Momentum ConservationIncluded In Fermat's Least Time Principle and Wave Mechanics

Francesco R. Ruggeri Hanwell, N.B. Sept. 20, 2022

According to (1) Fermat's minimum time principle is identical to Hyugen's wave principle. In a previous note (2) we considered two ray light systems (in particular Snell's law and reflection from a mirror moving at constant velocity v) and argued that Fermat's principle is equivalent to a hypothetical velocity approach where H sin(AA) = v(A relative) and H sin(BB = v(B relative). Here H is the hypothetical velocity which acts as a hypotenuse and AA and BB are the incident and reflected rays in the case of a moving mirror (measured relative to the normal) and the incident and refracted rays for Snell's law. The hypothetical velocity approach, however, is exactly in the form needed for conservation of momentum parallel to the medium surface which holds for both the moving mirror and Snell's law if: 1/wavelength(A) is proportional to 1/v(A relative) and the same result for B. This is equivalent to considering a fixed frequency f which applies to A and B (and is not 1/energy proportional to wavelength) so that vrel(A) = (same frequency) wavelength(A) and the same for B. As a result, conservation of momentum parallel to the medium surface is implicitly included in Fermat's principle.

In the case of quantum mechanics of a free particle represented by exp(ipx) where p is momentum, conservation of momentum is also built in because exp(ipx) is orthogonal to exp(ip1x). For <exp(ip1x) A W> where W is a wavelength only results consistent with conservation of momentum occur as is well-known in the literature. We argue that quantum mechanics of a free particle is statistical and linked to the idea of an equilibrium for which one does not consider time. Thus for exp(-iEt+ipx), t and x may be considered as independent. This implies that exp(ipx) must carry the idea of conservation of momentum by itself because one is not following a particle via x(t). "X' may take on any value so this is an equilibrium type of scenario and the orthogonality of different exp(ipx) vectors ensures conservation of momentum. Thus a wave approach again implicitly contains the notion of conservation of momentum.

Momentum Conservation

The idea of momentum and momentum conservation is present in Newtonian mechanics, but must be imposed as an independent condition. Furthermore, waves such as those on a string, sound and water waves may also be described in terms of Newtonian mechanics of the medium. As a result, one would not necessarily expect a wave description to include conservation of momentum. It is not even clear that a wave itself should have momentum. Long after Newton, Maxwell wrote his electromagnetic equations based on experimental results and discovered an equation for a wave which he considered to be light. In this case p=hbar/wavelength and E=pc emerge with p and E being average energies. Thus there is a link between the Newtonian concepts of momentum p and E and a wave. Furthermore it was already known that velocity = frequency* wavelength for a wave, so E and p (Newtonian concepts) now have extended meaning. Experimentally, the momentum of a photon may be measured as it strikes a thin foil. On the other hand, interference experiments were already done by Young around 1801.

Fermat's Principle of Least Time

Fermat's principle of least time may be applied to problems involving light. For example, one may derive Snell's law using this principle as well as a relationship between the angle of reflection in terms of the incident angle for light reflecting from a mirror moving at a constant velocity v as shown in (3). This same problem was solved by Einstein in the early 1900s using Lorentz transformations. In (1) it is stated that Fermat's principle is equivalent to Huygen's wave treatment. Thus rays which move in a specific direction are linked to a wave which is more isotropic. One may use wave considerations together with the reflected/incident angle relation from (3) to find a relationship between the incident wavelength and reflected from a mirror moving at a constant velocity v as is done in (4). Given that p (magnitude) = hbar/ wavelength one may obtain information about p and then E form E=pc.

Built In Conservation of Momentum

The results for incident and scattered wavelengths together with the Fermat's principle result for incident/scattered angles are compatible with the Newtonian concept of conservation of momentum parallel to the mirror surface i.e.

p(incident) sin(AA) = p (reflected) sin(BB) ((1))

As a result, a wave treatment together with Fermat's principle already includes the notion of momentum conservation. It does not have to be imposed a priori. This same observation applies to the derivation of Snell's law.

We ask: How does one see this the Fermat's principle formulation?

Fermat's Principle Includes Momentum Conservation

In (2) we argued that Fermat's principle is equivalent to a hypothetical velocity H approach. This same approach allows one to show how Lorentz transformations are linked to Fermat's principle as shown in (5). The hypothetical velocity approach treats H as a hypotenuse with a relative velocity being the adjacent of an angle 90-AA or 90-BB. Thus:

H sin(AA) = v (A relative) = c - v cos(A) and H sin(BB) = v(B relative) = c + v cos(BB) ((2))

((2)) applies to a moving mirror. For Snell's law:

H sin(AA) = c/n1 and H sin(BB) = c/n2 ((3))

sin(AA) and sin(BB) are present in ((1)), the conservation of momentum parallel to the mirror surface relation, thus from ((2)) one expects that:

1/ wavelength incident = 1/v(A relative)   and 1/ wavelength reflected = 1/v(B relative) ((4))

Where 1/wavelength = magnitude of momentum.

For Snell's law ((4)) is p(incident)/p(refracted) = n1/n2.

(Note we have already shown in previous notes that the results of wavelength/wavelength reflected and incident/reflected angle both given in (4) are consistent with ((2)) parallel momentum conservation.

((4)) seems to suggest that there is a hypothetical frequency which is the same for the reflected and incident ray causing ((4)) to hold.

Thus a main point we wish to make is that Fermat's principle seems to already include the notion of conservation of momentum parallel to a medium surface if one considers p=hbar/wavelength.

Quantum Mechanics

Fermat's principle of time, which is equivalent to Huygen's wave approach, applies to light which is a quantum object. In fact, Fermat's principle seems to link ray behaviour (which is similar to particle behaviour) to wave behaviour. An interesting point about wave behaviour is the idea that a wavelength applies in more than one dimension whereas a ray only moves in one. Thus a wave picture may be used to describe pictorially two slit interference for example.

One may note that Fermat's principle is linked to special relativity. For example, in (5) the hypothetical velocity approach (equivalent to Fermat's principle) is shown to be directly linked to Lorentz transformations. In special relativity -Et+px is an invariant for both light and particles with rest mass. P scales x and E scales time so this suggests periodic behaviour i.e. 1/p as a wavelength and E as a frequency.

We would like to suggest that the wave scenario seems to be more isotropic than the particle one even though a particle may be moving along a ray. In A= -Et+px, one may consider t and x as being independent. exp(ipx) and exp(-iEt) then satisfy eigenfunction equations:

id exp(-iEt) / dt = E exp(-iEt)   and -i d/dx exp(ipx) = p exp(ipx)   ((5))

exp(ipx) is a function which exists throughout a spatial region and is decoupled from time. This is reminiscent of a classical statistical mechanical equilibrium scenario. It also does not depend on time and is associated with filling phase space rather than tracing a particle through x(t). The periodicity of exp(ipx), however, is not present. We argue that an important feature of this periodicity is that it ensures that different exp(ipx) vectors are orthogonal using Integral dx exp(-ipx) exp(ipx) as the definition of an inner product. We argue that this is linked to conservation of momentum being implicitly part of this wave-probability description. For example, the inner product:

Integral dx exp(-ip1 x) exp(ipx) = 0 ((6))

suggests that a particle with momentum p cannot simply transform into a particle of momentum p1. For a more complex situation e.g. a wavefunction W(x)= Sum over p a(p)exp(ipx), an interaction described by V(x) a potential i.e. V(x) W(x) may create different exp(ipx) type vectors i.e.

V(x)W(x) = Sum over p b(p) exp(ipx)   ((7))

Integral exp(-ip1x) V(x)W(x) dx ((8)) automatically brings in conservation of momentum. This is already well known in the literature, but we wish to associate this idea of momentum conservation with that already contained in Fermat's principle. In previous notes we argued that different exp(ipx) vectors are orthogonal in order to establish the identity of p over space, but if p identifies a state then this seems to be associated with a kind of conservation i.e. the value of p does not change. The two ideas seem to be linked as seen in ((8)). Thus exp(ipx) statistically represents a particle of momentum p throughout x (with t out of the picture in say a bound state problem) and must establish its identity i.e. p does not change. Thus exp(ipx) is orthogonal to other exp(ip1 x)'s.

Conclusion

In conclusion, we argue that Fermat's principle of least time, associated with a wave description, can yield a relationship between incident wavelength and angle (AA) and reflected wavelength and angle (BB) for a moving mirror reflection problem as shown in (4). This relationship already contains the notion of conservation of momentum parallel to the mirror surface built in as one may show that p(incident) sin(AA) = p(refleted) sin(BB) is consistent with the results of (4). We argue that one may describe how this relationship arises from Fermat's principle by using the hypothetical velocity H approach described in (2) in particular H sin(AA) = v(relative A) and H sin(BB) = v(relative B). Thus for p=hbar/wavelength it seems that 1/v(relative A) = 1/ wavelength (A)   and 1/v(relative B) = 1/Wavelength(B). Given that v=f wavelength for a wave, it appears there is a common hypothetical frequency for A and B. These same arguments may be applied to Snell's law. Thus we stress that conservation of momentum parallel to the medium is already included in a wave treatment and does not have to be enforced as an independent condition.

What about quantum mechanics for a free particle? In such a case vectors exp(ipx) and exp(-iEt) represent the free particle. We suggest that x and t be decoupled as in a statistical equilibrium situation. Furthermore we argue that the orthogonality of various exp(ipx) vectors is linked to conservation of momentum. In particular if one has a wavefunction W(x) = Sum over p a(p)exp(ipx) and a potential V(x) then V(x)W(x) = Sum over p b(p) exp(ipx). The inner product: Integral dx exp(-ip1 x) V(x)W(x) ensures that momentum is conserved i.e. only the exp(ipx) term of V(x)W(x) is chosen. This is already well-known in the literature. exp(ipx) represents a state of momentum p throughout space. Thus p identifies this state which means that this p should be conserved. Thus there seems to be a link between quantum identity and conservation. The wave picture seems to have momentum conservation built-in and is more linked to a statistical equilibrium scenario than the classical x(t).

References

1.https://en.wikipedia.org/wiki/Fermat%27s_principle

2. Ruggeri, Francesco R. The Equivalence of Fermat's Least Time with a Hypothetical Velocity (preprint, zenodo, 2022)

3. Gjurchinovski, A. Einstein's Mirror and Fermat's Principle of least time (Am. J. of Phys 72 (10) Oct. 2004) http://muj.optol.cz/~richterek/data/media/ref_str/gjurchinovski2004.pdf

4.Gjurchinovski, A. The Doppler Effect from a uniformly moving mirror https://arxiv.org/pdf/physics/0409014.pdf

5. Ruggeri, Francesco R. Fermat's Principle, Hypothetical Velocity, Lorentz Transformations and a Moving Mirror (preprint, zenodo, 2022)