An apparent relation between the Newtonian gravitational constant and the Planck constant in a geometry-based energetic expression derived from a hypothetical system

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Abstract. The hypothetical system considered here comprises only the so-called internal energy, without free energy. Thus, in the canonical form, the partition function Z of the system has a unity value. As a first further specification, the system, in terms of energy distribution, exists in two states with the amounts of energy E_1 and $E_2 = 2E_1$. The mean energy of the system may thus be expressed as a weighted average of E_1 and E_2, i.e., $\sum_{i=1}^{N=2} p_i E_i$. Given $E_2 = 2E_1$ and $Z = 1$, the state probabilities $p_1 = \frac{1}{\varphi}$ and $p_2 = \frac{1}{\varphi^2}$ where the geometric-ratio-constant $\varphi = \frac{1+\sqrt{5}}{2}$. Because this condition corresponds to an asymmetric partition of a probability space into two portions with a ratio of φ, a thermodynamic solution of the expression for the mean energy is given by $-\frac{1}{\beta}\sum_{i=1}^{N=2}\frac{1}{\varphi^i} ln\frac{1}{\varphi^i} = \frac{1}{\beta}\left(1+\frac{1}{\varphi^2}\right) ln\varphi$. A hypothetical example of such portioned systems is described in the main text with more details. As a second further specification, the amount of mean energy in the system is defined as the average of comparable amounts of gravitational energy and photon energy. The relative portions of these two types of energy have the constant near-unity ratio that is the same as the ratio between the significant digits of the physical constants G and h. Under the condition that the amounts of energy specified in these two ways are equal at the presently testable precision, the following relation is reached: $G = \left(1 + \frac{1}{\varphi^2}\right) ln\varphi^2 \times 10^{-10}\ kg^{-1}\ m^3\ s^{-2} - h \times 10^{23}\ kg^{-2}\ m\ s^{-1}$, in which G is related to h by the mathematical constants 1, 2, e and φ, along with the necessary physical units and their associated exponential scalars. A calculation of G using this expression to the sixth significant digit from h of an exact value yields $6.67430 \times 10^{-11}\ kg^{-1}\ m^3\ s^{-2}$, which matches the accepted G that was recommended by CODATA in 2018. This calculation is readily verifiable. It will be interesting to learn the limit of precision at which the above apparent equality remains valid, or starts to break down, when an independently reproducible experimental estimate of G with more significant digits becomes available in the future.

Introduction

Gravitational and electromagnetic interactions are two types of fundamental interactions, which are commonly experienced in daily life. Derived from Newton's law, the gravitational (potential) energy between two objects with the masses M_1 and M_2 is given by

$$U = -G\frac{M_1 M_2}{r} \tag{1}$$

where G is the Newtonian constant of gravitation and r defines the distance between the centers of mass of those two objects. In the Planck relation, the electromagnetic energy carried by each photon is expressed as

$E = h\nu$ (2)

where h is Planck's constant and ν is the frequency of a photon. Thus, G and h are fundamental physical constants.

Accurate physical constants are important because the quantitative predictions of theories of physics cannot be more accurate than the numerical values of the constants that the theories contain and, furthermore, accurate numerical values of these constants are necessary for testing the overall consistency and correctness of the basic theories of physics [1]. The constant h has been accurately determined to the ninth significant digit and been fixed to an exact value of $6.62607015 \times 10^{-34}$ $(kg\ m^2\ s^{-1})$ with zero uncertainty by the Committee on Data of the International Science Council (CODATA) [2,3].

In contrast, the weakness of gravitational interactions and the inability to shield gravitational effects make it extremely challenging to determine G and, furthermore, there is no definitive relationship between G and any of other fundamental physical constants or theoretical prediction of the G value against which to test experimental results [4]. There have however been attempts to establish relations between G and other physical constants. For example, a recent theoretical study tried to relate G to h through other constants, in which the calculated value of G using the resulting preliminary formula is 6.79769×10^{-11} [5]. In the subsequent adjusting and refining steps, the scalars $2^{-\frac{1}{38}}$ and $2^{-\frac{1}{1.6\times10^5}}$ are used to numerically close the gap of 0.12339×10^{-11} between this calculated value and the generally accepted value of G (see below).

Experimentally, more than a dozen of G were determined in the last four decades; their values agree only on the first three significant digits 6.67 [4,6-21]. For higher accuracy, CODATA in 2018 used a pool of 16 sets of experimental measurements of G together as the basis of their most recently recommended G. This recommended G is of six significant digits, i.e., 6.67430 $(\times 10^{-11}\ kg^{-1}\ m^3\ s^{-2})$ [2,22], which have been generally accepted.

Below is a summary of a piece of self-assigned homework where a hypothetical system harboring gravitational energy and photon energy is evaluated (see reference #23 for the original version). This evaluation has led to an apparent relation (Eq. 47) with which all six accepted significant digits of G can be calculated from those of h (Eq. 48).

Evaluation of a hypothetical system

Specifications of the system

The hypothetical system is comprised of solely internal energy (U_{int}), without any free energy. As a first (further) specification, the system, in terms of energy distribution, has two accessible states with the energy levels of E_1 and E_2 where

$E_2 = 2E_1$ (3)

Under this first specification, the mean U_{int}, denoted as U_{int}^{1st}, may be calculated from E_1 and E_2 as an average weighted by the probabilities p_1 and p_2 of being in energy state 1 and state 2:

$$U_{int}^{1st} = \sum_{i=1}^{N=2} p_i E_i = p_1 E_1 + p_2 E_2 \quad (4)$$

A hypothetical example of this type of partitioned system is provided below in more physical details after introducing a second (further) specification.

In the canonical form, the partition function Z of the present system without any of the Helmholtz free energy should have a unity value such that

$$Z = \sum_{i=1}^{N=2} e^{-\beta E_i} = e^{-\beta E_1} + e^{-\beta E_2} = e^{-\beta E_1} + e^{-2\beta E_1} = e^{-\beta E_1} + \left(e^{-\beta E_1}\right)^2 = 1 \quad (5)$$

in which thermodynamic β corresponds to the inverse of the product of the Boltzmann constant k_B and thermodynamic temperature T in the unit K, $i.e.,\ \beta = \frac{1}{k_B T}$. The final equality in Eq. 5 has the same form as the quadratic equation from which the inverse of the golden ratio φ is solved:

$$\frac{1}{\varphi} + \frac{1}{\varphi^2} = 1 \quad (6)$$

where the roots are

$$e^{-\beta E_1} = \frac{1}{\varphi} = \frac{-1 \pm \sqrt{5}}{2} \quad (7)$$

For the positive root, the non-transcendental irrational number $\frac{1}{\varphi} = 0.618$ .... or $\varphi = 1.618$ ..., which is adopted here. Eq. 7 can be rearranged to

$$E_1 = -\frac{1}{\beta} ln \frac{1}{\varphi} = -\frac{1}{\beta_V} ln \frac{1}{\varphi}\ J \quad (8)$$

and E_2 is then given by

$$E_2 = -\frac{1}{\beta} ln \frac{1}{\varphi^2} = -\frac{1}{\beta_V} ln \frac{1}{\varphi^2}\ J = -\frac{2}{\beta_V} ln \frac{1}{\varphi}\ J \quad (9)$$

where

$$\frac{1}{\beta} = \frac{1}{\beta_V}\ J \quad (10)$$

For later operational convenience, the variable value of $\frac{1}{\beta}$ is represented by $\frac{1}{\beta_V}$, and in this inverse form, its unit is explicitly written as J.

According to the Maxwell-Boltzmann statistics, the probability of state i may be expressed as

$$p_i = \frac{1}{Z} e^{-\beta E_i} \quad (11)$$

Given $Z = 1$, $E_1 = -\frac{1}{\beta} ln \frac{1}{\varphi}$ (Eq. 8), and $E_2 = -\frac{1}{\beta} ln \frac{1}{\varphi^2}$ (Eq. 9)

$$p_1 = e^{-\beta E_1} = \frac{1}{\varphi} \tag{12}$$

and

$$p_2 = e^{-\beta E_2} = e^{-2\beta E_1} = \frac{1}{\varphi^2} \tag{13}$$

Their summation yields the expected unity value:

$$\sum_{i=1}^{N=2} \frac{1}{p^i} = \sum_{i=1}^{N=2} \frac{1}{\varphi^i} = \frac{1}{\varphi} + \frac{1}{\varphi^2} = 1 \tag{14}$$

Eq. 14 describes an asymmetric partition of a unit of a geometric object, or a probability space, by a ratio of φ into two portions defined by $\frac{1}{\varphi}$ and $\frac{1}{\varphi^2}$.

On the basis of Eqs. 8, 9, 12 and 13, Eq. 4 can be further expressed as U_{int}^{1st} being proportional to the sum of weighted two portions of the partitioned probability space in their natural logarithm forms ($ln\frac{1}{\varphi}$ and $ln\frac{1}{\varphi^2}$):

$$U_{int}^{1st} = -\frac{1}{\beta}\sum_{i=1}^{N=2} p_i ln\, p_i = -\frac{1}{\beta}\left(\frac{1}{\varphi} ln\frac{1}{\varphi} + \frac{1}{\varphi^2} ln\frac{1}{\varphi^2}\right) = \frac{1}{\beta_V}\left(1 + \frac{1}{\varphi^2}\right) ln\varphi\, J = \frac{6.65\ldots}{\beta_V} \times 10^{-1}\, J \tag{15}$$

where

$$-\left(\frac{1}{\varphi} ln\frac{1}{\varphi} + \frac{1}{\varphi^2} ln\frac{1}{\varphi^2}\right) = \left(\frac{1}{\varphi} + \frac{2}{\varphi^2}\right) ln\varphi = \left(\frac{1}{\varphi} + \frac{1}{\varphi^2} + \frac{1}{\varphi^2}\right) ln\varphi = \left(1 + \frac{1}{\varphi^2}\right) ln\varphi = 6.65 \ldots \times 10^{-1} \tag{16}$$

Thus, U_{int}^{1st}, defined according to the first specification, is related to a specific geometric partition, i.e., unity to $\frac{1}{\varphi}$ and $\frac{1}{\varphi^2}$.

As a second specification, in terms of mean U_{int}, denoted as U_{int}^{2nd}, the system harbors half of a specified amount of gravitational energy ($-U_G$) plus half of a specified amount of photon energy (E_h) themselves, or their quantitative equivalence, with a constant ratio between the two portions of energy:

$$U_{int}^{2nd} = -\frac{1}{2} U_G + \frac{1}{2} E_h = \frac{-U_G + E_h}{2} \tag{17}$$

These two portions of energy are treated as directly additive quantities in accordance with the law of conservation of energy, and expressed as their arithmetic average. Here, the expression of gravitational energy and photon energy in an additive manner is also, in part, inspired by the studies of Pound and Rebka [24,25], in which they theoretically and experimentally investigated the so-called gravitational shift of apparent photon frequency.

To conceptually relate the two specifications, a hypothetical example of the present type of system is described below, which contains two objects, each with a constant mass. Additionally, object 1 has a positive charge (Q_1^+) whereas object 2 has a negative change (Q_2^-). In state 1, the centers of mass or charge of two bodies are separated by a distance of r_1 whereas in state 2, they are separated by a distance of r_2, and furthermore $r_1 = 2r_2$. Thus, the amount of the gravitational energy and the electrical potential energy (U_e) in state 2 is twice as much as those in state 1, so is their sum, i.e., $E_2 = 2E_1$ as specified earlier.

Derived from the Coulomb's law, the expression of the electrical potential energy (U_e) is given by

$$U_e = k_e \frac{Q_1^+ Q_2^-}{r} \tag{18}$$

where k_e is the coulomb constant. The electrical potential energy is, in turn, represented here by that of the photon energy $h\nu_1$ in state 1 and $h\nu_2$ in state 2 where $\nu_2 = 2\nu_1$, or in terms of wavelength (λ), $\lambda_1 = 2\lambda_2$, analogous to the relation of $r_1 = 2r_2$. (A piece of self-assigned homework regarding an apparent quantitative relation between Newton's law of universal gravitation and Coulomb's law is summarized in reference #26.)

The effective mean inverse of the distance, denoted as $\frac{1}{r}$, should be related to the weighted average of $\frac{1}{r_1}$ and $\frac{1}{r_2}$:

$$\frac{1}{r} = p_1 \frac{1}{r_1} + p_2 \frac{1}{r_2} = p_1 \frac{1}{r_1} + p_2 \frac{1}{\frac{1}{2} r_1} = \frac{1}{\varphi r_1} + \frac{1}{\frac{1}{2}\varphi^2 r_1} \tag{19}$$

rearranged to

$$r_1 = \left(\frac{1}{\varphi} + \frac{1}{\frac{1}{2}\varphi^2}\right) r = \left(\frac{1}{\varphi} + \frac{1}{\varphi^2} + \frac{1}{\varphi^2}\right) r = \left(1 + \frac{1}{\varphi^2}\right) r \tag{20}$$

and then

$$r_2 = \frac{1}{2}\left(1 + \frac{1}{\varphi^2}\right) r \tag{21}$$

Similarly, λ_1 and λ_2 should be related to the effective mean λ such that

$$\lambda_1 = \left(\frac{1}{\varphi} + \frac{1}{\frac{1}{2}\varphi^2}\right) \lambda = \left(1 + \frac{1}{\varphi^2}\right) \lambda = \left(1 + \frac{1}{\varphi^2}\right) \frac{c}{\nu} \tag{22}$$

and

$$\lambda_2 = \frac{1}{2}\left(1 + \frac{1}{\varphi^2}\right) \lambda = \frac{1}{2}\left(1 + \frac{1}{\varphi^2}\right) \frac{c}{\nu} \tag{23}$$

where c is the speed of light in vacuum, and ν is the effective mean frequency. Note that $\left(1 + \frac{1}{\varphi^2}\right)$ is also part of Eq. 15 that expresses the mean energy of the system in terms of U_{int}^{1st}.

To define the quantities $\frac{M_1 M_2}{r}$ and ν underling $-U_G$ and E_h in terms of the second specification, first, their magnitude but not exact significant digits will be estimated, on the basis of the closeness of the arithmetic and geometric averages of $-U_G$ and E_h in comparable amounts, which would be equal only if the amounts of $-U_G$ and E_h were absolutely equal:

$$\frac{-U_G + E_h}{2} \approx \sqrt{-U_G E_h} \approx \frac{6.65}{\beta_V} \times 10^{-1}\, J \tag{24}$$

where $U_{int}^{2nd} = \frac{-U_G+E_h}{2}$ (Eq. 17) is intended to be equal to $U_{int}^{1st} = \frac{6.65}{\beta_V} \times 10^{-1}\, J$ (Eq. 15).

Solely for operational simplicity in this paragraph alone, in which the two types of average are further expressed individually, the significant digits of G and h are both approximated − temporarily − with the same value 6.65 such that

$$G \approx 6.65 \times 10^{-11}\, kg^{-1}\, m^{3}\, s^{-2} \tag{25}$$

and

$$h \approx 6.65 \times 10^{-34}\, kg\, m^2\, s^{-1} \tag{26}$$

Furthermore, the quantities $\frac{M_1 M_2}{r}$ and ν are expressed as

$$\frac{M_1 M_2}{r} = x\, kg^2\, m^{-1} \tag{27}$$

and

$$\nu = y\, s^{-1} \tag{28}$$

The arithmetic and geometric averages of $-U_G$ and E_h are then separately expressed as

$$\frac{-U_G+E_h}{2} = \frac{1}{2}\left(G\,\frac{M_1 M_2}{r} + h\nu\right) = \frac{1}{2}[Gx(kg^2\, m^{-1}) + hy(s^{-1})]$$
$$\approx \frac{1}{2}(6.65 \times 10^{-11} \times x + 6.65 \times 10^{-34} \times y)\, J = \frac{6.65}{2}(x \times 10^{-11} + y \times 10^{-34})\, J \tag{29}$$

and

$$\sqrt{-U_G E_h} = \sqrt{G\,\frac{M_1 M_2}{r}\,h\nu} = \sqrt{Gx(kg^2\, m^{-1}) \times hy(s^{-1})}$$
$$\approx \sqrt{6.65 \times 10^{-11} \times x \times 6.65 \times 10^{-34} \times y}\, J = 6.65\sqrt{xy \times 10^{-45}}\, J \tag{30}$$

On the one hand, relating Eqs. 24 and 29 as an "equality", solely for ease of estimating x and y, yields

$$\frac{6.65}{\beta_V} \times 10^{-1}\, J = \frac{6.65}{2}(x \times 10^{-11} + y \times 10^{-34})\, J \tag{31}$$

simplified to

$$\beta_V(x \times 10^{-10} + y \times 10^{-33}) - 2 = 0 \tag{32}$$

On the other hand, relating Eqs. 24 and 30 also as an "equality" gives

$$\frac{6.65}{\beta_V} \times 10^{-1}\, J = 6.65\sqrt{xy \times 10^{-45}}\, J \tag{33}$$

simplified to

$$x = \frac{1}{{\beta_V}^2 y} \times 10^{43} \tag{34}$$

Substituting Eq. 34 into Eq. 32 yields

$$\beta_V\left(\frac{1}{{\beta_V}^2 y}\times 10^{43}\times 10^{-10}+y\times 10^{-33}\right)-2=0 \tag{35}$$

rearranged to

$$\left(\beta_V y\times 10^{-\frac{33}{2}}-10^{\frac{33}{2}}\right)^2=0 \tag{36}$$

and then simplified to

$$y=\frac{1}{\beta_V}\times 10^{33} \tag{37}$$

Substituting Eq. 37 into Eq. 34 gives

$$x=\frac{1}{\beta_V}\times 10^{10} \tag{38}$$

The above estimates of x and y make an intuitive sense. For $U_{int}^{1st}=U_{int}^{2nd}$ under each condition defined by $\frac{1}{\beta_V}$, the values of $\frac{M_1 M_2}{r}$ and ν, which are multiplied by G and h to calculate the system's gravitational energy and photon energy, must include at least the unit-less variable $\frac{1}{\beta_V}$ multiplied with the exponential values 10^{10} and 10^{33}, respectively. These latter two exponential values are necessary to bring the two respective types of energy underlying U_{int}^{2nd} to the magnitude of U_{int}^{1st}, because U_{int}^{1st} has a multiplying exponential factor of 10^{-1} (Eq. 15) whereas G has 10^{-11} and h has 10^{-34}. Thus, based on the two estimates given by Eqs. 37 and 38, $-U_G$ and E_h are tentatively expressed as

$$-U_G=G\frac{M_1 M_2}{r}=Gx\ kg^2\ m^{-1}=G\frac{1}{\beta_V}\times 10^{10}\ kg^2\ m^{-1} \tag{39}$$

where

$$\frac{M_1 M_2}{r}=\frac{1}{\beta_V}\times 10^{10}\ kg^2\ m^{-1} \tag{40}$$

and

$$E_h=h\nu=hy\ s^{-1}=h\frac{1}{\beta_V}\times 10^{33}\ s^{-1} \tag{41}$$

where

$$\nu=\frac{1}{\beta_V}\times 10^{33}\ s^{-1} \tag{42}$$

Substituting Eqs. 39 and 41 into Eq. 17 gives

$$U_{int}^{2nd}=\frac{-U_G+E_h}{2}=\frac{1}{2}\left(G\frac{1}{\beta_V}\times 10^{10}\ kg^2\ m^{-1}+h\frac{1}{\beta_V}\times 10^{33}\ s^{-1}\right) \tag{43}$$

where $-U_G$ and E_h have comparable but not identical relative contributions to U_{int}^{2nd} with a constant ratio of 1.01. Given this slight inequality between $-U_G$ and E_h, which stems from that between the significant digits of G and h, the arithmetic average of $-U_G$ and E_h is, by definition, not the same as, but slightly greater than, their geometric average. (Another piece of self-assigned homework involving the geometric average is summarized in reference #27.)

Up to this point, the information regarding the significant digits has not been used to specify the amount of energy in the system. Thus, a specific scaling factor (denoted as ζ here) may, in principle, be needed to numerically equalize U_{int}^{1st} and U_{int}^{2nd} at a specified level of precision.

Evaluation of the numerical relation between U_{int}^{1st} and U_{int}^{2nd}

To estimate the factor ζ, the ratio of U_{int}^{2nd} to U_{int}^{1st} is calculated in accordance with Eqs. 15 and 43:

$$\zeta = \frac{U_{int}^{2nd}}{U_{int}^{1st}} = \frac{\frac{1}{2}\left(G\frac{1}{\beta_V}\times 10^{10}\ kg^2\ m^{-1} + h\frac{1}{\beta_V}\times 10^{33}\ s^{-1}\right)}{\frac{1}{\beta_V}\left(1+\frac{1}{\varphi^2}\right) ln\varphi\ kg\ m^2\ s^{-2}} \quad (44)$$

After $\frac{1}{\beta_V}$ being cancelled out, the ratio is calculated as

$$\zeta = \frac{U_{int}^{2nd}}{U_{int}^{1st}} = \frac{\frac{1}{2}\left(G\times 10^{10}\ kg^2\ m^{-1} + h\times 10^{33}\ s^{-1}\right)}{\left(1+\frac{1}{\varphi^2}\right) ln\varphi\ kg\ m^2\ s^{-2}} = \frac{\frac{1}{2}(6.67430+6.626070)\times 10^{-1} J}{(1+0.3819660)\times 0.4812118\ J} = 1.00000 \quad (45)$$

which is limited to the sixth significant digit by the precision of G. At this precision, $\zeta = \frac{U_{int}^{2nd}}{U_{int}^{1st}}$ has an apparent unity value. One implication of this surprising, but arguably desirable, finding is that G might be calculated from h that has an accepted exact value of $6.62607015 \times 10^{-34}$ ($kg\ m^2\ s^{-1}$) with zero uncertainty [2,3].

An apparent relation for calculating G from h

To calculate G, Eq. 45 is rearranged first to

$$\frac{1}{2}(G \times 10^{10}\ kg^2\ m^{-1} + h \times 10^{33}\ s^{-1}) = \left(1 + \frac{1}{\varphi^2}\right) ln\varphi\ kg\ m^2\ s^{-2} \quad (46)$$

and then to

$$G = \left(1 + \frac{1}{\varphi^2}\right) ln\varphi^2 \times 10^{-10}\ kg^{-1}\ m^3\ s^{-2} - h \times 10^{23}\ kg^{-2}\ m\ s^{-1} \quad (47)$$

In Eq. 47, G is related to h by the mathematical constants 1, 2, e and φ along with the necessary physical units and their associated exponential scalars. For direct comparison, G is calculated to the sixth significant digit from h (see Appendix for detailed calculation):

$$G_{cal-6} = 6.67430 \times 10^{-11}\ kg^{-1}\ m^3\ s^{-2} \quad (48)$$

Without any further surprise, G_{cal-6} matches the accepted $G = 6.67430 \times 10^{-11}\ kg^{-1}\ m^3\ s^{-2}$, recommended by CODATA in 2018 [2,22].

Discussion

The mean internal energy U_{int} of the present hypothetical system is defined in two distinct ways. In one way, this energy is defined as the average amount of gravitational energy and photon energy with the constant near-unity ratio that is the same as the ratio between the significant digits of G and h (Eq. 43). In principle, this sum of energy does not have to be, literally, the specified gravitational energy and phonon energy themselves, but be that of equivalent amounts of energy that may, e.g., be transformed from or to them. In another way, the energy in terms of its distribution is effectively defined by a geometric model, in which a probability space is partitioned in a ratio specified by the geometric constant golden-ratio φ. The resulting geometry-based expression $-\sum_{i=1}^{N=2}\frac{1}{\varphi^i}ln\frac{1}{\varphi^i}=\left(1+\frac{1}{\varphi^2}\right)ln\varphi$ is linked by $\frac{1}{\beta}$, defined here also as $\frac{1}{\beta_V}$ in the unit J, to the energetic expression in terms of gravitational energy and photon energy. Given in Eq. 40 or 42, the value of $\frac{1}{\beta_V}$ is related to that of $\frac{M_1M_2}{r}$ or ν by a factor of 10^{10} or 10^{33}, the value of $\frac{1}{\beta}$, $\frac{M_1M_2}{r}$ or ν here predicts those of the remaining two quantities.

At the currently testable precision, i.e., the sixth significant digit limited by that of G, both of those ways lead to the same amount of energy so that $U_{int}^{1st}=U_{int}^{2nd}$ (Eq. 45). Thus, the process of the partition underlying U_{int}^{1st} could create a system with the energy U_{int}^{2nd}. In accordance with the first law of thermodynamics, the amount of energy of U_{int}^{2nd} would be just enough to eliminate the partition, creating or restoring the unpartitioned unity probability space, at the expense of an increase in entropy elsewhere, a condition demanded by the second law of thermodynamics.

The combination of Eqs.16 and 45 yields

$$-\left(\frac{1}{\varphi}ln\frac{1}{\varphi}+\frac{1}{\varphi^2}ln\frac{1}{\varphi^2}\right)=\left(1+\frac{1}{\varphi^2}\right)ln\varphi\ =\frac{1.00000}{2}(G\times10^{10}\ kg^2\ m^{-1}+h\times10^{33}\ s^{-1})\ J^{-1} \quad (49)$$

Numerically (see Eq. 45), this apparent equality reflects an extremely good approximation of the arithmetic average of the significant digits of G and h by those of the value of $-\left(\frac{1}{\varphi}ln\frac{1}{\varphi}+\frac{1}{\varphi^2}ln\frac{1}{\varphi^2}\right)$, a form of expression of a geometry-based model where a probability space of a unity value is partitioned by a ratio of φ into the two portions with the values of $\frac{1}{\varphi}$ and $\frac{1}{\varphi^2}$, values that correspond to about 62% and 38%, respectively.

For reference, Eq. 46 or 49 may be scaled to the condition that $\nu=1\ s^{-1}$:

$$\frac{1}{2}(G\times10^{-23}\ kg^2\ m^{-1}+h\times1\ s^{-1})=-k_BT\left(\frac{1}{\varphi}ln\frac{1}{\varphi}+\frac{1}{\varphi^2}ln\frac{1}{\varphi^2}\right)=10^{-33}\left(1+\frac{1}{\varphi^2}\right)ln\varphi\ \ J \quad (50)$$

The temperature would be calculated from $10^{-33}\ J$ as

$$T=\frac{10^{-33}\ J}{k_B}=\frac{10^{-33}\ J}{1.380649\times10^{-23}\ J\ K^{-1}}=7.242971\times10^{-11}K \quad (51)$$

which is an extremely cold temperature about $70\ pK$.

Moreover, should the apparent equality between G and h hold also at higher precision, G would be numerically related to h at an even greater number of significant digit. As a further exercise, G is calculated using Eq. 47 to the ninth significant digit from all nine significant digits of h [2,3]:

$$G_{cal-9} = 6.67429758 \times 10^{-11}\ kg^{-1}\ m^{3}\ s^{-2} \tag{52}$$

The details of this calculation are provided in the Appendix below. It will be interesting to learn the extent to which the apparent equality (Eq. 46 or 47) does or does not hold, when an independently reproducible and more precise experimental estimate of G becomes available in the future.

Appendix

Details of calculation of G to the ninth significant digit from h

$$G = \left(1+\frac{1}{\varphi^2}\right) ln\varphi^2 \times 10^{-10} kg^{-1}\, m^3\, s^{-2} - h \times 10^{23} kg^{-2}\, m\, s^{-1}$$

$$= 13.30036773 \times 10^{-11} kg^{-1}\, m^3\, s^{-2} - 6.62607015 \times 10^{-34} kg\, m^2\, s^{-1} \times 10^{23} kg^{-2}\, m\, s^{-1}$$

$$= (13.30036773 - 6.62607015) \times 10^{-11} kg^{-1}\, m^3\, s^{-2}$$

$$= 6.67429758 \times 10^{-11} kg^{-1}\, m^3\, s^{-2} \quad (53)$$

where

$$\varphi = 1.6180339887 \quad (54)$$

$$\varphi^2 = 2.6180339887 \quad (55)$$

$$\frac{1}{\varphi^2} = 0.38196601125 \quad (56)$$

$$ln\varphi = 0.48121182506 \quad (57)$$

$$ln\varphi^2 = 0.96242365012 \quad (58)$$

$$\left(1+\frac{1}{\varphi^2}\right) ln\varphi^2 \times 10 = 13.30036773 \quad (59)$$

$$h = 6.62607015 \times 10^{-34}\, kg\, m^2\, s^{-1} \quad (60)$$

$$G = 6.67430 \times 10^{-11} kg^{-1}\, m^3\, s^{-2} \quad (61)$$

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