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Matrix continued fractions and Expansions of the Error Function

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Abstract. In this paper we recall some results and some criteria on the convergence of matrix continued fractions. The aim of this paper is to give some properties and results of continued fractions with matrix arguments. Then we give continued fraction expansions of the error function $\text{erf}(A)$ where A is a matrix. At the end, some numerical examples illustrating the theoretical results are discussed.

1 Introduction and motivation

The theory of continued fractions has been a topic of great interest over the last two hundred years. The basic idea of this theory over real numbers is to give an approximation of various real numbers by the rationals. A continued fraction is an expression obtained through an iterative process of representing a number as the sum of its integer part and the reciprocal of another number, then writing this other number as the sum of its integer part and another reciprocal, and so on. One of the main reasons why continued fractions are so useful in computations is that they often provide representations for transcendental functions that are much more generally valid than the classical representation by, say, the power series. Further, in the convergent case, continued fraction expansions have the advantage that they converge more rapidly than other numerical algorithms.

Recently, the extension of continued fraction theory from real numbers to the matrix case has seen several developments and interesting applications (see [1], [3], [6]). Since

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calculations involving matrix valued functions with matrix arguments are feasible with large computers, it will be an interesting attempt to develop such a matrix theory. In this direction, and generally in a Banach space, few convergence results on non-commutative continued fraction are known.

Two theorems are stated in [10], where Wynn reviews many aspects of the theory of continued fractions, whose elements do not commute under a multiplication law. In Banach space, extensions of Worpitsky's have been proven by Haydan [2] and Negoescu [8].

In [9], the authors give several convergence criteria on non-commutative continued fractions whose arguments are $m \times m$ matrices of the form $K(B_n/A_n)$.

The error function erf is a special function that is important since it appears in the solutions of diffusion problems in heat, mass and momentum transfer, probability theory, theory of errors and various branches of mathematical physics. The closely related Fresnel integrals, which are fundamental in the theory of optics, can be derived directly from the error function.

2 Preliminaries and notations

Throughout this paper, we denote by $\mathcal{M}_m$ the set of $m \times m$ real (or complex) matrices endowed with the subordinate matrix infinity norm defined by,

$$\forall A = (a_{i,j}),\ A \in \mathcal{M}_m,\ \|A\| = \max_{1\leq i\leq m} \sum_{j=1}^{m} |a_{i,j}|.$$

This norm satisfies the inequality

$$\|AB\| \leq \|A\|\|B\|\,.$$

Let $A \in \mathcal{M}_m$, A is said to be positive semi-definite (resp. positive definite) if A is symmetric and

$$\forall x \in \mathbb{R}^m, (Ax, x) \geq 0 \text{ (resp. } \forall x \in \mathbb{R}^m,\ x \neq 0,\ (Ax, x) > 0)$$

where $(\cdot,\cdot)$ denotes the standard scalar product of $\mathbb{R}^m$ defined by

$$\forall x = (x_1, \ldots, x_m) \in \mathbb{R}^m, \forall y = (y_1, \ldots, y_m) \in \mathbb{R}^m : (x, y) = \sum_{i=1}^{m} x_i y_i\,.$$

For any $A, B \in \mathcal{M}_m$ with B invertible, we write $A/B := B^{-1}A$, in particular, if $A = I$, where I is the m^{th} order identity matrix, then we write $I/B = B^{-1}$. It is clear that for any invertible matrix C, we have

$$\frac{CA}{CB} = \frac{A}{B}\,.$$

Definition 2.1. Let $(A_n)_{n\geq 0}, (B_n)_{n\geq 1}$ be two nonzero sequences of $\mathcal{M}_m$. The continued fraction of (A_n) and (B_n), denoted by $K(B_n/A_n)$, is the quantity

$$A_0 + \cfrac{B_1}{A_1 + \frac{B_2}{A_{2+\dots}}} = \left[A_0; \frac{B_1}{A_1}, \frac{B_2}{A_2}, \dots\right].$$

Sometimes, we use the notation $\left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{+\infty}$ or $K(B_n/A_n)$, where

$$\left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{n} = \left[A_0; \frac{B_1}{A_1}, \frac{B_2}{A_2}, \dots, \frac{B_n}{A_n}\right].$$

The fractions $\frac{B_n}{A_n}$ and $\frac{P_n}{Q_n} := \left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{n}$ are called, respectively, the n^{th} partial quotient and the n^{th} convergent of the continued fraction $K(B_n/A_n)$.

When $B_n = I$ for all $n \geq 1$, then $K(I/A_n)$ is called an ordinary continued fraction. The following proposition gives an adequate method to calculate $K(B_n/A_n)$.

Proposition 2.2. *The elements $(P_n)_{n\geq -1}$ and $(Q_n)_{n\geq -1}$ of the n^{th} convergent of $K(B_n/A_n)$ are given by the relationships*

$$\begin{cases} P_{-1} = I, & P_0 = A_0 \\ Q_{-1} = 0, & Q_0 = I \end{cases} \quad \textit{and} \quad \begin{cases} P_n = A_n P_{n-1} + B_n P_{n-2} \\ Q_n = A_n Q_{n-1} + B_n Q_{n-2} \end{cases} \quad , n \geq 1\,.$$

Proof. This can be done by induction. □

The proof of the next Proposition is elementary and we leave it to the reader.

Proposition 2.3. *For any two matrices C and D with C invertible, we have*

$$C\left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{n} D = \left[CA_0D; \frac{B_1 D}{A_1 C^{-1}}, \frac{B_2 C^{-1}}{A_2}, \frac{B_k}{A_k}\right]_{k=3}^{n}. \tag{1}$$

The continued fraction $K(B_n/A_n)$ converges in $\mathcal{M}_m$ if the sequence

$$(F_n) = \left(\frac{P_n}{Qn}\right) = (Q_n^{-1} P_n)$$

converges in $\mathcal{M}_m$ in the sense that there exists a matrix $F \in \mathcal{M}_m$ such that

$$\lim_{n\to+\infty} ||F_n - F|| = 0.$$

In the opposite case, we say that $K(B_n/A_n)$ is divergent. It is clear that

$$\frac{P_n}{Q_n} = A_0 + \sum_{i=1}^{n}\left(\frac{P_i}{Q_i} - \frac{P_{i-1}}{Q_{i-1}}\right), \tag{2}$$

and from (2), we see that the continued fraction $K(B_n/A_n)$ converges in $\mathcal{M}_m$ if and only if the series $\sum\limits_{n=1}^{+\infty} (\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}})$ converges in $\mathcal{M}_m$.

Definition 2.4. Let (A_n), (B_n), (C_n) and (D_n) be four sequences of matrices. We say that the continued fractions $K(B_n/A_n)$ and $K(D_n/C_n)$ are equivalent if we have $F_n = G_n$ for all $n \geq 1$, where F_n and G_n are the n^{th} convergent of $K(B_n/A_n)$ and $K(D_n/C_n)$ respectively.

The following lemma characterizes equivalence of continued fractions.

Lemma 2.5 ([4]). *Let (r_n) be a non-zero sequence of real numbers. The continued fractions*

$$\left[a_0; \frac{r_1 b_1}{r_1 a_1}, \frac{r_2 r_1 b_2}{r_2 a_2}, \ldots, \frac{r_n r_{n-1} b_n}{r_n a_n}, \ldots\right] \text{ and } \left[a_0; \frac{b_1}{a_1}, \frac{b_2}{a_2}, \ldots, \frac{b_n}{a_n}, \ldots\right]$$

are equivalent.

We also recall the following Lemma. From the expansion of a function given by its Taylor series, we give the expansion in continued fractions of the series that was established by Euler.

Lemma 2.6 ([5]). *Let f be a function with Taylor series expansion $f(x) = \sum_{n=0}^{+\infty} c_n x^n$ in $D \subset R$. Then, the expansion in continued fraction of $f(x)$ is*

$$\begin{aligned} f(x) &= \left[c_0; \frac{c_1 x}{1}, \frac{-c_2 x}{c_1 + c_2 x}, \frac{-c_1 c_3 x}{c_2 + c_3 x}, \ldots \frac{-c_{n-2} c_n x}{c_{n-1} + c_n x}, \ldots\right] \\ &= \left[\frac{c_0}{1}, \frac{-c_1 x}{c_0 + c_1 x}, \frac{-c_0 c_2 x}{c_1 + c_2 x}, \frac{-c_1 c_3 x}{c_2 + c_3 x}, \ldots, \frac{-c_{n-2} c_n x}{c_{n-1} + c_n x}, \ldots\right]. \end{aligned}$$

Remark 2.7. Let (A_n) and (B_n) be two sequences of $\mathcal{M}_m$. Then we notice that we can write the first convergents of the continued fraction $K(B_n/A_n)$ by:

$$F_1 = A_0 + A_1^{-1} B_1 = A_0 + (B_1^{-1} A_1)^{-1}.$$

$$F_2 = A_0 + (A_1 + A_2^{-1} B_2)^{-1} B_1 = A_0 + (B_1^{-1} A_1 + (B_2^{-1} A_2 B_1)^{-1})^{-1}.$$

If we put, $A_1^* = B_1^{-1} A_1$ and $A_2^* = B_2^{-1} A_2 B_1$, we have

$$F_1 = A_0 + \frac{I}{A_1^*}, F_2 = A_0 + \frac{I}{A_1^* + \frac{I}{A_2^*}}.$$

Generally, we prove by a recurrence that if we put for all $k \geq 1$,

$$A_{2k}^* = (B_{2k} \ldots B_2)^{-1} A_{2k} B_{2k-1} \ldots B_1$$

and

$$A_{2k+1}^* = (B_{2k+1} \ldots B_1)^{-1} A_{2k+1} B_{2k} \ldots B_2,$$

then the continued fractions $A_0 + K(B_n/A_n)$ and $A_0 + K(I/A_n^*)$ are equivalent.

So, the convergence of one of these continued fractions implies the convergence of the other continued fraction.

Theorem 2.8 ([12]). *Let all the elements of $A_n (n = 1, 2, \ldots)$ be positive, namely, A_n are positive matrices for all n, then the matrix continued fraction $K(I/A_n)$ converges if and only if $\sum\limits_{n=0}^{+\infty} ||A_n|| = \infty$.*

We will use the following Theorem to prove our main result.

Theorem 2.9 ([9]). *Let (A_n), (B_n) be two sequences of $\mathcal{M}_m$. If*

$$||(B_{2k-2} \ldots B_2)^{-1} A_{2k-1}^{-1} B_{2k-1} \ldots B_1|| \leq \alpha$$

and

$$||(B_{2k-1} \ldots B_1)^{-1} A_{2k}^{-1} B_{2k} \ldots B_2)|| \leq \beta$$

for all $k \geq 1$, where $0 < \alpha < 1, 0 < \beta < 1$ and $\alpha\beta \leq 1/4$, then the continued fraction $K(Bn/An)$ converges in $\mathcal{M}_m$.

We need to present the following Proposition:

Proposition 2.10 ([7]). *Let $C \in \mathcal{M}_m$ such that $||C|| < 1$, then the matrix $I - C$ is invertible and we have*

$$||(I - C)^{-1}|| \leq \frac{1}{1 - ||C||}. \tag{3}$$

To end this section, we give the following Theorem.

Theorem 2.11 ([11]). *If the function $f(x)$ can be expanded in a power series in the circle $|x - x_0| < r$, namely*

$$f(x) = \sum_{p=0}^{+\infty} \alpha_p (x - x_0)^p, \tag{4}$$

then this expansion remains valid when the scalar argument x is replaced by a matrix A whose characteristic values lie within the circle of convergence.

3 Main results

3.1 The real case

In mathematics, the error function, also called the Gauss error function, is a special function of sigmoid shape that occurs in probability, statistics and partial differential equations describing diffusion. It is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, \mathrm{d}t, \quad |x| < \infty$$

where the coefficient in front of the integral normalizes $\operatorname{erf}(+\infty) = 1$. A plot of $\operatorname{erf}(x)$ over the range $-3 \leq x \leq 3$ is shown as follows.

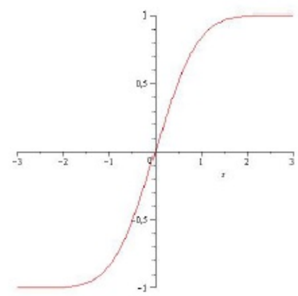

Figure 1: plot of function error.

The power series expansion for the error function is given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)n!} x^{2n+1}. \tag{5}$$

Accordingly, we have

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^n x^{n+1}}{(2n+1)n!} x^n. \tag{6}$$

Lemma 3.1. *Let x be a real number. Then the continued fraction expansion of the error function is*

$$\operatorname{erf}(x) = \left[0; \frac{(2/\sqrt{\pi})x}{1}, \frac{x^2}{3-x^2}, \frac{-(n-1)(2n-1)^2 x^2}{(-1)^{n-1}(n(2n+1)-(2n-1)x^2)}\right]_{n=2}^{+\infty}. \tag{7}$$

Proof. We use Lemma 2.6 for the function

$$g(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{n+1}}{(2n+1)n!} x^n, \quad c_n = \frac{(-1)^n x^{n+1}}{(2n+1)n!}.$$

So, we have

$$\frac{c_0}{1} = \frac{x}{1}, \quad \frac{-c_1 x}{c_0 + c_1 x} = \frac{\frac{x^3}{3}}{\frac{3x-x^3}{3}}, \quad \frac{-c_0 c_2 x}{c_1 + c_2 x} = \frac{\frac{-x^5}{10}}{\frac{-10x^2+3x^4}{30}}.$$

For $n \geq 3$, we get

$$\begin{aligned} c_{n-2} c_n x &= \frac{(-1)^{n-2} x^{n-2+1}}{(2(n-2)+1)(n-2)!} \cdot \frac{(-1)^n x^{n+1}}{(2n+1)n!} x \\ &= \frac{x^{2n+1}}{n(n-1)(2n+1)(2n-3)((n-2)!)^2}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} c_{n-1}+c_n x &= \frac{(-1)^{n-1}x^{n-1+1}}{(2(n-1)+1)(n-1)!}+\frac{(-1)^n x^{n+1}}{(2n+1)n!}x \\ &= \frac{(-1)^{n-1}x^n}{(n-1)!}\left(\frac{1}{2n-1}+\frac{-x^2}{n(2n+1)}\right) \\ &= \frac{(-1)^{n-1}(n(2n+1)-(2n-1)x^2)}{(2n+1)(2n-1)n!}x^n \,. \end{aligned}$$

Then, we obtain

$$\frac{-c_{n-2}.c_n x}{c_{n-1}+c_n x}=\frac{\frac{-x^{2n+1}}{n(n-1)(2n+1)(2n-3)((n-2)!)^2}}{\frac{(-1)^{n-1}(n(2n+1)-(2n-1)x^2)}{(2n+1)(2n-1)n!}x^n}.$$

Therefore, the continued fraction expansion of $\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}g(x)$ is

$$\begin{aligned} \mathrm{erf}(x) &= \left[0;\frac{b_n}{a_n}\right]_{n=1}^{+\infty} \\ &= \left[0;\frac{(2/\sqrt{\pi})x}{1},\frac{\frac{x^3}{3}}{\frac{3x-x^3}{3}},\frac{\frac{-x^5}{10}}{\frac{-10x^2+3x^4}{30}},\frac{\frac{-x^{2n+1}}{n(n-1)(2n+1)(2n-3)((n-2)!)^2}}{\frac{(-1)^{n-1}(n(2n+1)-(2n-1)x^2)}{(2n+1)(2n-1)n!}x^n}\right]_{n=3}^{+\infty}. \end{aligned}$$

Let us define the sequence $(r_n)_{n\geq 1}$ by

$$\begin{cases} r_1=1 \\ r_n=\frac{(2n-1)(2n-3)((n-1)!)}{x^{n-1}}, \text{ for } n\geq 2. \end{cases}$$

Then, we have

$$\begin{cases} \dfrac{r_1 b_1}{r_1 a_1}=\dfrac{(2/\sqrt{\pi})x}{1}, \\ \dfrac{r_1 r_2 b_2}{r_2 a_2}=\dfrac{x^2}{3-x^2}, \\ \dfrac{r_n r_{n+1} b_{n+1}}{r_{n+1}a_{n+1}}=\dfrac{-(n-1)(2n-1)^2x^2}{(1-)^{n-1}(n(2n+1)-(2n-1)x^2)} \text{ for } n\geq 2. \end{cases}$$

By applying the result of Lemma 2.5 to the sequence $(r_n)_{n\geq 1}$, we obtain

$$\begin{aligned} \mathrm{erf}(x) &= \left[0;\frac{(2/\sqrt{\pi})x}{1},\frac{x^2}{3-x^2},\frac{-9x^2}{-10+3x^2},\frac{-(n-1)(2n-1)^2x^2}{(-1)^{n-1}(n(2n+1)-(2n-1)x^2)}\right]_{n=3}^{+\infty} \\ &= \left[0;\frac{(2/\sqrt{\pi})x}{1},\frac{x^2}{3-x^2},\frac{-(n-1)(2n-1)^2x^2}{(-1)^{n-1}(n(2n+1)-(2n-1)x^2)}\right]_{n=2}^{+\infty} \end{aligned}$$

and the proof is complete. □

3.2 The matrix case

According to Theorem 2.11, we have

Definition 3.2. Let A be a matrix in $\mathcal{M}_m$. Then we define the error function by the expression

$$\operatorname{erf}(A) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)n!} A^{2n+1}. \tag{8}$$

Now, we treat the matrix case,

Theorem 3.3. *Let A be a matrix in $\mathcal{M}_m$, such that $||A|| = \alpha$, where $0 < \alpha < \frac{1}{2}$. The continued fraction*

$$\left[0; \frac{(2/\sqrt{\pi})A}{I}, \frac{A^2}{3I - A^2}, \frac{-(n-1)(2n-1)^2 A^2}{(-1)^{n-1}(n(2n+1)I - (2n-1)A^2)}\right]_{n=2}^{+\infty}$$

converges in $\mathcal{M}_m$. Furthermore, this continued fraction represents $\operatorname{erf}(A)$. *So*

$$\operatorname{erf}(A) = \left[0; \frac{(2/\sqrt{\pi})A}{I}, \frac{A^2}{3I - A^2}, \frac{-(n-1)(2n-1)^2 A^2}{(-1)^{n-1}(n(2n+1)I - (2n-1)A^2)}\right]_{n=2}^{+\infty}.$$

Proof. We study the convergence of the continued fraction $K(B_k/A_k)$ with

$$\begin{cases} A_1 = I, A_2 = 3I - A^2, \\ B_1 = (2/\sqrt{\pi})A, B_2 = A^2, \end{cases}$$

and for $k \geq 3$, we have:

$$\begin{cases} A_k = (-1)^{k-2}((k-1)(2k-1)I - (2k-3)A^2), \\ B_k = -(k-2)(2k-3)^2 A^2, \end{cases}$$

we check that the conditions of Theorem 2.9 are satisfied:

$$\begin{aligned} B_{2k-2} \dots B_2 &= \pm((2k-2)-2)(2(2k-2)-3)^2 \dots (4-2)(2 \cdot 4 - 3)^2 A^{2(k-1)} \\ &= \pm(2k-4)(4k-7)^2 \dots 50 A^{2(k-1)}, \\ A_{2k-1}^{-1} &= -((2k-2)(4k-3)I - (4k-5)A^2)^{-1} \end{aligned}$$

and

$$\begin{aligned} &B_{2k-1}B_{2k-3} \dots B_1 \\ &\quad = \pm(2k-1-2)(2(2k-1)-3)^2 \dots (3-2)(2 \cdot 3 - 3)^2 \cdot (2/\sqrt{\pi})A^{2k-1} \\ &\quad = \pm(2/\sqrt{\pi})(2k-3)(4k-5)^2 \cdot \dots \cdot 9A^{2k-1}. \end{aligned}$$

Then, we have:

$$\begin{aligned}
&\|(B_{2k-2}\dots B_2)^{-1}A_{2k-1}^{-1}B_{2k-1}B_{2k-3}\dots B_1\|\\
&=\left\|\frac{1}{(2k-4)(4k-7)^2\dots 50}A^{-2(k-1)}((2k-2)(4k-3)I-(4k-5)A^2)^{-1}(2/\sqrt{\pi})(2k-3)(4k-5)^2\dots 9A^{2(k-1)+1}\right\|\\
&\le\frac{(2/\sqrt{\pi})(2k-3)(4k-5)^2\dots 9}{(2k-4)(4k-7)^2\dots 50}\|A^{-2(k-1)}((2k-2)(4k-3)I-(4k-5)A^2)^{-1}A^{2(k-1)+1}\|.
\end{aligned}$$

Now, the matrices $((2k-2)(4k-3)I-(4k-5)A^2)^{-1}$ and $A^{-2(k-1)}$ commute, so the above inequality becomes

$$\begin{aligned}
&\left\|(B_{2k-2}\dots B_2)^{-1}A_{2k-1}^{-1}B_{2k-1}B_{2k-3}\dots B_1\right\|\\
&\qquad\qquad\le\frac{(2/\sqrt{\pi})(2k-3)(4k-5)^2\dots 9}{(2k-4)(4k-7)^2\dots 50}\left\|(I-\frac{(4k-5)}{(2k-2)(4k-3)}A^2)^{-1}A\right\|.
\end{aligned}$$

By Proposition 2.10 and the fact that $||A||<1/2$, we obtain

$$\left\|(I-\frac{(4k-3)}{(2k-1)(4k-1)}A^2)^{-1}\right\|\le\frac{1}{1-\left\|\frac{(4k-5)}{(2k-2)(4k-3)}A^2\right\|}<1$$

It implies that for all sufficiently large k, we get

$$||B_{2k-2}\dots B_2)^{-1}A_{2k-1}^{-1}B_{2k-1}B_{2k-3}\dots B_1||\le||A||=\alpha<1/2.$$

To prove the second inequality of Theorem 2.9, we have

$$\begin{aligned}
&(B_{2k-1}B_{2k-3}\dots B_1)^{-1}A_{2k}^{-1}B_{2k}\dots B_2\\
&\qquad=\frac{(2k-2)(4k-3)^2\dots 50}{(2/\sqrt{\pi})(2k-3)(4k-5)^2\dots 9}\frac{A^{2(k-1)+2}}{A^{2(k-1)+1}}\left(I-\frac{4k-3}{(2k-1)(4k-1)}A^2\right)^{-1}
\end{aligned}$$

Again using the fact that the matrices $\left(I-\frac{4k-3}{(2k-1)(4k-1)}A^2\right)^{-1}$ and A^{2k-1} commute, the Proposition 2.10 and passing to the norm, we get

$$||(B_{2k-1}B_{2k-3}\dots B_1)^{-1}A_{2k}^{-1}B_{2k}\dots B_2||\le||A||=\alpha<1/2$$

which completes the proof. □

4 Numerical applications

This section will provide some numerical data to illustrate the preceding results. The focus will be on two cases:

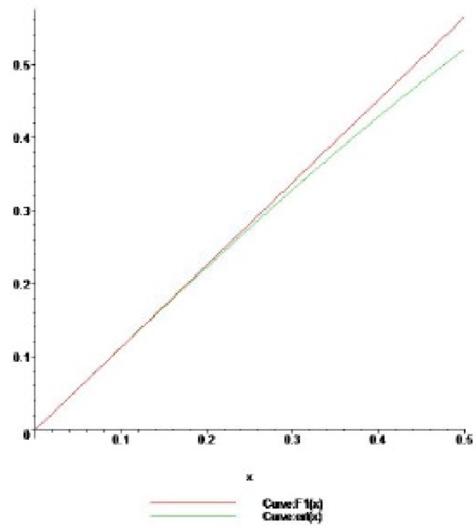

Figure 2: Iteration 1.

4.1 Real case:

- The following table clarifies the differences between $\operatorname{erf}(x)$ and its first convergents when applying Lemma 3.1.

x	$(\operatorname{erf} - F_1)(x)$	$(\operatorname{erf} - F_2)(x)$	$(\operatorname{erf} - F_3)(x)$	$(\operatorname{erf} - F_4)(x)$	$(\operatorname{erf} - F_5)(x)/\operatorname{erf}(x)$
0.005	-0.47015e-7	0.1e-11	0.1e-11	0.1e-11	0.1e-11
0.05	-0.00004698055	0.3525e-7	-0.2e-10	0.1e-10	0
0.075	-0.15841090e-3	0.26742e-6	-0.35e-9	-0.1e-10	-0.1e-10
0.1	-0.3750007e-3	0.11257e-5	-0.27e-8	0	0
0.15	-0.12609036e-2	0.85229e-5	-0.457e-7	0.3e-9	0
0.2	-0.29732442e-2	0.357669e-4	-0.3412e-6	0.27e-8	0
0.25	-0.57684016e-2	0.1085733e-3	-0.16201e-5	0.196e-7	-0.4e-9
0.3	-0.98869906e-2	0.2684219e-3	-0.57742e-5	0.1014e-6	-0.12e-8
0.35	-0.155506548e-1	0.5757641e-3	-0.168817e-4	0.4037e-6	-0.81e-8
0.4	-0.229593118e-1	0.11127771e-2	-0.426832e-4	0.13344e-5	-0.351e-7
0.45	-0.322889054e-1	0.19856118e-2	-0.965652e-4	0.38255e-5	-0.1275e-6

  We can clearly see that F_5 is approximately the exact value of $\operatorname{erf}(x)$.

- The following graphics illustrate the approximations of $\operatorname{erf}(x)$ in terms of continued fractions.

After the first iteration, the convergence of $(F_n(x))$ to $\operatorname{erf}(x)$ is very rapid. It is hard to distinguish between the curve of the convergents and that of $\operatorname{erf}(x)$, for $0 \leq x \leq 0.5$.

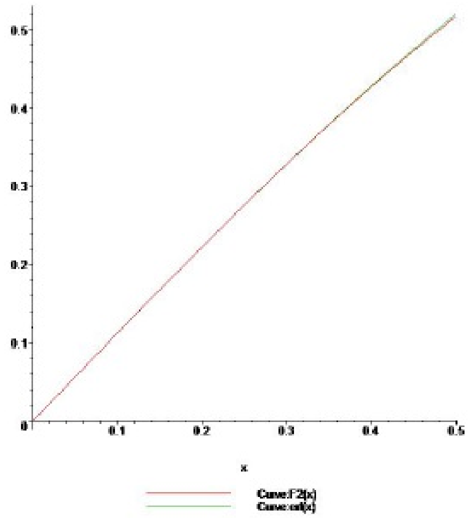

Figure 3: Iteration 2.

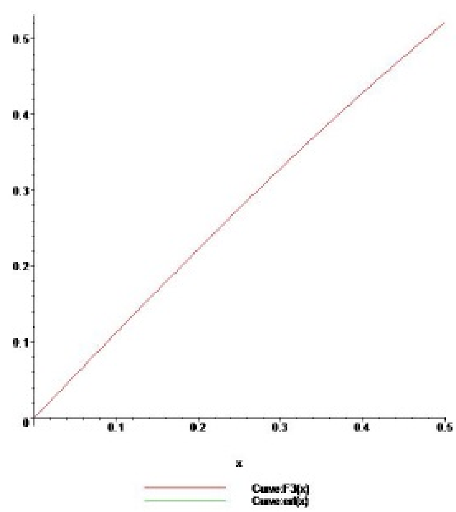

Figure 4: Iteration 3.

4.2 Matrix case:

Example 4.1. Let A be a matrix such that

$$A = \begin{pmatrix} \frac{1}{3} & \frac{1}{17} \\ \\ \frac{-2}{23} & \frac{1}{11} \end{pmatrix}$$

The value of $\mathrm{erf}(A)$ is given by

$$\mathrm{erf}(A) = \begin{pmatrix} 0.3640064111 & 0.06327099117 \\ \\ -0.09353103045 & 0.1032532354 \end{pmatrix}.$$

Using the expansion of Theorem 3.3, we can obtain the following convergents of $\mathrm{erf}(A)$:

$$F_1 = \begin{pmatrix} 0.3636532973 & 0.06317676555 \\ \\ -0.09339174038 & 0.1032884453 \end{pmatrix}.$$

$$F_2 = \begin{pmatrix} 0.3640145285 & 0.06327316896 \\ \\ -0.09353424976 & 0.1032523777 \end{pmatrix}.$$

$$F_3 = \begin{pmatrix} 0.3640062588 & 0.06327095029 \\ \\ -0.09353097002 & 0.1032532515 \end{pmatrix}.$$

$$F_4 = \begin{pmatrix} 0.3640064133 & 0.06327099181 \\ \\ -0.09353103134 & 0.1032532351 \end{pmatrix}.$$

$$F_5 = \begin{pmatrix} 0.3640064109 & 0.06327099114 \\ \\ -0.09353103039 & 0.1032532353 \end{pmatrix}.$$

Example 4.2. Let A be a matrix such that

$$A = \begin{pmatrix} \frac{1}{15} & \frac{1}{9} & 0 \\ \\ 0 & \frac{1}{20} & 0 \\ \\ \frac{1}{7} & 0 & \frac{1}{5} \end{pmatrix}.$$

The value of $\mathrm{erf}(A)$ is given by

$$\mathrm{erf}(A) = \begin{pmatrix} 0.07511398139 & -0.1249466906 & 0 \\ \\ 0 & 0.05637197780 & 0 \\ \\ 0.1581306512 & 0.0018630324 & 0.2227025892 \end{pmatrix}.$$

We calculate $\mathrm{erf}(A)$ by using the expansion given in Theorem 3.3. The first convergents are

$$F_1 = \begin{pmatrix} 0.07511383296 & -0.1249459359 & 0 \\ 0 & 0.05637194255 & 0 \\ 0.1580924886 & 0.001890582379 & 0.2226668223 \end{pmatrix}.$$

$$F_2 = \begin{pmatrix} 0.07511398151 & -0.1249466915 & 0 \\ 0 & 0.05637197782 & 0 \\ 0.1581310167 & 0.001862762625 & 0.2227029304 \end{pmatrix}.$$

$$F_3 = \begin{pmatrix} 0.07511398140 & -0.1249466906 & 0 \\ 0 & 0.05637197779 & 0 \\ 0.1581306484 & 0.001863034561 & 0.2227025865 \end{pmatrix}.$$

$$F_4 = \begin{pmatrix} 0.07511398140 & -0.1249466906 & 0 \\ 0 & 0.05637197779 & 0 \\ 0.1581306513 & 0.001863032439 & 0.2227025892 \end{pmatrix}.$$

$$F_5 = \begin{pmatrix} 0.07511398140 & -0.1249466906 & 0 \\ 0 & 0.05637197779 & 0 \\ 0.1581306512 & 0.001863032453 & 0.2227025892 \end{pmatrix}.$$

Example 4.3. Let A be a matrix such that

$$A = \begin{pmatrix} 0.1 & -0.02 & 0 & 0 & 0 \\ 0 & 0.008 & 0 & 0 & 0 \\ 0.015 & -0.075 & 0.025 & -0.09 & 0 \\ 0.001 & 0 & 0 & 0.05 & 0 \\ 0.002 & 0 & 0 & 0.05 & 0.002 \end{pmatrix}.$$

The value of $\mathrm{erf}(A)$ is given by

$$\mathrm{erf}(A)=\begin{pmatrix} 0.1124632135 & -0.02248610288 & 0 & 0.000007479569022 & 0.00001649965871 \\ -0.000002023944323 & 0.009027140204 & 0 & -0.00002255022425 & 0.02256695121 \\ 0.01685888359 & -0.08458907646 & 0.02820360331 & -0.1013779889 & 0.00002199040709 \\ 0.001121818739 & 0.000001184002249 & 0 & 0.05637197730 & 1.495913804\,10^{-7} \\ 0.002246257465 & 0.000002023944322 & 0 & 0.05637002255 & 0.002257054843 \end{pmatrix}.$$

Now, let us apply the expansion of Theorem 3.3 to obtain the following approximations of $\mathrm{erf}(A)$

$$F_1=\begin{pmatrix} 0.1128379167 & -0.02256758334 & 0 & 0 & 0 \\ 0 & 0.009027033336 & 0 & 0 & 0.02256758334 \\ 0.01692568750 & -0.08462843752 & 0.02820947918 & -0.1015541250 & 0 \\ 0.001128379167 & 0 & 0 & 0.05641895835 & 0 \\ 0.002256758334 & 0 & 0 & 0.05641895835 & 0.002256758334 \end{pmatrix},$$

$$F_2=\begin{pmatrix} 0.1124620912 & -0.02248585859 & 0 & 0.000007522527778 & 0.00001654956112 \\ -0.000002031082501 & 0.009027141660 & 0 & -0.00002256758334 & 0.02256695145 \\ 0.01685868999 & -0.08458902888 & 0.02820360220 & -0.1013778158 & 0.00002200339376 \\ 0.001121796955 & 0.000001188559389 & 0 & 0.05637194255 & 1.504505556\,10^{-7} \\ 0.002246223786 & 0.000002031082501 & 0 & 0.05636998669 & 0.002257056226 \end{pmatrix},$$

$$F_3=\begin{pmatrix} 0.1124632161 & -0.02248610346 & 0 & 0.000007479459802 & 0.00001649953872 \\ -0.000002023926862 & 0.009027140199 & 0 & -0.00002255019727 & 0.02256695120 \\ 0.01685888405 & -0.08458907661 & 0.02820360330 & -0.1013779889, & 0.00002199031849] \\ 0.001121818792 & 0.000001183991006 & 0 & 0.05637197734 & 1.495891960\,10^{-7} \\ 0.002246257546 & 0.000002023926862 & 0 & 0.05637002258 & 0.002257054839 \end{pmatrix},$$

$$F_4=\begin{pmatrix} 0.1124632135 & -0.02248610288 & 0 & 0.000007479574705 & 0.00001649965756 \\ -0.000002023944490 & 0.009027140205 & 0 & -0.00002255020712 & 0.02256695120 \\ 0.01685888359 & -0.08458907650 & 0.02820360330 & -0.1013779888 & 0.00002199033965 \\ 0.001121818739 & 0.000001184002463 & 0 & 0.05637197732 & 1.495914941\,10^{-7} \\ 0.002246257465 & 0.000002023944490 & 0 & 0.05637002257 & 0.002257054844 \end{pmatrix},$$

$$F_5 = \begin{pmatrix} 0.1124632135 & -0.02248610288 & 0 & 0.000007479574479 & 0.00001649965732] \\ -0.000002023944456 & 0.009027140205 & 0 & -0.00002255020711 & 0.02256695120 \\ 0.01685888360 & -0.08458907650 & 0.02820360330 & -0.1013779888 & 0.00002199033962 \\ 0.001121818739 & 0.000001184002441 & 0 & 0.05637197732 & 1.495914896\,10^{-7} \\ 0.002246257465 & 0.000002023944456 & 0 & 0.05637002257 & 0.002257054844 \end{pmatrix}.$$

In the examples above, we can clearly see that F_5 is approximately the exact value of $\mathrm{erf}(A)$. This shows the importance of the continued fractions approach.

References

[1] Cuyt, A.; Petersen, V.B.; Verdonk, B.; Waadeland, H.; Jones, W.B.: *Handbook of continued fractions for special functions*. Springer, New York (2008).

[2] Hayden, T.L.: Continued fractions in Banach spaces. *Rocky Mt J Math* 4 (1974) 367–369.

[3] Golub, G.H.; Van Loan, Ch.F.: *Matrix computations*. Johns Hopkins University Press, Baltimore, MD (1996).

[4] Jones, W.B.; Thron W.J.: Continued fractions, analytic theory and applications. 11 (1980) .

[5] Khovanskii, A.N.: *The Application of Continued Fractions and Their Generalizations to Problems in Approximation Theory*. Groningen, Netherlands: P. Noordhoff (1963).

[6] Lorentzen, L.; Waadeland, H.: Continued fractions with applications. 3 (1992) .

[7] Murphy, G.J.: *C^*-algebras and operator theory*. Academic Press, Inc., Boston, MA (1990).

[8] Negoescu, N.: Convergence theorems on non-commutative continued fractions. *Mathematica - Rev. Anal. Numer. Theor. Approx.* 5 (2) (1976) 165–180.

[9] Raissouli, M.; Kacha, A.: Convergence of matrix continued fractions. *Linear Algebra Appl.* 320 (1-3) (2000) 115–129.

[10] Wynn, P.: One some recent developments in the theory of continued fractions. *Journal of the Society for Industrial and Applied Mathematics: Series B, Numerical Analysis* 1 (1964) 177–197.

[11] Gantmacher, F.R.: *The Theory of Matrices,*. Chelsea, New York, 1959. Elsevier Science Publishers (1992).

[12] Zhao, H.; Zhu, G.: Matrix-valued continued fractions. *J. Approx. Theory* 120 (2003) 136–152.

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