The Distribution Of Prime Numbers And The Continued Fraction

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Abstract. In this paper, we discovered a new sequence contains only ones and the prime numbers, wich can be calculated in two different ways that give the same result, the first way using the greatest common divisor (gcd), the second way consisting of using the denominator of the continued fraction defined by

$$\frac{mb(n-3)-nb(n-4)}{n(m-n+2)-m}=\cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{4-\cfrac{5}{\ddots\,(n-1)-\cfrac{n}{m}}}}}$$

Our sequence defined by

$$a_m(n)=\frac{|n(m-n+2)-m|}{gcd\big(n(m-n+2)-m,mb(n-3)-nb(n-4)\big)}$$

Where $|x|$ denotes the absolute value of x.

1. Introduction

A continued fraction is an expression of the form

$$a_0+\cfrac{b_0}{a_1+\cfrac{b_1}{a_2+\cfrac{b_2}{\ddots}}}$$

Other notation

$$a_0+\frac{b_0}{a_1+}\,\frac{b_1}{a_2+}\,\frac{b_2}{a_3+}\cdots$$

Where a_i and b_i are either rational numbers or real numbers.

The distribution of prime numbers has been analyzed for a formula helpful in generating the prime numbers or testing if the given numbers is prime. In this paper, we present some known formulas.

Mills showed that there exists a real number $A>1$ such that $f(n)=\left[\,A^{3^n}\,\right]$ is a prime number for any integers n, approximately A=1.306377883863,.. (see A051021). The first few values

$f(n)=$ {2, 11, 1361, 2521008887, 1602223620400981813183132 0183,..}, (see A051254)

Euler's quadratic polynomial n^2+n+41 is prime for all n between 0 and 39, however, it is not prime for all integers.

The Rowland sequence of prime numbers composed entirely of 1's and primes, the sequence defined by the recurrence relation

$$r(n) = r(n-1) + \gcd(n, r(n-1)); \;\; r(1) = 7$$

The sequence of differences $r(n+1) - r(n)$

1, 1, 1, 5, 3, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 23, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,1, 1, 1, 47, 3, 1, 5, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 101, 3, 1, 1, 7, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 13, 1, 1, 1, 1, 1, 1, 1, 1, 1,.. (see A132199).

For more details and formulas see [1] and [2]. In this paper, we present an interesting sequence which plays the same role as Rowland's sequence composed of a prime number or 1. Moreover, our sequence gives all distinct prime numbers in order (conjecture 2 except the prime numbers 2 and 3).

In this paper, we use the recursive formula defined by

$$b(n) = (n+2)\big(b(n-1) - b(n-2)\big)$$

With the starting conditions $b(-1) = 0$ and $b(0) = 1$. Some few values of $b(n)$

0, 1, 3, 8, 25, 102, 539,3496, 26613, 231170, 2250127, 24227484, …(see A051403)

Also, we use the recursive formula is as follows

$$c(n) = (n+2)\big(c(n-1) - c(n-2)\big)$$

With the starting conditions $c(1) = 1$ and $c(2) = 4$. Some few values of $c(n)$

1, 4, 15, 66, 357, 2328, 17739, 154110, 1500081, 16151652, 190470423,…

In this section, we give an explicit formula for the the continued fraction in the following theorem

Theorem 1. For all integers $n \geq 3$. The continued fraction

$$\frac{mb(n-3) - nb(n-4)}{n(m-n+2) - m} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5}{\ddots \, (n-1) - \cfrac{n}{m}}}}} \qquad (1)$$

Where m is a polynomial in term n.

Proof. Let

$$a_1 = 2a_2 - 3a_3 \,; \; a_2 = 3a_3 - 4a_4 \,; \, a_3 = 4a_4 - 5a_5 \,; \; a_4 = 5a_5 - 6a_6$$

Then we have

$$\frac{a_2}{a_1} = \frac{a_2}{2a_2 - 3a_3} = \frac{1}{\frac{2a_2 - 3a_3}{a_2}} = \frac{1}{2 - \frac{3a_3}{a_2}} = \cfrac{1}{2 - \cfrac{3}{\frac{3a_3 - 4a_4}{a_3}}}$$

$$= \cfrac{1}{2-\cfrac{3}{3-\cfrac{4a_4}{a_3}}} = \cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{\cfrac{4a_4-5a_5}{a_4}}}} = \cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{4-\cfrac{5a_5}{a_4}}}}$$

After some simplification, we find

$$\frac{a_2}{a_1} = \cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{4-\cfrac{5}{\ddots \, (n-1)-\cfrac{na_n}{a_{n-1}}}}}} \qquad (2)$$

From (1) and (2), we have

$$ma_n = a_{n-1} \qquad (3)$$

We write a_1 in terms of a_{n-1} and a_n

$$a_1 = 2a_2 - 3a_3 = \cdots = (n-1)a_{n-1} - (n^2-2)a_n \qquad (4)$$

Substituting (3) into (4), we find

$$a_1 = (n(m-n+2)-m)a_n$$

Using the same procedure for a_2, we have

$$a_2 = 3a_3 - 4a_4 = 8a_4 - 15a_5 = 25a_5 - 48a_6 = \cdots$$

We observe that

$$a_2 = b(n-3)a_{n-1} - nb(n-4)a_n \qquad (5)$$

Substtiting (3) into (5), we get

$$a_2 = \big(mb(n-3) - nb(n-4)\big)a_n$$

Returning to (2), we obtain

$$\frac{a_2}{a_1} = \frac{mb(n-3)-nb(n-4)}{n(m-n+2)-m} = \cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{4-\cfrac{5}{\ddots \, (n-1)-\cfrac{n}{m}}}}} \qquad (6)$$

This complet the proof.

Theorem 2. The denominator of the continued fraction can be expressed as follows

$$n(m-n+2)-m = 2\big(mb(n-3)-nb(n-4)\big) - 3\big(mc(n-3)-nc(n-4)\big)$$

Proof. Similarly, using the same procedure as that of proving the theorem 1.

We have

$$a_3 = 4a_4 - 5a_5 = 15a_5 - 24a_6 = 66a_6 - 105a_7 = \cdots$$

We observe that

$$a_3 = c(n-3).a_{n-1} - nc(n-4).a_n \qquad (7)$$

Substituting (3) into (7), we find

$$a_3 = \big(mc(n-3) - nc(n-4)\big)a_n$$

Then, we have

$$a_1 = 2a_2 - 3a_3$$

$$(n(m-n+2)-m)a_n = \big[2\big(mb(n-3) - nb(n-4)\big) - 3\big(mc(n-3) - nc(n-4)\big)\big].a_n$$

Then, we get

$$n(m-n+2) - m = 2\big(mb(n-3) - nb(n-4)\big) - 3\big(mc(n-3) - nc(n-4)\big)$$

This complet the proof.

The sequence which is actually important is the next one.

2. The sequence of the unreduced denominator of the continued fraction

we can obtain the sequence of the unreduced denominator of the continued fraction as follows

$$a_m(n) = \frac{|n(m-n+2)-m|}{gcd\big(n(m-n+2)-m, mb(n-3) - nb(n-4)\big)}$$

Where gcd(x, y) denotes the greatest common divisor of x and y.

Conjecture 2.1. For all integers $n \geq 3$ and $m = n+1$. The continued fraction

$$\frac{b(n-2)+b(n-3)}{2n-1} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5}{\ddots \, (n-1) - \cfrac{n}{n+1}}}}} ; n \geq 3$$

The sequence of the unreduced denominator is as follows

$$a(n) = \frac{2n-1}{gcd\big(2n-1, b(n-2)+b(n-3)\big)} ; n \geq 2$$

The values of $a(n)$

3, 5, 7, 3, 11, 13, 1, 17, 19, 1, 23, 1, 1, 29, 31, 1, 1, 37, 1, 41, 43, 1, 47, 1, 1, 53, 1, 1, 59, 61, 1, 1, 67, 1, 71, 73, 1, 1, 79, 1, 83, 1, 1, 89, 1, 1, 1, 97, 1, 101, 103, 1, 107, 109, 1, 113, 1, 1, 1, 1, 1, 1, 127, 1, 131, 1, 1, 137, 139, 1, 1, 1, 1, 149, 151, 1, 1, 157, 1, 1, 163, 1, 167,…

For $n \geq 2$, $a(n) = 2n-1$ if $2n-1$ is prime (except for n=5), 1 otherwise .

Every term of this sequence is either a prime number or 1.

Conjecture 2.2. For all integers $n \geq 4$ and $m = n-3$. The continued fraction

$$\frac{3b(n-3)-b(n-2)}{2n-3}=\cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{4-\cfrac{5}{\ddots\cfrac{}{(n-1)-\cfrac{n}{n-3}}}}}}\;;n\geq 4$$

The sequence of the unreduced denominator is as follows

$$a(n)=\frac{2n-3}{gcd\big(2n-3,3b(n-3)-b(n-2)\big)};n\geq 2$$

The values of $a(n)$

1, 1, 5, 7, 1, 11, 13, 1, 17, 19, 1, 23, 1, 1, 29, 31, 1, 1, 37, 1, 41, 43, 1, 47, 1, 1, 53, 1, 1, 59, 61, 1, 1, 67, 1, 71, 73, 1, 1, 79, 1, 83, 1, 1, 89, 1, 1, 1, 97, 1, 101, 103, 1, 107, 109, 1, 113, 1, 1, 1, 1, 1, 1, 127, 1, 131, 1, 1, 137, 139, 1, 1, 1, 1, 149, 151, 1, 1, 157, 1, 1, 163, 1, 167,…

For $n\geq 4,\ a(n)=2n-3$ if $2n-3$ is prime, 1 otherwise .

This sequence finds all odd prime numbers of the form $2n-3$ (except for the prime 3) in order.

Conjecture 2.3. For all integers $n\geq 3$ and $m=-1$. The continued fraction

$$\frac{b(n-3)+nb(n-4)}{n^2-n-1}=\cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{4-\cfrac{5}{\ddots\cfrac{}{(n-1)-\cfrac{n}{-1}}}}}}\;;n\geq 3$$

The sequence of the unreduced denominator is as follows

$$a(n)=\frac{n^2-n-1}{gcd(n^2-n-1,\ b(n-3)+nb(n-4))}\ ;\ for\ n\geq 2$$

The values of $a(n)$

1, 5, 11, 19, 29, 41, 11, 71, 89, 109, 131, 31, 181, 19, 239, 271, 61, 31, 379, 419, 461, 101, 29, 599, 59, 701, 151, 811, 79, 929, 991, 211, 59, 41, 1259, 1, 281, 1481, 1559, 149, 1721, 1, 61, 1979, 2069, 2161, 1, 2351, 79, 2549, 241, 1, 2861, 2969, 3079, 3191,…(see A356247)

We conjectured that :

∗ Every term of this sequence is either a prime number or 1.

∗ Except for 5, the primes all appear exactly twice, such that

$$a(n)=a(a(n)-n+1)$$

Consequently, let us consider the values of n and m such that we get:

$$a(n)=a(m)=n+m-1$$

And

$$a(n) = a(m) = \gcd(n^2 - n - 1, m^2 - m - 1)$$

Conjecture 2.4. For all integers $n \geq 3$ and $m = -2$. The continued fraction

$$\frac{2b(n-3) + nb(n-4)}{n^2 - 2} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5}{\ddots (n-1) - \cfrac{n}{-2}}}}} \; ; \; for\ n \geq 3$$

The expression of the sequence $a(n)$ is as follows

$$a(n) = \frac{n^2 - 2}{gcd\left(n^2 - 2,\ 2b(n-3) + nb(n-4)\right)} \; ; \; for\ n \geq 3$$

The values of $a(n)$.

7, 7, 23, 17, 47, 31, 79, 7, 17, 71, 167, 97, 223, 127, 41, 23, 359, 199, 439, 241, 31, 41, 89, 337, 727, 1, 839, 449, 137, 73, 1087, 577, 1223, 647, 1367, 103, 1, 47, 73, 881, 1, 967, 1, 151, 2207, 1151, 2399, 1249, 113, 193, 401, 1, 3023, 1567, 191, 41, 71…

The sequence $a(n)$ takes only 1's and primes.

Conjecture 2.5. For all integers $n \geq 3$ and $m = n + 2$. The continued fraction

$$\frac{(n+1)b(n-3) - b(n-4) - (n-1)b(n-5)}{3n - 2} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5}{\ddots (n-1) - \cfrac{n}{n+2}}}}}$$

The expression of the sequence $a(n)$ is as follows

$$a(n) = \frac{3n - 2}{gcd\left(3n - 2,\ (n+1)b(n-3) - b(n-4) - (n-1)b(n-5)\right)} \; ; \; for\ n \geq 3$$

The values of $a(n)$ for $n \geq 3$

7 , 5, 13, 2, 19, 11, 5, 1, 31, 17, 37, , 1, 43, 23, 1, 1, 1, 29, 61, 1, 67, 1, 73, 1, 79, 41, 1, 1, 1, 47, 97, 1, 103, 53, 109, 1, 1, 59, 1, 1, 127, 1, 1, 1, 139, 71, 1, 1, 151, 1, 157, 1, 163, 83, 1, 1, 1, 89, 181, 1, 1, 1, 193, 1, 199, 101, 1, 1, 211,…

The sequence $a(n)$ contains only ones and the primes.

Conjecture 2.6. For all integers $n \geq 3$ and $m = n + 3$. The continued fraction

$$\frac{(n+2)b(n-3) - b(n-4) - (n-1)b(n-5)}{4n - 3} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5}{\ddots (n-1) - \cfrac{n}{n+3}}}}}$$

The expression of the sequence $a(n)$ is as follows

$$a(n) = \frac{4n-3}{gcd\big(4n-3,\ (n+2)b(n-3) - b(n-4) - (n-1)b(n-5)\big)} \ ;\ for\ n \geq 3$$

The values of $a(n)$ for $n \geq 3$

3, 13, 17, 7, 5, 29, 11, 37, 41, 1, 7, 53, 19, 61, 1, 23, 73, 1, 1, 1, 89, 31, 97, 101, 1, 109, 113, 1, 1, 1, 43, 1, 137, 47, 1, 149, 1, 157, 1, 1, 1, 173, 59, 181, 1, 1, 193, 197, 67, 1, 1, 71, 1, 1, 1, 229, 233, 79, 241, 1, 83, 1, 257, 1, 1, 269, 1, 277,…

The sequence $a(n)$ takes only 1's and primes.

Remark

There is many sequence that contains only 1's and the primes for various values of m.

Acknowldgements

I would like to thank Bill Mceachen for the numerous comments and suggestions. Thanks go also to Jon. E. Schoenfield, Alois. P. Heinz, Michael De Vlieger and the other editor-in-chief of the on-line encyclopedia of integers sequences (Oeis).

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