The Holy Grail of Mathematics

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Abstract

A trustworthy proof for the Riemann hypothesis has been considered as the Holy Grail of Mathematics by several authors. The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Let Q be the set of prime numbers q_n satisfying the inequality $\prod_{q \leq q_n} \frac{q}{q-1} > e^{\gamma} \cdot \log \theta(q_n)$ with the product extending over all prime numbers q that are less than or equal to q_n, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, $\theta(x)$ is the Chebyshev function and $\log$ is the natural logarithm. If the Riemann hypothesis is false, then there are infinitely many prime numbers q_n outside and inside of Q. In this note, we obtain a contradiction when we assume that there are infinitely many prime numbers q_n outside of Q. By reductio ad absurdum, we prove that the Riemann hypothesis is true.

Keywords: Riemann hypothesis, Nicolas inequality, Prime numbers, Chebyshev function

MSC Classification: 11M26 , 11A41 , 11A25

1 Introduction

The Riemann hypothesis is the assertion that all non-trivial zeros have real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. It was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems. This is one of the Clay Mathematics Institute's Millennium Prize Problems. In mathematics, the Chebyshev function

$\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers q that are less than or equal to x, where log is the natural logarithm. We say that $\mathsf{Nicolas}(q_n)$ holds provided that

$$\prod_{q \leq q_n} \frac{q}{q-1} > e^{\gamma} \cdot \log \theta(q_n),$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and q_n is the nth prime number. Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true. However, there are some implications in case of the Riemann hypothesis might be false.

Proposition 1 *If the Riemann hypothesis is false, then there are infinitely many prime numbers p_n such that $\mathsf{Nicolas}(p_n)$ holds and there are infinitely many prime numbers q_n such that $\mathsf{Nicolas}(q_n)$ fails (i.e. $\mathsf{Nicolas}(q_n)$ does not hold) [1, Theorem 3 (c) pp. 376].*

For $x \geq 2$, the function $u(x)$ is defined as follows [1, pp. 379]:

$$u(x) = \sum_{q > x} \left(\log(\frac{q}{q-1}) - \frac{1}{q} \right).$$

It is known the value of $H = \gamma - B$ such that $B \approx 0.26149$ is the Meissel-Mertens constant [2, (17.) pp. 54]. Franz Mertens obtained some important results about the constant H.

Proposition 2 *We have [2, pp. 52]:*

$$\sum_{q \leq x} \log(\frac{q}{q-1}) = \sum_{q \leq x} \frac{1}{q} + H - u(x).$$

Putting all together yields the proof of the Riemann hypothesis.

2 Known Inequalities

Proposition 3 *For $x > 0$ [3, pp. 1]:*

$$\frac{x}{x+1} < \log(1+x).$$

Proposition 4 *For $x \geq -1$ and $r > 1$ [3, pp. 1]:*

$$(1+x)^r \geq 1 + r \cdot x.$$

3 New Inequalities

Lemma 5 *For $x \geq 2$:*

$$\frac{1}{x} < \log(\frac{x}{x-1}).$$

Proof We have

$$\begin{aligned} \log(\frac{x}{x-1}) &= \log(1 + \frac{1}{x-1}) \\ &> \frac{\frac{1}{x-1}}{\frac{1}{x-1} + 1} \\ &= \frac{1}{(x-1) \cdot (\frac{1}{x-1} + 1)} \\ &= \frac{1}{1 + (x-1)} \\ &= \frac{1}{x} \end{aligned}$$

by Proposition 3. □

Lemma 6 *For $y > x > e$ and $z = \frac{y}{x}$:*

$$\frac{\log y}{\log x} > z^{\frac{1}{z \cdot \log x}}.$$

Proof We have $y = x + \varepsilon$ for $\varepsilon > 0$. We obtain that

$$\begin{aligned} \frac{\log y}{\log x} &= \frac{\log(x + \varepsilon)}{\log x} \\ &= \frac{\log \left(x \cdot (1 + \frac{\varepsilon}{x})\right)}{\log x} \\ &= \frac{\log x + \log(1 + \frac{\varepsilon}{x})}{\log x} \\ &= 1 + \frac{\log(1 + \frac{\varepsilon}{x})}{\log x} \end{aligned}$$

and

$$\begin{aligned} z &= \frac{y}{x} \\ &= \frac{x + \varepsilon}{x} \\ &= 1 + \frac{\varepsilon}{x}. \end{aligned}$$

We need to show that

$$1 + \frac{\log(1 + \frac{\varepsilon}{x})}{\log x} > \left(1 + \frac{\varepsilon}{x}\right)^{\frac{1}{z \cdot \log x}}$$

which is the same as

$$\left(1+\frac{\log(1+\frac{\varepsilon}{x})}{\log x}\right)^{z\cdot\log x} > (1+\frac{\varepsilon}{x}).$$

We know that

$$\begin{aligned}\left(1+\frac{\log(1+\frac{\varepsilon}{x})}{\log x}\right)^{z\cdot\log x} &\geq 1+(z\cdot\log x)\cdot\frac{\log(1+\frac{\varepsilon}{x})}{\log x}\\ &= 1+z\cdot\log(1+\frac{\varepsilon}{x})\end{aligned}$$

by Proposition 4. It is enough to show that

$$1+z\cdot\log(1+\frac{\varepsilon}{x}) > 1+\frac{\varepsilon}{x}$$

that is equivalent to

$$z > \frac{\frac{\varepsilon}{x}}{\log(1+\frac{\varepsilon}{x})}.$$

Let's define $w=\frac{\varepsilon}{x}$. Hence,

$$\begin{aligned}\frac{\frac{\varepsilon}{x}}{\log(1+\frac{\varepsilon}{x})} &= \frac{w}{\log(1+w)}\\ &< \frac{w}{\frac{w}{w+1}}\\ &= 1+w\\ &= 1+\frac{\varepsilon}{x}\\ &= z\end{aligned}$$

by Proposition 3. □

4 Main Theorem

Theorem 7 *The Riemann hypothesis is true.*

Proof Suppose that the Riemann hypothesis is false. Consequently, there are infinitely many prime numbers q_n such that $\mathsf{Nicolas}(q_n)$ fails by Proposition 1. Let's take a large enough prime number q_k such that $\mathsf{Nicolas}(q_k)$ fails. For the same reason, there are infinitely many prime numbers p_n such that $\mathsf{Nicolas}(p_n)$ holds by Proposition 1. Let's take a prime number $q_{k+j} > q_k$ such that $\mathsf{Nicolas}(q_{k+j})$, $j \geq 1$ and

$$\frac{q_{k+1}\cdot\log z}{z\cdot\log\theta(q_k)} \geq j$$

hold at the same time, where $z=\frac{\theta(q_{k+j})}{\theta(q_k)}$. Since $\mathsf{Nicolas}(q_k)$ fails, then

$$\prod_{q\leq q_k}\frac{q}{q-1} \leq e^{\gamma}\cdot\log\theta(q_k).$$

Let's apply the logarithm, then

$$\sum_{q\leq q_k}\log(\frac{q}{q-1}) \leq \gamma+\log\log\theta(q_k).$$

That's the same as

$$\sum_{q \leq q_k} \frac{1}{q} + H - u(q_k) \leq \gamma + \log\log\theta(q_k)$$

by Proposition 2. Let's add

$$\sum_{q_k < q \leq q_{k+j}} \left(\frac{2}{q} - \log(\frac{q}{q-1})\right)$$

to

$$\sum_{q \leq q_k} \frac{1}{q} + H - u(q_k)$$

and obtain that

$$\sum_{q \leq q_{k+j}} \frac{1}{q} + H - u(q_{k+j})$$

because of

$$\left(\sum_{q \leq q_k} \frac{1}{q}\right) + \left(\sum_{q_k < q \leq q_{k+j}} \frac{1}{q}\right) = \sum_{q \leq q_{k+j}} \frac{1}{q}$$

and

$$-u(q_k) + \sum_{q_k < q \leq q_{k+j}} \left(\frac{1}{q} - \log(\frac{q}{q-1})\right) = -u(q_{k+j}).$$

As a consequence, we have

$$\sum_{q \leq q_{k+j}} \frac{1}{q} + H - u(q_{k+j}) \leq \gamma + \log\log\theta(q_k) + \sum_{q_k < q \leq q_{k+j}} \left(\frac{2}{q} - \log(\frac{q}{q-1})\right).$$

Since $\mathsf{Nicolas}(q_{k+j})$ holds, then

$$\gamma + \log\log\theta(q_{k+j}) < \sum_{q \leq q_{k+j}} \frac{1}{q} + H - u(q_{k+j})$$

by Proposition 2. We notice that

$$\begin{aligned}\sum_{q_k < q \leq q_{k+j}} \left(\frac{2}{q} - \log(\frac{q}{q-1})\right) &= \sum_{q_k < q \leq q_{k+j}} \left(\frac{1}{q} - \log(\frac{q}{q-1})\right) + \sum_{q_k < q \leq q_{k+j}} \frac{1}{q} \\ &< \sum_{q_k < q \leq q_{k+j}} \frac{1}{q} \\ &\leq \frac{j}{q_{k+1}}\end{aligned}$$

by Lemma 5, since $\frac{1}{q} - \log(\frac{q}{q-1}) < 0$ for every prime q. In this way, we have

$$\log\log\theta(q_{k+j}) - \log\log\theta(q_k) < \frac{j}{q_{k+1}}.$$

That is equivalent to

$$q_{k+1} \cdot \log \frac{\log\theta(q_{k+j})}{\log\theta(q_k)} < j.$$

We know that

$$\frac{\log\theta(q_{k+j})}{\log\theta(q_k)} > z^{\frac{1}{z \cdot \log\theta(q_k)}}$$

by Lemma 6. Consequently, we obtain that

$$q_{k+1} \cdot \log z^{\frac{1}{z \cdot \log \theta(q_k)}} < j$$

which is

$$\frac{q_{k+1} \cdot \log z}{z \cdot \log \theta(q_k)} < j.$$

Hence, we obtain a contradiction under the assumption that $\mathsf{Nicolas}(q_k)$ fails and the possible existence of the prime number q_{k+j} such that $\mathsf{Nicolas}(q_{k+j})$, $j \geq 1$ and

$$\frac{q_{k+1} \cdot \log z}{z \cdot \log \theta(q_k)} \geq j$$

hold at the same time, where $z = \frac{\theta(q_{k+j})}{\theta(q_k)}$. The study of this arbitrary large enough prime number q_k reveals that the existence of such prime q_{k+j} is never possible under the assumption that the Riemann hypothesis is false. This contradicts the fact that we cannot always guarantee the non-existence of such prime number q_{k+j} when $\mathsf{Nicolas}(q_k)$ fails for an arbitrary and large enough prime q_k. By reductio ad absurdum, we prove that the Riemann hypothesis is true. □

5 Conclusions

Practical uses of the Riemann hypothesis include many propositions that are known to be true under the Riemann hypothesis and some that can be shown to be equivalent to the Riemann hypothesis. Indeed, the Riemann hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf hypothesis, the Large Prime Gap Conjecture, etc. Certainly, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics in general.

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Declarations

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