The Distribution Of Prime Numbers
And The Continued Fractions

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Abstract. In this paper, we present a new sequence containing only ones and the prime numbers, which can be calculated in two different ways, the first way using the greatest common divisor (gcd) and Kurepa left factorial function, the second way consisting of using the denominator of the continued fraction defined by

$$\frac{mb(n-3)-nb(n-4)}{n(m-n+2)-m} = \cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{4-\cfrac{5}{\ddots\, (n-1)-\cfrac{n}{m}}}}}$$

Our sequence defined by

$$a_m(n) = \frac{|n(m-n+2)-m|}{gcd\big(n(m-n+2)-m, mb(n-3)-nb(n-4)\big)}$$

Where $|x|$ denotes the absolute value of x.

Keywords. Prime numbers, continued fraction, left factorial, sequence.

1. Introduction and preliminaries

A continued fraction is an expression of the form

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{\ddots}}}$$

Other notation

$$a_0 + \frac{b_0}{a_1 +}\,\frac{b_1}{a_2 +}\,\frac{b_2}{a_3 +}\cdots$$

Where a_i and b_i are either rational numbers, real numbers or complex numbers. For more details see **[3].**

In 1971, Kurepa introduced the left factorial function, with the symbol $!n$. For more details and formulas see [4], the Kurepa function is defined by

$$K(0) = 0\,, \qquad K(n) = !\,n = \sum_{i=1}^{n-1} i!\,, \qquad n \in \mathbb{N}$$

In this paper, We establish a connection between the left factorial function of Kurepa $K(n)$ and the continued fraction in the theorem (1.3). We define the recursive formula

$$b(n) = (n+2)\big(b(n-1) - b(n-2)\big)$$

Such that

$$K(n) = !\,n = \frac{2b(n-1)}{n+1}$$

With the initial conditions $b(-1) = 0$ and $b(0) = 1$. A few values of $b(n)$

0, 1, 3, 8, 25, 102, 539,3496, 26613, 231170, 2250127, 24227484, …(see A051403)

Similarly, we define the second recursive formula

$$c(n) = (n+2)\big(c(n-1) - c(n-2)\big)$$

With the initial conditions $c(1) = 1$ and $c(2) = 4$. A few values of $c(n)$

1, 4, 15, 66, 357, 2328, 17739, 154110, 1500081, 16151652, 190470423,…

The objective of this paper is to construct a new sequence for the distribution of prime numbers which takes only ones and primes in order. The distribution of prime numbers has been analyzed for a formula helpful in generating the prime numbers or testing if the given numbers is prime. In this paper, we present some known formulas.

Mills showed that there exists a real number $A > 1$ such that $f(n) = \left[\, A^{3^n} \,\right]$ is a prime number for any integers n, approximately A=1.306377883863,.. (see A051021). The first few values

$f(n) =$ {2, 11, 1361, 2521008887, 1602223620400981813183132018 3,..}, (see A051254)

Euler's quadratic polynomial $n^2 + n + 41$ is prime for all n between 0 and 39, however, it is not prime for all integers.

In 2008, Rowland introduce an explicit sequence that contain only ones and primes, the sequence defined by the recurrence relation

$$r(n) = r(n-1) + \gcd(n\,, r(n-1)); \;\; r(1) = 7$$

The sequence of differences $r(n+1) - r(n)$

1, 1, 1, 5, 3, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 23, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,1, 1, 1, 47, 3, 1, 5, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 101, 3, 1, 1, 7, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 13, 1, 1, 1, 1, 1, 1, 1, 1, 1,.. (see A132199).

For more details and formulas see [1] and [2]. In this paper, we present an interesting sequence which plays the same role as Rowland's sequence composed of a prime number or 1. Moreover, our sequence gives all distinct prime numbers in order.

In this section, we give an explicit formula for the continued fraction in the following theorem

Theorem 1.1. For all integers $n \geq 3$. The continued fraction

$$\frac{mb(n-3)-nb(n-4)}{n(m-n+2)-m} = \cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{4-\cfrac{5}{\ddots \atop (n-1)-\cfrac{n}{m}}}}} \tag{1}$$

Where m is a polynomial in term n.

Proof. Let

$$a_1 = 2a_2 - 3a_3\ ;\ \ a_2 = 3a_3 - 4a_4\ ;\ a_3 = 4a_4 - 5a_5\ ;\ \ a_4 = 5a_5 - 6a_6$$

Then we have

$$\frac{a_2}{a_1} = \frac{a_2}{2a_2 - 3a_3} = \cfrac{1}{\cfrac{2a_2 - 3a_3}{a_2}} = \cfrac{1}{2 - \cfrac{3a_3}{a_2}} = \cfrac{1}{2 - \cfrac{3}{\cfrac{3a_3 - 4a_4}{a_3}}}$$

$$= \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4a_4}{a_3}}} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{\cfrac{4a_4 - 5a_5}{a_4}}}} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5a_5}{a_4}}}}$$

After some simplification, we find

$$\frac{a_2}{a_1} = \cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{4-\cfrac{5}{\ddots \atop (n-1)-\cfrac{na_n}{a_{n-1}}}}}} \tag{2}$$

From (1) and (2), we have

$$ma_n = a_{n-1} \tag{3}$$

We write a_1 in terms of a_{n-1} and a_n

$$a_1 = 2a_2 - 3a_3 = \cdots = (n-1)a_{n-1} - (n^2-2)a_n \tag{4}$$

Substituting (3) in (4), we find

$$a_1 = (n(m-n+2)-m)a_n$$

Using the same procedure for a_2, we have

$$a_2 = 3a_3 - 4a_4 = 8a_4 - 15a_5 = 25a_5 - 48a_6 = \cdots$$

We observe that

$$a_2 = b(n-3)a_{n-1} - nb(n-4)a_n \tag{5}$$

Substituting (3) in (5), we get

$$a_2 = \big(mb(n-3) - nb(n-4)\big)a_n$$

Returning to (2), we obtain

$$\frac{a_2}{a_1} = \frac{mb(n-3) - nb(n-4)}{n(m-n+2) - m} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5}{\ddots \atop (n-1) - \cfrac{n}{m}}}}} \qquad (6)$$

This complete the proof.

Theorem 1.2. For all integers $n \geq 3$. The denominator of the continued fraction is as follows

$$n(m-n+2) - m = 2\big(mb(n-3) - nb(n-4)\big) - 3\big(mc(n-3) - nc(n-4)\big)$$

Where m is a polynomial in term n.

Proof. Similarly, using the same procedure as that of proving the theorem 1.1

We have

$$a_3 = 4a_4 - 5a_5 = 15a_5 - 24a_6 = 66a_6 - 105a_7 = \cdots$$

We observe that

$$a_3 = c(n-3).a_{n-1} - nc(n-4).a_n \qquad (7)$$

Substituting (3) in (7), we find

$$a_3 = \big(mc(n-3) - nc(n-4)\big)a_n$$

Then, we have

$$a_1 = 2a_2 - 3a_3$$

$$(n(m-n+2) - m)a_n = \big[2\big(mb(n-3) - nb(n-4)\big) - 3\big(mc(n-3) - nc(n-4)\big)\big].a_n$$

Then, we get

$$n(m-n+2) - m = 2\big(mb(n-3) - nb(n-4)\big) - 3\big(mc(n-3) - nc(n-4)\big)$$

This completes the proof.

Theorem 1.3. For all integers $n \geq 3$. The continued fraction

$$\frac{2.\big(mb(n-3) - nb(n-4)\big)}{n(m-n+1)} = \cfrac{1}{1 - \cfrac{1}{2 - \cfrac{2}{3 - \cfrac{3}{\ddots \atop (n-1) - \cfrac{n-1}{m}}}}} \qquad (8)$$

Where m is a polynomial in term n.

Proof. Similarly, Using the same procedure of proof the theorem (1.1)

Putting

$a_1 = a_2 - a_3$; $a_2 = 2a_3 - 2a_4$; $a_3 = 3a_4 - 3a_5$; $a_4 = 4a_5 - 4a_6$;…

And we obtain the desired result.

Remark 1

For $m = n$, the Kurepa left factorial function is as continued fraction

$$K(n) = !n = \cfrac{1}{1 - \cfrac{1}{2 - \cfrac{2}{3 - \cfrac{3}{\ddots \atop (n-1) - \cfrac{n-1}{n}}}}}$$

The more interesting sequence follows

2. The sequence $a_m(n)$

The sequence of the unreduced denominator of the continued fraction (theorem 1.1) is as follows

$$a_m(n) = \frac{|n(m-n+2) - m|}{gcd\big(n(m-n+2) - m, mb(n-3) - nb(n-4)\big)}$$

Where gcd(x, y) denotes the greatest common divisor of x and y.

3. Main results

In this section we present some new results for our sequence in the following conjectures

Conjecture 3.1. For all integers $n \geq 2$ and $m = n + 1$. The sequence of the unreduced denominator is as follows

$$a(n) = \frac{2n-1}{gcd\big(2n-1, b(n-2) + b(n-3)\big)}$$

The values of $a(n)$

3, 5, 7, 3, 11, 13, 1, 17, 19, 1, 23, 1, 1, 29, 31, 1, 1, 37, 1, 41, 43, 1, 47, 1, 1, 53, 1, 1, 59, 61, 1, 1, 67, 1, 71, 73, 1, 1, 79, 1, 83, 1, 1, 89, 1, 1, 1, 97, 1, 101, 103, 1, 107, 109, 1, 113, 1, 1, 1, 1, 1, 1, 127, 1, 131, 1, 1, 137, 139, 1, 1, 1, 1, 149, 151, 1, 1, 157, 1, 1, 163, 1, 167,…

The sequence $a(n)$ takes only 1's and primes in order (except for the prime $a(5) = 3$). We have verified this conjecture to $n = 10000$ by using the denominator of the continued fraction in the theorem (1.1) for $m = n + 1$. The checking was facilitated by the following observation: except for n=5, $a(n) = 2n - 1$ if $2n - 1$ is prime, 1 otherwise.

Conjecture 3.2. For all integers $n \geq 2$ and $m = n - 3$. The sequence of the unreduced denominator is as follows

$$a(n) = \frac{2n-3}{gcd\big(2n-3, 3b(n-3) - b(n-2)\big)}$$

The values of $a(n)$

1,   1, 5, 7, 1, 11, 13, 1, 17, 19, 1, 23, 1, 1, 29, 31, 1, 1, 37, 1, 41, 43, 1, 47, 1, 1, 53, 1, 1, 59, 61, 1, 1, 67, 1, 71, 73, 1, 1, 79, 1, 83, 1, 1, 89, 1, 1, 1, 97, 1, 101, 103, 1, 107, 109, 1, 113, 1, 1, 1, 1, 1, 1, 127, 1, 131, 1, 1, 137, 139, 1, 1, 1, 1, 149, 151, 1, 1, 157, 1, 1, 163, 1, 167,…

We conjecture that :

∗ Every term of this sequence is either a prime number or 1.

∗ Except for the primes 2 and 3, the primes all appear in this sequence in order.

On the other hand the conjecture verified for $n \leq 10000$. The checking was facilitated by the following observation: for $n \geq 4,\ a(n) = 2n - 3$ if $2n - 3$ is prime, 1 otherwise .

Conjecture 3.3. For all integers $n \geq 3$ and $m = -1$. The sequence of the unreduced denominator is as follows

$$a(n) = \frac{n^2 - n - 1}{gcd(n^2 - n - 1,\ b(n-3) + nb(n-4))}$$

The values of $a(n)\ \ for\ n \geq 2$

1, 5, 11, 19, 29, 41, 11, 71, 89, 109, 131, 31, 181, 19, 239, 271, 61, 31, 379, 419, 461, 101, 29, 599, 59, 701, 151, 811, 79, 929, 991, 211, 59, 41, 1259, 1, 281, 1481, 1559, 149, 1721, 1, 61, 1979, 2069, 2161, 1, 2351, 79, 2549, 241, 1, 2861, 2969, 3079, 3191,…(see A356247)

We conjecture that :

∗ Every term of this sequence is either a prime number or 1.

∗ Except for 5, the primes terms all appear exactly twice, such that

$$a(n) = a(a(n) - n + 1)$$

Consequently, let us consider the values of n and m such that we get:

$$a(n) = a(m) = n + m - 1$$

And

$$a(n) = a(m) = \gcd(n^2 - n - 1, m^2 - m - 1)$$

We have verified this conjecture to $n = 10001$ (Bill Mceachen and Michael De Vlieger verification) by using the denominator of the continued fraction in the theorem (1.1) for $m = -1$.

Conjecture 3.4. For all integers $n \geq 3$ and $m = -2$. The expression of the sequence $a(n)$ is as follows

$$a(n) = \frac{n^2 - 2}{gcd\left(n^2 - 2,\ 2b(n-3) + nb(n-4)\right)}$$

The values of $a(n)$.

7, 7, 23, 17, 47, 31, 79, 7, 17, 71, 167, 97, 223, 127, 41, 23, 359, 199, 439, 241, 31, 41, 89, 337, 727, 1, 839, 449, 137, 73, 1087, 577, 1223, 647, 1367, 103, 1, 47, 73, 881, 1, 967, 1, 151, 2207, 1151, 2399, 1249, 113, 193, 401, 1, 3023, 1567, 191, 41, 71…

The sequence $a(n)$ takes only 1's and primes.

Conjecture 3.5. For all integers $n \geq 3$ and $m = n + 2$. The expression of the sequence $a(n)$ is as follows

$$a(n) = \frac{3n - 2}{gcd\big(3n - 2,\ (n + 1)b(n - 3) - b(n - 4) - (n - 1)b(n - 5)\big)}$$

The values of $a(n)$

7 , 5, 13, 2, 19, 11, 5, 1, 31, 17, 37, 1, 43, 23, 1, 1, 1, 29, 61, 1, 67, 1, 73, 1, 79, 41, 1, 1, 1, 47, 97, 1, 103, 53, 109, 1, 1, 59, 1, 1, 127, 1, 1, 1, 139, 71, 1, 1, 151, 1, 157, 1, 163, 83, 1, 1, 1, 89, 181, 1, 1, 1, 193, 1, 199, 101, 1, 1, 211,…

The sequence $a(n)$ contain only ones and the primes.

Conjecture 3.6. For all integers $n \geq 3$ and $m = n + 3$. The expression of the sequence $a(n)$ is as follows

$$a(n) = \frac{4n - 3}{gcd\big(4n - 3,\ (n + 2)b(n - 3) - b(n - 4) - (n - 1)b(n - 5)\big)}$$

The values of $a(n)$

3, 13, 17, 7, 5, 29, 11, 37, 41, 1, 7, 53, 19, 61, 1, 23, 73, 1, 1, 1, 89, 31, 97, 101, 1, 109, 113, 1, 1, 1, 43, 1, 137, 47, 1, 149, 1, 157, 1, 1, 1, 173, 59, 181, 1, 1, 193, 197, 67, 1, 1, 71, 1, 1, 1, 229, 233, 79, 241, 1, 83, 1, 257, 1, 1, 269, 1, 277,…

The sequence $a(n)$ takes only 1’s and primes.

Remark 2

Obviously many more such conjectures can be formulated.

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