Prescribed Performance Regulation in the presence of Unknown Time-Varying Delays ⋆

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Abstract: We design a state-feedback controller to impose prescribed performance attributes on the output regulation error for uncertain nonlinear systems, in the presence of unknown time-varying delays appearing both to the state and control input signals, provided that an upper bound on those delays is known. The proposed controller achieves pre-specified minimum convergence rate and maximum steady-state error, and keeps bounded all signals in the closed-loop. We proved that the error is confined strictly within a delayed version of the constructed performance envelope, that depends on the difference between the actual state delay and its corresponding upper bound. Nevertheless, the maximum value of the output regulation error at steady-state remains unaltered, exactly as pre-specified by the constructed performance functions. Furthermore, the controller does not incorporate knowledge regarding the nonlinearities of the controlled system, and is of low-complexity in the sense that no hard calculations (analytic or numerical) are required to produce the control signal. Simulation results validate the theoretical findings.

Keywords: prescribed performance control, delays, uncertain nonlinear systems, regulation.

1. INTRODUCTION

In networked control systems (NCSs) the plant and the controller operate remotely and share information over digital communication networks. The operation of the underlying network besides many benefits (e.g., low cost, easy installation and maintenance, high reliability), inevitably introduces to the control architecture a number of significant technological constraints that formulate challenging control problems. Network induced delays, which occur in both the sensor and the control input channels, is one of the most recognized issues, mainly owing to the severe consequences on the closed-loop system operation, ranging from performance degradation to instability.

Control input delays have been thoroughly examined in (Fischer et al. (2013); Li et al. (2014); Bekiaris-Liberis and Krstic (2017); Obuz et al. (2017); Zuo et al. (2017); Mazenc and Malisoff (2017); Bresch-Pietri et al. (2018); Ran et al. (2020)), and state measurement delays in (Chakrabarty et al. (2018); Sanz et al. (2019)). However, the concurrent appearance of state and input delays is typical in NCSs. In this direction, significant progress has been reported in (Karafyllis and Krstic (2012); Selivanov and Fridman (2016); Zhou et al. (2017); Battilotti (2019); Weston and Malisoff (2019)). A further challenging aspect related to delays is the consideration of uncertainty in the delay values. Unknown control input delays are considered in (Li et al. (2014); Obuz et al. (2017)), and unknown state delays in (Selivanov and Fridman (2016)). All above works consider, however, fully known system dynamics, an assumption rarely met in practice. Incorporating uncertain dynamics in the analysis is considered in (Fischer et al. (2013); Obuz et al. (2017); Ran et al. (2020)), restricted, however, exclusively to control input delays.

Despite the progress in the field, the majority of existing results address the stabilization problem, while performance issues are typically overlooked. In this direction, model predictive control was employed in (Li and Shi (2013); Sun et al. (2016)). However, performance is introduced via minimizing certain indices, which are not directly connected with the actual system response in terms of maximum overshoot, minimum convergence rate, and maximum steady-state error, making the selection of the aforementioned metrics in advance, infeasible.

In a delay-free setup, two methodologies exist to *a priori* guarantee those performance-related metrics; funnel control (FC) approach (Ilchmann et al. (2002); Hackl et al. (2013); Berger et al. (2018)), and prescribed performance control (PPC) methodology (Bechlioulis and Rovithakis (2008, 2014); Bechlioulis and Rovithakis (2017); Bikas and Rovithakis (2019)). Steps towards introducing delays while guaranteeing pre-specified performance are reported

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in (Liberzon and Trenn (2013); Bikas and Rovithakis (2019)), where FC and PPC approaches were followed, respectively, to address constant and known delays. Recently in (Bikas and Rovithakis (2021)), the PPC framework was extended for time-varying delays with the additional possibility of having uncertainty exclusively on the control input delay.

In this paper, we utilize the PPC approach to address unknown time-varying delays in both state and control input signals provided that an upper bound on those delays is known, while guaranteeing pre-specified performance of the output regulation error in terms of minimum convergence rate and maximum steady-state-error [1]. We prove that the output regulation error is confined strictly within a delayed version of the constructed performance envelope, that depends on the difference between the actual state delay and its corresponding upper bound. Nevertheless, the controller guarantees that the maximum error at steady-state is achieved exactly as pre-specified by the constructed performance functions. On the other hand, the requested performance is established irrespectively of the uncertainty on the control input delay. Further, we guarantee that all signals in the closed-loop remain bounded. Finally, the control scheme we propose inherits the *low-complexity* property of the PPC controllers, as it does not incorporate any prior knowledge regarding the controlled system nonlinearities, and does not utilize approximation/adaptive techniques to acquire such knowledge; resulting in a control scheme which does not involve hard calculations (analytic or numerical) to produce the control signal.

The rest of the paper is organized as follows. Section 2 states the problem. The proposed control scheme is presented in Section 3, and the main results are proved in Section 4. In Section 5 simulation studies verify the theoretical findings, and in Section 6 we conclude the paper.

2. PROBLEM FORMULATION

Consider nonlinear systems of the following form

$$\dot{x}_i = x_{i+1}, \tag{1a}$$
$$\dot{x}_n = f(\bar{x},t) + g(\bar{x},t)u(t-\tau_u(t)), \tag{1b}$$

where $\bar{x} = [x_1 \dots x_n]^T \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, $x_1 \in \mathbb{R}$ is the output, $\tau_u(t)$ is a time-varying control input delay satisfying $0 \le \tau_u(t) \le \bar{\tau}$ for all $t \ge -2\bar{\tau}$ for some known $\bar{\tau} > 0$, and $f(\bar{x},t)$, $g(\bar{x},t) : \mathbb{R}^n \times \mathbb{R}_{\ge -2\bar{\tau}} \to \mathbb{R}$ are nonlinear functions locally Lipschitz in $\bar{x}$ and piecewise-continuous in t with unknown analytical expressions. Further, assume that the state measurements are also subject to time-varying delay denoted by $\tau_s(t)$, satisfying $0 \le \tau_s(t) \le \bar{\tau}$ for all $t \ge -2\bar{\tau}$.

In this work we consider the problem of regulating the output $x_1(t)$ to a constant desired value denoted by $x_d \in \mathbb{R}$, and define the output regulation error as

$$e(t) = x_1(t) - x_d. \tag{2}$$

Assumption 1. The function $g(\bar{x},t)$ is either strictly positive or strictly negative for all $\bar{x}(t) \in \mathbb{R}^n$ and $t \ge -2\bar{\tau}$, and its sign denoted by $\mathrm{sgn}(g)$ is considered known.

[1] In this work we disregarded possible restrictions on the maximum overshoot to simplify notation.

Assumption 2. The delays $\tau_s(t)$, $\tau_u(t)$ are unknown though continuously differentiable functions of time, satisfying $\dot{\tau}_s(t), \dot{\tau}_u(t) \in \mathcal{L}_\infty$ with $\dot{\tau}_u(t) < 1$ and $\dot{\tau}_s(t) \le \bar{\dot{\tau}}_s$ for some unknown constant $\bar{\dot{\tau}}_s < 1$.

Assumption 3. The initial solution of (1) on $[-2\bar{\tau}, 0]$ denoted by $\varphi = [\varphi_1 \dots \varphi_n]^T \triangleq \bar{x}(t)$ exists, is bounded and considered available for control design. Moreover, we assume that $u(t) = 0$, for all $t \le t_0$, with $t_0 \in [-\bar{\tau}, 0]$ denoting the time instant when the controller is initiated.

Remark 1. Assumption 1 is standard and imposes a strong controllability condition on (1) (Bechlioulis and Rovithakis (2011)). In Assumption 2, by the boundedness requirement imposed on $\dot{\tau}_s(t), \dot{\tau}_u(t)$, the first-in/first-out principle is guaranteed (Mazenc and Malisoff (2017); Bresch-Pietri et al. (2018)). Assumption 3 is common in literature (Karafyllis and Krstic (2012); Zhou et al. (2017); Ran et al. (2020)) and is necessary to guarantee a well-defined closed-loop system for all $t \ge 0$.

Remark 2. Owing to the uncertainty in the state measurement delay, the analysis of this work is restricted to the regulation problem. Extension of this work to the problem of tracking a time-varying desired trajectory with prescribed performance is important and demanding, and deserves further investigation.

The problem addressed in this paper reads as follows.

Prescribed Performance Regulation in the presence of Unknown Delays (PPC-UD) Problem: Consider system (1), with state measurement and control input delays $\tau_s(t)$ and $\tau_u(t)$, respectively, a desired output value x_d, and Assumptions 1-3. Design a state-feedback controller to meet the following objectives:

- all signals in the closed-loop are bounded,
- system output x_1 is regulated to the desired value x_d, exhibiting a pre-specified maximum steady-state error and convergence rate no less than a predetermined value,
- the derived controller should be of *low-complexity* by satisfying the following constraints: i) it should not incorporate knowledge regarding the controlled system nonlinearities; ii) it should not employ approximation/adaptive structures to acquire such knowledge; iii) no hard calculations (analytic or numerical) should be required to produce the control signal.

3. CONTROLLER DESIGN

Select any strictly increasing function $\mathrm{T}(s) : (-1,1) \to \mathbb{R}$, satisfying $\lim_{s\to 1^-} \mathrm{T}(s) = +\infty$ and $\lim_{s\to -1^+} \mathrm{T}(s) = -\infty$. The following serves as a feasible candidate selection:

$$\mathrm{T}(s) = \ln\left(\frac{1+s}{1-s}\right). \tag{3}$$

Further, by Assumption 3, we straightforwardly derive constants $\bar{\varphi}_i > 0$, $i = 1,\dots,n$, such that $|\varphi_i(t)| \le \bar{\varphi}_i$, for all $t \in [-2\bar{\tau}, 0]$. Moreover, it holds that $t_0 - \tau_s(t_0) \ge -2\bar{\tau}$, and $t_0 - 2\bar{\tau} \ge -3\bar{\tau}$, for any $t_0 \in [-\bar{\tau}, 0]$. Therefore, as $\bar{x}(t) = \varphi(t)$ for all $t \in [-2\bar{\tau}, 0]$ and $u(t) = 0$ for all $t \le t_0$ by Assumption 3, we guarantee that $x_i(t-\tau_s(t))$, $i = 1,\dots,n$, and $\int_{t-2\bar{\tau}}^{t} u(\theta)d\theta$, are well-defined at $t = t_0$.

To solve the *PPC-UD Problem* we propose for all $t \ge t_0$:

Step 1: Select the exponentially decaying output performance functions

$$\rho_1(t) = \left(\rho_1^0 - \rho_1^\infty\right) e^{-\lambda_1 t} + \rho_1^\infty, \tag{4}$$

with parameters satisfying $\rho_1^\infty > 0$, $\lambda_1 \geq 0$ and

$$\rho_1^0 > \frac{\bar{\varphi}_1 + |x_d| + (e^{\lambda_1 \bar{\tau}} - 1)\rho_1^\infty}{e^{\lambda_1 \bar{\tau}}}. \tag{5}$$

Choose a control gain $k_1 > 0$, and design the first intermediate control signal as

$$a_1(x_1, t) = -k_1 \mathrm{T}\left(\frac{e(t - \tau_s(t))}{\rho_1(t - \bar{\tau})}\right). \tag{6}$$

Further, by Assumption 3 and (6), we guarantee that $|a_1(t)| \leq \bar{a}_1^{t_0} \triangleq k_1 \left|\mathrm{T}\left(\frac{\bar{\varphi}_1 + |x_d|}{\rho_1^\infty}\right)\right| > 0$, for all $t \in [t_0, 0]$.

Step i ($i = 2, \dots, n-1$): Select the performance function

$$\rho_i(t) = \left(\rho_i^0 - \rho_i^\infty\right) e^{-\lambda_i t} + \rho_i^\infty, \tag{7}$$

with parameters satisfying $\rho_i^\infty > 0$, $\lambda_i \geq 0$ and

$$\rho_i^0 > \frac{\bar{\varphi}_i + \bar{a}_{i-1}^{t_0} + (e^{\lambda_i \bar{\tau}} - 1)\rho_i^\infty}{e^{\lambda_i \bar{\tau}}}. \tag{8}$$

Choose a control gain $k_i > 0$, and design the intermediate control signals as

$$a_i(x_1, \dots, x_i, t) = -k_i \mathrm{T}\left(\frac{x_i(t - \tau_s(t)) - a_{i-1}(t)}{\rho_i(t - \bar{\tau})}\right). \tag{9}$$

Further, we deduce $|a_i(t)| \leq \bar{a}_i^{t_0} \triangleq k_i \left|\mathrm{T}\left(\frac{\bar{\varphi}_i + \bar{a}_{i-1}^{t_0}}{\rho_i^\infty}\right)\right| > 0$, for all $t \in [t_0, 0]$.

Step n: Select the performance functions

$$\rho_n(t) = \left(\rho_n^0 - \rho_n^\infty\right) e^{-\lambda_n t} + \rho_n^\infty, \tag{10}$$

with parameters satisfying $\rho_n^\infty > 0$, $\lambda_n \geq 0$ and

$$\rho_n^0 > \frac{\bar{\varphi}_n + \bar{a}_{n-1}^{t_0} + (e^{\lambda_n \bar{\tau}} - 1)\rho_n^\infty}{e^{\lambda_n \bar{\tau}}}. \tag{11}$$

Choose a control gain $k_n > 0$, and design the control input as

$$u(\bar{x}, t) = -\mathrm{sgn}(g) k_n \mathrm{T}\left(\zeta(t)\right), \tag{12}$$

where

$$\zeta(t) = \frac{x_n(t - \tau_s(t)) + \mathrm{sgn}(g)\int_{t-2\bar{\tau}}^{t} u(\theta)d\theta - a_{n-1}(t)}{\rho_n(t - \bar{\tau})}. \tag{13}$$

Remark 3. The selection of $\rho_1(t)$ in (4), directly introduces the required performance attributes on the output regulation error. Indeed, it is not difficult to verify that if

$$|e(t)| < \rho_1(t - (\bar{\tau} - \tau_s(t))) \leq \rho_1(t - \bar{\tau}),\ \forall t \geq 0, \tag{14}$$

then $x_1(t)$ is regulated at x_d, exhibiting a pre-specified maximum steady-state error given by ρ_1^∞, and a predetermined minimum convergence rate given by $e^{-\lambda_1(t-\bar{\tau})}$.

Remark 4. Continuing the discussion of Remark 3, $e(t)$ will be proved to evolve strictly within $(-\rho_1(t - (\bar{\tau} - \tau_s(t))), \rho_1(t - (\bar{\tau} - \tau_s(t))))$, representing a delayed version of the constructed performance envelope $(-\rho_1(t), \rho_1(t))$. Therefore, owing to (4) and (14), the minimum rate of convergence is obtained by $e^{-\lambda_1(t-(\bar{\tau}-\tau_s(t)))}$. Hence, for large values of $\bar{\tau}$ and for $\tau_s(t) = 0$, we conclude that $(-\rho_1(t - \bar{\tau}), \rho_1(t - \bar{\tau}))$ is the most delayed performance envelope that can be realized, which, owing to the fact that $\tau_s(t)$ is unknown, it represents the performance envelope that can be *a priori* specified, thus, leading to minimum convergence rate $e^{-\lambda_1(t-\bar{\tau})}$. On the other hand, for any $\bar{\tau}$, $\tau_s(t)$, we have $\lim_{t\to+\infty} \rho_1(t - (\bar{\tau} - \tau_s(t))) = \lim_{t\to+\infty} \rho_1(t) = \rho_1^\infty$, which implies that at steady-state $e(t)$ is guaranteed to evolve strictly within $(-\rho_1^\infty, \rho_1^\infty)$. Finally, we stress that the requested performance is achieved irrespectively of the uncertainty on the control input delay.

Remark 5. The proposed controller (4)-(13) does not incorporate prior knowledge regarding the controlled system nonlinearities. Further, approximation structures or adaptive techniques are avoided, and no hard calculations (analytic or numerical) are used to produce the control signal. Hence, the controller satisfies all design constraints requested by the *PPC-UD Problem*, and therefore, it constitutes a *low-complexity* control solution.

4. MAIN RESULTS

Let us introduce the change of coordinates

$$z_1 = \frac{e(t - \tau_s(t))}{\rho_1(t - \bar{\tau})}, \tag{15a}$$

$$z_i = \frac{x_i(t - \tau_s(t)) - a_{i-1}(t)}{\rho_i(t - \bar{\tau})},\ i = 2, \dots, n-1, \tag{15b}$$

$$z_n = \zeta(t), \tag{15c}$$

and define the transformed errors

$$\epsilon_i = \mathrm{T}(z_i),\ i = 1, \dots, n. \tag{16}$$

By employing (15), (16), the control signals (6), (9) and (12) become

$$a_i = -k_i \epsilon_i,\ i = 1, \dots, n-1, \tag{17a}$$

$$u = -\mathrm{sgn}(g) k_n \epsilon_n. \tag{17b}$$

The time derivative of (15) in view of (17) yields

$$\begin{aligned} \dot{z}_1 &\triangleq h_1(z_1, z_2, t) \\ &= \frac{1}{\rho_1(t - \bar{\tau})}\Big[(1 - \dot{\tau}_s) z_2 \rho_2(t - \bar{\tau}) \\ &\qquad - z_1 \dot{\rho}_1(t - \bar{\tau}) - (1 - \dot{\tau}_s) k_1 \epsilon_1\Big], \end{aligned} \tag{18a}$$

$$\begin{aligned} \dot{z}_i &\triangleq h_i(z_1, \dots, z_{i+1}, t) \\ &= \frac{1}{\rho_i(t - \bar{\tau})}\Big[(1 - \dot{\tau}_s) z_{i+1} \rho_{i+1}(t - \bar{\tau}) \\ &\qquad - z_i \dot{\rho}_i(t - \bar{\tau}) - \dot{a}_{i-1} - (1 - \dot{\tau}_s) k_i \epsilon_i\Big], \end{aligned} \tag{18b}$$

$$\begin{aligned} \dot{z}_n &\triangleq h_n(z_1, \dots, z_n, t) \\ &= \frac{1}{\rho_n(t - \bar{\tau})}\Big[(1 - \dot{\tau}_s) f(\bar{x}, t - \tau_s(t)) - z_n \dot{\rho}_n(t - \bar{\tau}) \\ &\qquad + (1 - \dot{\tau}_s) g(\bar{x}, t - \tau_s(t)) u(t - \tau_s(t) - \tau_u(t)) \\ &\qquad - \dot{a}_{n-1} - \mathrm{sgn}(g) u(t - 2\bar{\tau}) - k_n \epsilon_n\Big]. \end{aligned} \tag{18c}$$

Further, define $\bar{z} = [z_1 \dots z_n]^T \in \mathbb{R}^n$. The $\bar{z}$-coordinate closed-loop system can be written in compact form as

$$\dot{\bar{z}} = h(\bar{z}, t) = [h_1 \dots h_n]^T \in \mathbb{R}^n. \tag{19}$$

Finally, define the open set $\Omega_z = (-1, 1)^n \subset \mathbb{R}^n$.

The main results of this work are summarized in the following theorem.

Theorem 1. *Consider system* (1)*, with state measurement and control input delays* $\tau_s(t)$, $\tau_u(t) \geq 0$*, respectively, a* *desired output regulation value x_d, and Assumptions 1-3. The controller* (4)-(13) *guarantees the solution of the PPC-UD Problem.*

Proof. The set Ω_z is non-empty and open. Further, owing to Assumption 3, $x_i(t-\tau_s(t))$, $i=1,\dots,n$, are well-defined at $t=0$, and by recalling (5), (8) and (11), we obtain that $\bar{z}(0)\in\Omega_z$. Therefore, owing to the continuity of the right-hand sides of (18), the existence and uniqueness of solutions of (19) over a maximal time interval $[0,t_f)$ for some $t_f\in(0,+\infty]$, is established (Khalil, 2001, Theorem 3.1). Hence, $\bar{z}(t)\in\Omega_z$ for all $[0,t_f)$, and the signals (16) are well-defined for all $[0,t_f)$.

In what follows, a recursive step-like procedure is adopted to prove that all signals in the closed-loop remain bounded and that $\bar{z}(t)$ evolves strictly within a compact subset of Ω_z for all $t\in[0,t_f)$.

Step 1 ($t\in[0,t_f)$): Consider the positive definite and radially unbounded function $V_1=\frac{1}{2}\epsilon_1^2$. The time derivative of V_1 in view of (18a) yields

$$\dot{V}_1=\epsilon_1 r_1\Big[(1-\dot{\tau}_s)z_2\rho_2(t-\bar{\tau})-z_1\dot{\rho}_1(t-\bar{\tau})-(1-\dot{\tau}_s)k_1\epsilon_1\Big], \tag{20}$$

where $r_1=\frac{2}{(1-z_1^2)\rho_1(t-\bar{\tau})}$. At this point, notice that $1-\dot{\tau}_s\geq 1-\bar{\dot{\tau}}_s>0$ owing to Assumption 2, and $\rho_1(t-\bar{\tau})>0$, $\bar{z}(t)\in\Omega_z$, for all $[0,t_f)$. Thus, we deduce that $r_1>0$, for all $[0,t_f)$. Further, $\dot{\rho}_1(t-\bar{\tau}),\rho_2(t-\bar{\tau})$ are bounded by construction, z_1,z_2 are bounded as $\bar{z}\in\Omega_z$, and $1-\dot{\tau}_s$ is bounded owing to Assumption 2. Therefore, we derive the existence of a constant $\bar{F}_1>0$, satisfying

$$\Big|(1-\dot{\tau}_s)z_2\rho_2(t-\bar{\tau})-z_1\dot{\rho}_1(t-\bar{\tau})\Big|\leq\bar{F}_1. \tag{21}$$

Consequently, we arrive at

$$\dot{V}_1\leq|\epsilon_1|r_1\left(\bar{F}_1-(1-\bar{\dot{\tau}}_s)\,k_1|\epsilon_1|\right), \tag{22}$$

which is negative provided $|\epsilon_1|>\frac{\bar{F}_1}{(1-\bar{\dot{\tau}}_s)k_1}$. Therefore,

$$|\epsilon_1|\leq\bar{\epsilon}_1\triangleq\max\left\{|\epsilon_1(0)|,\frac{\bar{F}_1}{(1-\bar{\dot{\tau}}_s)\,k_1}\right\}. \tag{23}$$

Taking the inverse of the T-function we obtain

$$-1<\mathrm{T}^{-1}(-\bar{\epsilon}_1)\leq z_1\leq\mathrm{T}^{-1}(\bar{\epsilon}_1)<1. \tag{24}$$

In addition, owing to (17a) and (24), a_1 and r_1 are also bounded. Finally, notice that by the continuity of $h_1(z_1,z_2,t)$ in (18a) and the Extreme Value Theorem, there exist constant $\bar{h}_1>0$ such that $|h_1(z_1,z_2,t)|\leq\bar{h}_1$. Hence, employing (17a), we conclude $|\dot{a}_1|=\rho_1(t-\bar{\tau})r_1k_1|h_1(z_1,z_2,t)|\leq\bar{\dot{a}}_1$ for some constant $\bar{\dot{a}}_1>0$, implying the boundedness of $\dot{a}_1$.

Step i ($i=2,\dots,n-1$, $t\in[0,t_f)$): Consider the positive definite and radially unbounded function $V_i=\frac{1}{2}\epsilon_i^2$. Repeating the line of proof of *Step 1*, we guarantee the existence of constants $\bar{\epsilon}_i>0$ such that

$$-1<\mathrm{T}^{-1}(-\bar{\epsilon}_i)\leq z_i\leq\mathrm{T}^{-1}(\bar{\epsilon}_i)<1, \tag{25}$$

and similarly with *Step 1* by recalling (17a), we deduce the boundedness of $\dot{a}_i$.

Step n ($t\in[0,t_f)$): Consider the positive definite and radially unbounded function $V_n=\frac{1}{2}\epsilon_n^2$. The time derivative of V_n in view of (18c) yields

$$\begin{aligned}\dot{V}_n=\epsilon_n r_n\Big[&(1-\dot{\tau}_s)f(\bar{x},t-\tau_s(t))-z_n\dot{\rho}_n(t-\bar{\tau})\\&+(1-\dot{\tau}_s)g(\bar{x},t-\tau_s(t))u(t-\tau_s(t)-\tau_u(t))\\&-\dot{a}_{n-1}-\mathrm{sgn}(g)u(t-2\bar{\tau})-k_n\epsilon_n\Big],\end{aligned} \tag{26}$$

where $r_n=\frac{2}{(1-z_n^2)\rho_n(t-\bar{\tau})}$. Identically with *Step 1*, we deduce $r_n>0$.

To proceed, the following statement is necessary.

Proposition 1. *For all $t\in[0,t_f)$, there exist constant $\bar{u}>0$ satisfying $|u(t-\tau_s(t)-\tau_u(t))|\leq\bar{u}$ for all $\tau_s(t),\tau_u(t)>0$, and $|u(t-2\bar{\tau})|\leq\bar{u}$.*

Proof. To prove Proposition 1, we employ the fact that $\bar{z}(t)$ evolves continuously in Ω_z for all $t\in[0,t_f)$, i.e., $z_n(t)\in(-1,1)$. Let $\lim_{t\to t_f}|z_n(t)|=1$. By the continuity of $z_n(t)$, we directly deduce the existence of arbitrarily small constants $\beta,\delta>0$ such that $\beta<\lim_{t\to t_f}(\tau_s(t)+\tau_u(t))$, and $|z_n(t)|\leq|z_n(t_f-\beta)|<1-\delta$, for all $t\in[0,t_f-\beta]$. Thus, we conclude that $\lim_{t\to t_f}(t-\tau_s(t)-\tau_u(t))=t_f-\tau_s(t_f)-\tau_u(t_f)<t_f-\beta$, which guarantees that $t-\tau_s(t)-\tau_u(t)<t_f-\beta$, for all $t\in[0,t_f)$. Consequently, $|z_n(t-\tau_s(t)-\tau_u(t))|<1-\delta$, for all $t\in[0,t_f)$, which by (16), (17b) implies the the boundedness of $u(t-\tau_s(t)-\tau_u(t))$ for all $t\in[0,t_f)$. Similarly, we deduce $|z_n(t-2\bar{\tau})|<1-\delta$, for all $t\in[0,t_f)$, which implies the boundedness of $u(t-2\bar{\tau})$ for all $t\in[0,t_f)$. On the other hand, if $|z_n(t)|<1$, for all $t\in[0,t_f)$, we straightforwardly deduce that $u(t-\tau_s(t)-\tau_u(t))$ and $u(t-2\bar{\tau})$ are bounded for all $t\in[0,t_f)$. Therefore, we guarantee the existence of a constant $\bar{u}>0$ satisfying $|u(t-\tau_s(t)-\tau_u(t))|\leq\bar{u}$ for all $\tau_s(t),\tau_u(t)>0$, and $|u(t-2\bar{\tau})|\leq\bar{u}$, for all $t\in[0,t_f)$, completing the proof of Proposition 1. □

We continue the proof of Theorem 1 by distinguishing two cases.

Case A ($\tau_s(t)+\tau_u(t)=0$): In this case it holds that $\dot{\tau}_s=0$, and (26) becomes

$$\begin{aligned}\dot{V}_n=\epsilon_n r_n\Big[&f(\bar{x},t)-z_n\dot{\rho}_n(t-\bar{\tau})\\&-\dot{a}_{n-1}-\mathrm{sgn}(g)u(t-2\bar{\tau})-(|g|+1)k_n\epsilon_n\Big].\end{aligned} \tag{27}$$

By the continuity of $f(\bar{x},t)$, $g(\bar{x},t)$, application of the Extreme Value Theorem yields the existence of constants $\bar{f}>0$ and $\bar{g}>0$, such that $|f(\bar{x},t)|\leq\bar{f}$ and $|g(\bar{x},t)|\leq\bar{g}$. Furthermore, by Assumption 1, we conclude the existence of a constant $g^*>0$ such that $0<g^*\leq|g(\bar{x},t)|$. Moreover, $\dot{\rho}_n(t-\bar{\tau})$ is bounded by construction, $\dot{a}_{n-1}$ is proven bounded in *Step* $n-1$, and z_n is bounded as $\bar{z}\in\Omega_z$. Utilizing the preceding analysis and Proposition 1, we guarantee the existence of a constant $\bar{F}_n^A>0$, satisfying

$$\begin{aligned}\big|&f(\bar{x},t)-z_n\dot{\rho}_n(t-\bar{\tau})\\&-\dot{a}_{n-1}-\mathrm{sgn}(g)u(t-2\bar{\tau})\big|\leq\bar{F}_n^A.\end{aligned} \tag{28}$$

Therefore, (27) becomes

$$\dot{V}_n\leq|\epsilon_n|r_n\left(\bar{F}_n^A-(g^*+1)k_n|\epsilon_n|\right), \tag{29}$$

which is negative provided $|\epsilon_n|>\frac{\bar{F}_n^A}{(g^*+1)k_n}$. Consequently,

$$|\epsilon_n|\leq\bar{\epsilon}_n^A\triangleq\max\left\{|\epsilon_n(0)|,\frac{\bar{F}_n^A}{(g^*+1)k_n}\right\}. \tag{30}$$

Taking the inverse of the T-function we obtain

$$-1 < \mathrm{T}^{-1}(-\bar{\epsilon}_n^A) \leq z_n \leq \mathrm{T}^{-1}(\bar{\epsilon}_n^A) < 1. \tag{31}$$

Case B ($\tau_s(t) + \tau_u(t) > 0$): By recalling the arguments of *Case A* and employing Proposition 1, we guarantee the existence of a constant $\bar{F}_n^B > 0$, satisfying

$$\begin{aligned} &\big|(1-\dot{\tau}_s)(f(\bar{x}, t-\tau_s(t))) - z_n\dot{\rho}_n(t-\bar{\tau}) \\ &\quad + (1-\dot{\tau}_s)g(\bar{x}, t-\tau_s(t))u(t-\tau_s(t)-\tau_u(t)) \\ &\quad - \dot{a}_{n-1} - \mathrm{sgn}(g)u(t-2\bar{\tau})\big| \leq \bar{F}_n^B. \end{aligned} \tag{32}$$

Therefore, (26) becomes

$$\dot{V}_n \leq |\epsilon_n| r_n \left(\bar{F}_n^B - k_n|\epsilon_n|\right), \tag{33}$$

which is negative provided $|\epsilon_n| > \frac{\bar{F}_n^B}{k_n}$. Consequently,

$$|\epsilon_n| \leq \bar{\epsilon}_n^B \triangleq \max\left\{|\epsilon_n(0)|, \frac{\bar{F}_n^B}{k_n}\right\}. \tag{34}$$

Taking the inverse of the T-function we obtain

$$-1 < \mathrm{T}^{-1}(-\bar{\epsilon}_n^B) \leq z_n \leq \mathrm{T}^{-1}(\bar{\epsilon}_n^B) < 1. \tag{35}$$

Combining (31) and (35) we conclude that

$$-1 < \mathrm{T}^{-1}(-\bar{\epsilon}_n) \leq z_n \leq \mathrm{T}^{-1}(\bar{\epsilon}_n) < 1, \tag{36}$$

where $\bar{\epsilon}_n \triangleq \max\left\{\bar{\epsilon}_n^A, \bar{\epsilon}_n^B\right\}$. Further, by recalling (17b), we straightforwardly deduce the boundedness of u, and consequently of $\int_{t-2\bar{\tau}}^{t} u(\theta)d\theta$, for all $t \in [0, t_f)$. To conclude, owing to (24), (25), (36), and the analysis presented in *Step* 1-*Step n*, we conclude that $\bar{z}(t)$ evolves strictly within a compact subset of Ω_z for all $t \in [0, t_f)$, Therefore, following standard arguments (Khalil, 2001, Theorem 3.3), the solution is extended to $t_f = +\infty$. Finally, recalling (15), we obtain that for all $t \geq 0$,

$$|e(t)| < \rho_1^*(t), \tag{37a}$$

$$|x_i(t) - a_{i-1}(t+\tau_s(t))| < \rho_i^*(t), \tag{37b}$$

$$|x_n(t) - w(t)| < \rho_n^*(t), \tag{37c}$$

where $\rho_i^*(t) = \rho_i(t - (\bar{\tau} - \tau_s(t)))$, $i - 1, \ldots, n$, and $w(t) = a_{n-1}(t+\tau_s(t)) - \mathrm{sgn}(g)\int_{t+\tau_s(t)-2\bar{\tau}}^{t+\tau_s(t)} u(\theta)d\theta$. The above result implies that prescribed performance of the output regulation error is achieved in the sense that was clarified in Remark 3, and that all signals in the closed-loop remain bounded for all $t \geq 0$, thus completing the proof of Theorem 1. □

5. SIMULATION RESULTS

To verify the theoretical findings we perform simulation studies on a single-link robotic manipulator. Denoting with q_1 [rad] and q_2 [rad/s] the link position and velocity, respectively, the aforementioned system has the following form

$$\dot{q}_1 = q_2, \tag{38a}$$

$$\dot{q}_2 = -\frac{\mathrm{g}}{m}\sin(q_1) - \frac{\mathrm{c}}{m}q_2 + \frac{1}{m}u(t-\tau_u(t)), \tag{38b}$$

where $u(t - \tau_u(t))$ [N m] is the delayed control input torque and q_1 is the output of the system. Further, the measurements of q_1 and q_2 are subject to delay $\tau_s(t)$. The system parameters are $m = 5$ [N m s^2/rad], $\mathrm{g} = 10$ [N m] and $\mathrm{c} = 1$ [N m s/rad]. The system initially rests at $q_1(t) = \frac{\pi}{14}$, $q_2(t) = 0$, for all $t \in [-2\bar{\tau}, 0]$, which results in $\bar{\varphi}_1 = \frac{\pi}{14}$ and $\bar{\varphi}_2 = 0$. The desired value for the output is given by $q_d = \frac{\pi}{8}$ [rad]. The required performance indices are given by $\rho_1^\infty = 0.075$, and $\lambda_1 = 4$.

To illustrate the influence of the delays on the closed-loop system performance, we consider two different scenarios regarding the values of $\tau_s(t)$, $\tau_u(t)$ and $\bar{\tau}$, summarized in Table 1. We stress that only $\bar{\tau}$ is available for control design. The remaining controller parameters for both scenarios are selected as $\rho_1^0 = 2$ satisfying (5) for any $\bar{\tau}$ of Table 1, and moreover $k_1 = 1.25$, and $\rho_2^0 = 5$ satisfying (11) for any $\bar{\tau}$ of Table 1. In addition, $\rho_2^\infty = 0.75$, $\lambda_2 = 2$, and $k_2 = 75$.

Table 1. Delay functions

Scenario	$\tau_s(t)$	$\tau_u(t)$	$\bar{\tau}$
1	$0.01(1-\sin(\pi t))$	$0.01(1+\cos(\pi t))$	0.02
2	$0.02(1-\sin(\pi t))$	$0.02(1+\cos(\pi t))$	0.04

Simulation results are presented in Fig. 1. For both scenarios of Table 1 we illustrate: i) the output regulation error $e(t)$ alongside the corresponding performance bounds $\rho_1^*(t)$, $-\rho_1^*(t)$, ii) the intermediate error $q_2(t) - w(t)$ alongside the corresponding performance bounds $\rho_2^*(t)$, $-\rho_2^*(t)$, and iii) the produced control input torque $u(t)$. It is clearly shown that the increase of the delay values results in producing control input torque with larger magnitude and more intense high-frequency content. Nevertheless in both scenarios the quality of the evolution of the output regulation error within the performance envelope is preserved.

6. CONCLUSION

We proposed a controller for uncertain nonlinear systems, to achieve performance attributes on the output regulation error in terms of minimum convergence rate and maximum steady-state error, when both state and control input signals are subject to unknown and time-varying delays. State delay uncertainty dictate that the error is confined within a delayed version of the constructed performance envelope affecting the obtained minimum convergence rate, however, the maximum error at steady-state is obtained exactly as specified by the pre-selected performance functions. We validated the theoretical results by simulation studies on a single-link robotic manipulator.

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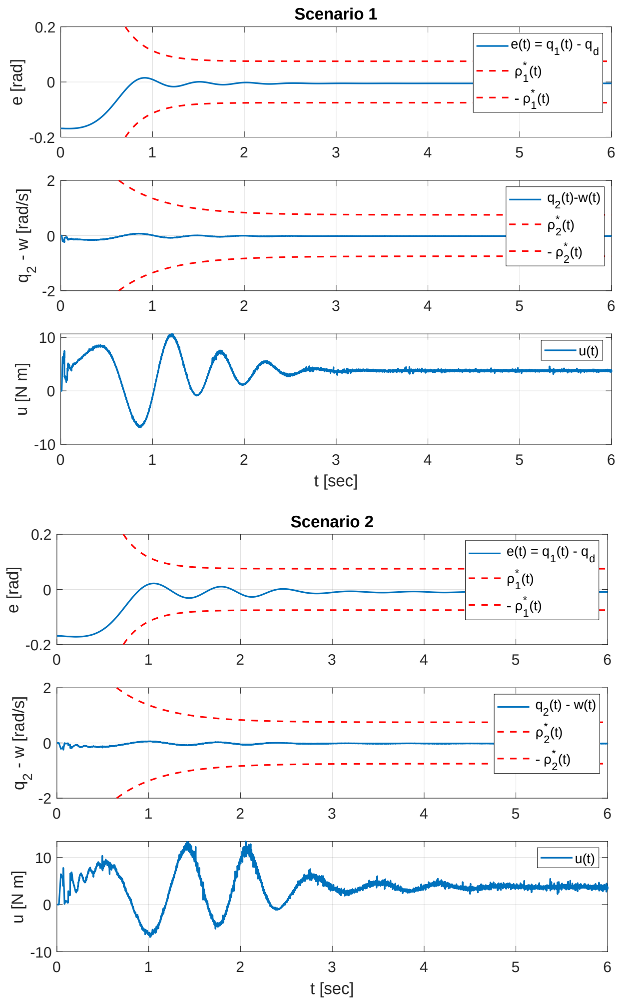

Fig. 1. Performance of (38) for Scenarios 1 and 2.

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