The Distribution Of Prime Numbers And The Continued Fraction

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Abstract. In this paper, we discovered a new sequence contains only ones and the prime numbers, wich can be calculated in two different ways that give the same result, the first way is using the greatest common divisor (gcd) and the second way, we can obtain this sequence from the denominator of the continued fraction defined by

$$\frac{B_m(n)}{A_m(n)} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5}{\ddots \, (n-1) - \cfrac{n}{m}}}}}$$

Our sequence sequence defined by

$$a_m(n) = \frac{|A_m(n)|}{gcd\big(A_m(n), B_m(n)\big)}$$

Where $|x|$ denotes the absolute value of x.

Introduction

A continued fraction is an expression of the form

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{\ddots}}}$$

Other notation

$$a_0 + \frac{b_0}{a_1 +}\,\frac{b_1}{a_2 +}\,\frac{b_2}{a_3 +}\cdots$$

Where a_i and b_i are real numbers.

The distribution of prime numbers has been analyzed for a formula helpful in generating the prime numbers or testing if the given numbers is prime. In this paper, we present some known formulas.

Mill's showed that there exists a real number $A > 1$ such that $f(n) = \left[\, A^{3^n} \,\right]$ is a prime number for any integers n, approximately A=1.306377883863,.. (see A051021). The first few values

$f(n)$ = {2, 11, 1361, 2521008887, 16022236204009818131831320183,..}, (see A051254)

Euler's quadratic polynomial $n^2 + n + 41$ is prime for all n between 0 and 39, however, it is not prime for all integers.

The Rowland sequence of prime numbers composed entirely of 1's and primes, the sequence defined by the recurrence relation

$$r(n) = r(n-1) + \gcd(n, r(n-1)); \;\; r(1) = 7$$

The sequence of differences $r(n+1) - r(n)$

1, 1, 1, 5, 3, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 23, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,1, 1, 1, 47, 3, 1, 5, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 101, 3, 1, 1, 7, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 13, 1, 1, 1, 1, 1, 1, 1, 1, 1,.. (see A132199).

For more details and formulas see [1] and [2]. In this paper, we present an interesting sequence which plays the same role as Rowland's sequence composed by a prime number or 1. Moreover, our sequence gives all prime numbers in order.

In this paper, we use the recursive formula defined by

$$b(n) = (n+2)\big(b(n-1) - b(n-2)\big)$$

With the starting conditions $b(-1) = 0$ and $b(0) = 1$.

Conjecture 1. The continued fraction

$$\frac{b(n-2)+b(n-3)}{2n-1} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5}{\ddots\, (n-1) - \cfrac{n}{n+1}}}}}; n \geq 3$$

The following expression finds all odd prime numbers

$$a(n) = \frac{2n-1}{gcd\big(2n-1, b(n-2)+b(n-3)\big)}; n \geq 2$$

Where gcd(x, y) denotes the greatest common divisor of x and y.

The values of $a(n)$

3, 5, 7, 3, 11, 13, 1, 17, 19, 1, 23, 1, 1, 29, 31, 1, 1, 37, 1, 41, 43, 1, 47, 1, 1, 53, 1, 1, 59, 61, 1, 1, 67, 1, 71, 73, 1, 1, 79, 1, 83, 1, 1, 89, 1, 1, 1, 97, 1, 101, 103, 1, 107, 109, 1, 113, 1, 1, 1, 1, 1, 1, 127, 1, 131, 1, 1, 137, 139, 1, 1, 1, 1, 149, 151, 1, 1, 157, 1, 1, 163, 1, 167…

Every term of this sequence is either a prime number or 1.

Conjecture 2. The continued fraction

$$\frac{3b(n-3)-b(n-2)}{2n-3} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5}{\ddots\, (n-1) - \cfrac{n}{n-3}}}}} ; n \geq 4$$

The following expression finds all odd prime numbers (except for 3) in order.

$$a(n) = \frac{2n-3}{gcd\big(2n-3, b(n-2) - 3b(n-3)\big)}; n \geq 2$$

The values of $a(n)$

1, 1, 5, 7, 1, 11, 13, 1, 17, 19, 1, 23, 1, 1, 29, 31, 1, 1, 37, 1, 41, 43, 1, 47, 1, 1, 53, 1, 1, 59, 61, 1, 1, 67, 1, 71, 73, 1, 1, 79, 1, 83, 1, 1, 89, 1, 1, 1, 97, 1, 101, 103, 1, 107, 109, 1, 113, 1, 1, 1, 1, 1, 1, 127, 1, 131, 1, 1, 137, 139, 1, 1, 1, 1, 149, 151, 1, 1, 157, 1, 1, 163, 1, 167…

the sequence $a(n)$ takes only 1's and primes

Conjecture 3. The continued fraction

$$\frac{b(n-3) + nb(n-4)}{n^2 - n - 1} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5}{\ddots \, (n-1) - \cfrac{n}{n-(n+1)}}}}} ; n \geq 3$$

The following expression finds all odd prime numbers which ends with a 1 or 9 (except for 5)

$$a(n) = \frac{n^2 - n - 1}{gcd(n^2 - n - 1, \; b(n-3) + nb(n-4))} \;\; ; \; for \; n \geq 2$$

The values of $a(n)$

1, 5, 11, 19, 29, 41, 11, 71, 89, 109, 131, 31, 181, 19, 239, 271, 61, 31, 379, 419, 461, 101, 29, 599, 59, 701, 151, 811, 79, 929, 991, 211, 59, 41, 1259, 1, 281, 1481, 1559, 149, 1721, 1, 61, 1979, 2069, 2161, 1, 2351, 79, 2549, 241, 1, 2861, 2969, 3079, 3191,…(see A356247)

We conjectured that :

∗ Every term of this sequence is either a prime number or 1.

∗ Except for 5, the primes all appear exactly twice, such that

$$a(n) = a(a(n) - n + 1)$$

Consequently, let us consider the values of n and m such that we get:

$$a(n) = a(m) = n + m - 1$$

And

$$a(n) = a(m) = \gcd(n^2 - n - 1, m^2 - m - 1)$$

Conjecture 4. The continued fraction

$$\frac{2b(n-3)+nb(n-4)}{n^2-2}=\cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{4-\cfrac{5}{\ddots\,(n-1)-\cfrac{n}{n-(n+2)}}}}}$$

The expression of the sequence $a(n)$ as follow

$$a(n)=\frac{n^2-2}{gcd\left(n^2-2,\ 2b(n-3)+nb(n-4)\right)}\ ;\ for\ n\geq 2$$

The values of $a(n)$.

7, 7, 23, 17, 47, 31, 79, 7, 17, 71, 167, 97, 223, 127, 41, 23, 359, 199, 439, 241, 31, 41, 89, 337, 727, 1, 839, 449, 137, 73, 1087, 577, 1223, 647, 1367, 103, 1, 47, 73, 881, 1, 967, 1, 151, 2207, 1151, 2399, 1249, 113, 193, 401, 1, 3023, 1567, 191, 41, 71…

the sequence $a(n)$ takes only 1's and primes.

Generalisation

The unreduced denominator $a_m(n)$ can be calculated by using the denominator of the continued fraction as follow

$$\frac{mb(n-3)-nb(n-4)}{n(m-n+2)-m}=\cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{4-\cfrac{5}{\ddots\,(n-1)-\cfrac{n}{m}}}}}$$

Where n is a positive integers and m is a real number.

we can obtain the sequence of the unreduced denominator of the continued fraction as follow

$$a_m(n)=\frac{|n(m-n+2)-m|}{gcd\left(n(m-n+2)-m, mb(n-3)-nb(n-4)\right)}$$

For $m=n+1$, we obtain the sequence in the conjecture 1.

For $m=n-3$, we find the sequence in the conjecture 2.

For $m=-1$, we find the sequence in the conjecture 3.

For $m=-2$, we obtain the sequence in the conjecture 4.

References

[**1**] Eric S. Rowland, A Natural Prime-Generating Recurrence, Journal of Integer Sequences, Vol. 11 (2008).

[**2**] Benoit Cloitre, 10 conjectures in additive number theory, https://arxiv.org/abs/1101.4274

[**3**] N. J. A. Sloane, The On-line Encyclopedia of integers sequences, https://oeis.org