.857
Green	222.0376°	0.4547°	99.796
Cyan	222.1956°	0.2966°	99.867
Orange	222.522°	0.0299°	99.987
Purple	222.419°	0.0737°	99.967

Table 10: Analysis of φ angles.

8 It all works together: e

The approximations for e are not all as "clean" or "simple" as for φ or $\sqrt{\pi}$, but they are there. 1000e is 2718.3, which is a little awkward... we can round down to 2718, or get better accuracy by rounding to the nearest half, 2718.5. The designers did both.

8.1 e in cubits

Firstly, in cubits, we have the shepherd's crook in Figure 26:

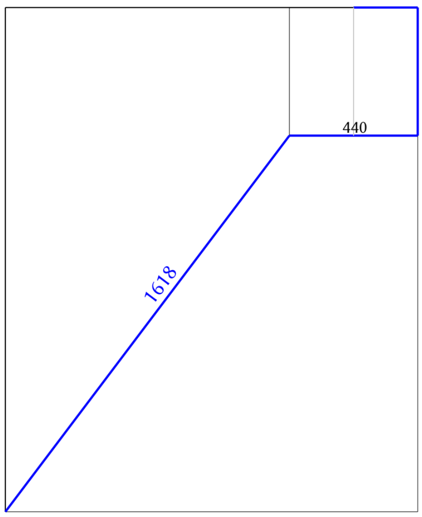

Figure 26: e in cubits

Proof: 1618 + 440 + 440 + 220 = 2718 = 1000e.

8.2 e in metres

For better accuracy, we need to use 440 and 280. This is Khufu's base and height, or we can use the diagonal on Menkuare, as that is also 280:

$\sqrt{201^2 + 195^2} = 280$

Then we add the two Legon diagonals to the 440 and 280.

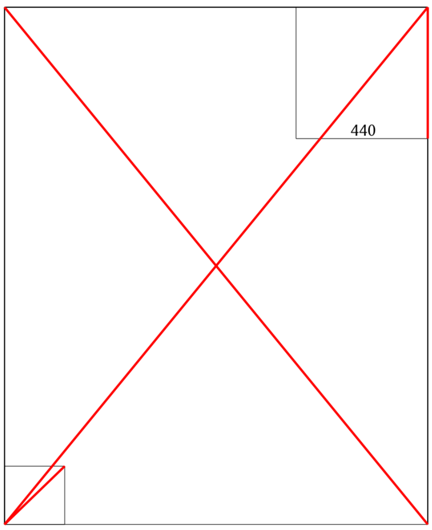

Figure 27: e in metres

Proof: 2236 + 2236 + 440 + 280 = 5192. Convert to metres: 5192 × 0.5236 = 2718.5 = 1000e.

8.3 e Line ratio

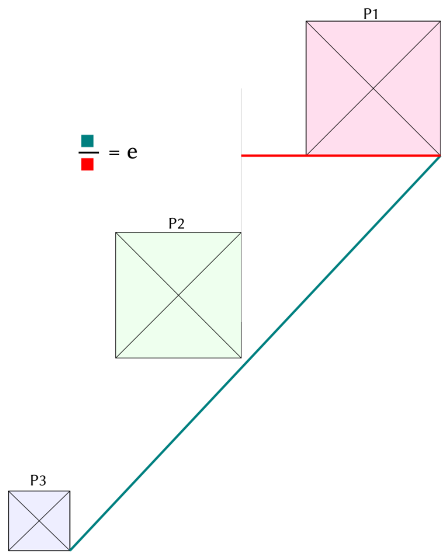

Figure 28: e ratio

$$\frac{1772}{440+212} = \frac{1772}{652} = 2.718 = \mathrm{e}$$

8.4 e Areas

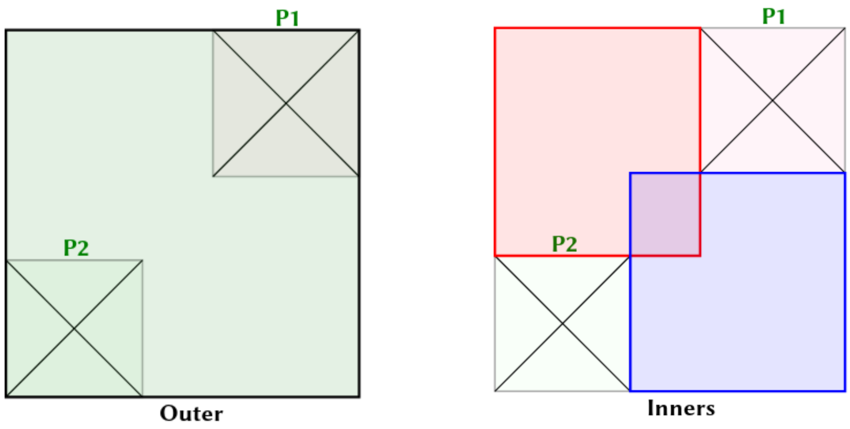

Table 11: e areas.

$$\frac{2\times\text{Outer}}{\text{sum Inners}} = \frac{2\times(411+212+440)\times(440+251+411)}{(411+212)\times(440+251)+(212+440)\times(251+411)} = \frac{2342852}{862117} = 2.71756 = 2.718 = \mathrm{e}$$

An alternative.

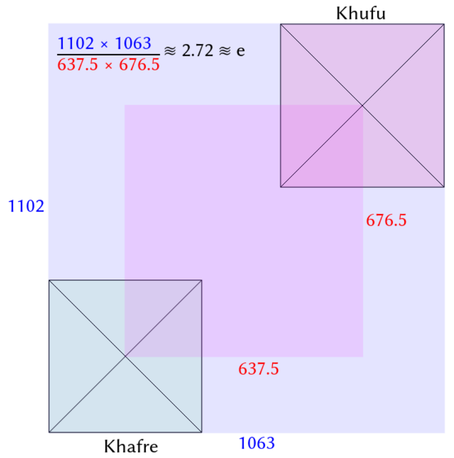

Figure 29: e area ratio

$$\frac{1106 \times 1063}{637.5 \times 676.5} = 2.716232 = 99.9246\% \text{ of e}$$

9 It all works together: ρ

9.1 ρ^3 Line ratio

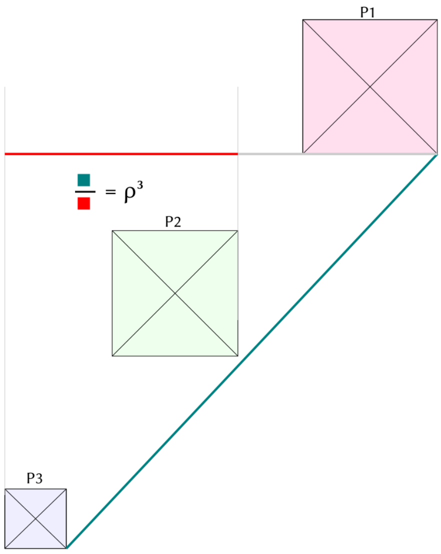

Figure 30: ρ^3 ratio

$$\frac{1772}{201 + 150 + 411} = \frac{1772}{762} = 2.325 = \rho^3$$

10 It all works together: Square roots

10.1 $\sqrt{2}$

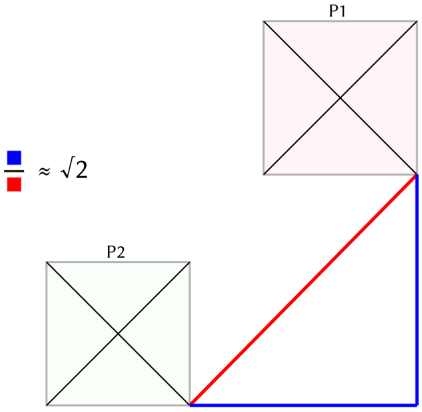

Figure 31: $\sqrt{2}$

Red: $\sqrt{(251+411)^2+(212+440)^2)} = 929.1652167$
Blue: $251+411+212+440 = 1314$
$\frac{\text{Blue}}{\text{Red}} = \frac{1314}{929.1652167} = 1.41417 = 1.4142 = \sqrt{2}$

11 It all works together: Combinations

11.1 ρ and π

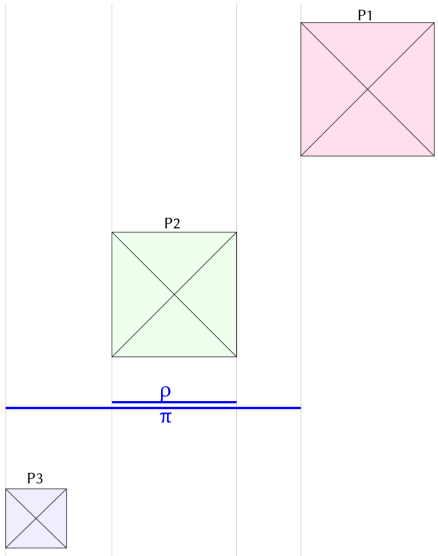

Figure 32: ρ and π

ρ line: 411
π line: $201+150+411+212 = 974$

Ratio of ρ line to π line: $\frac{411}{974} = 0.42197 = 0.422$

Ratio of ρ to π: $\frac{\rho}{\pi} = \frac{1.3247}{3.1416} = 0.42166 = 0.422$

11.2 $\pi\varphi$

This suggests that the site was designed before the location was chosen. It assumes that P1 and P2 are on the same base level. We have a vertical right-angled triangle, from the centre base of P1 to the centre base of P2, with the vertical part being P1's height.

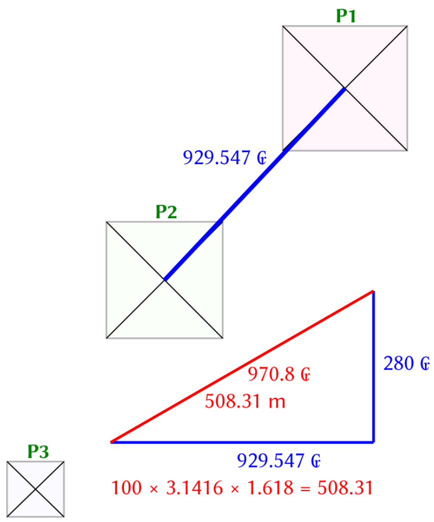

Figure 33: $\pi\varphi$

The distance from P1 centre to P2 centre is

$$\sqrt{(220 + 251 + 205.5)^2 + (220 + 212 + 205.5)^2} = \sqrt{864058.5}$$

Then the hypotenuse from P1 summit to P2 base centre is

$$\sqrt{864058.5 + 280^2} = 970.8030181$$

Convert ₢ to metres: $970.8030181 \times 0.5236 = 508.31246 = 508.31$

This is $100\pi\varphi$: $100 \times 3.1416 \times 1.618 = 508.31$.

12 Discussion

Adopting Legon's proposed layout, and making the necessary adjustments to Menkaure's footprint, leads to a coherent design. The design produces a multitude of π and other favoured irrationals approximations, many of which fail if you increase or decrease the numerator and/or the denominator by 1. The distances need to be quite precise, and at the same time work in multiple different relationships. The designers were clearly highly skilled.

The presence of π, φ and e, for example, clearly calls the whole "4th Dynasty" narrative into question. Either Giza is not 4th Dynasty, or we need to revise our history of mathematics, and then explain how this knowledge was lost by the time of the middle kingdom.

Khufu's base size of 440 cubits is key to making many of the relationships work, including 1000φ and thus the neat 2236 + 2236 + 1414 + 1732 - 1618 = 6000 ₢ = 1000π metres calculation.

List of Figures

List of Tables

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