Fermat's Principle and Lorentz Transformations for a Moving Mirror

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In (1) it is stated that Fermat's principle of least time applied to light is equivalent to a statement that light travels along a geodesic i.e. is a fundamental principle. In fact, this principle seems to be used frequently (i.e. light travels in straight lines ignoring curved space) without being explicitly called Fermat's principle. In (2) it is shown that one may solve for the reflected angle in terms of the incident angle for a light ray reflecting from a mirror moving at constant velocity v. It is stated that this allows one to find such a relationship without using Lorentz transformations.

In this note we argue that Lorentz transformations follow from using Fermat's principle in two frames. The first frame is that of the lab and is used in (2) to directly find the reflected angle in terms of the incident angle. The second frame is the moving frame such that one has a stationary mirror. Again Fermat's principle applies. One may deduce the Lorentz transformation by assuming that c is the same in all frames (Einstein's assumption) and that x in the direction perpendicular to motion is unchanged in different frames. This has already been done by Einstein in 1905 (3).

In this note we argue that conservation of momentum perpendicular to frame motion suggests that not only x, but perpendicular momentum is unchanged by frame motion. Consider a frame moving in the x direction. In this frame light is sent in the y direction and hits a wall and bounces back and forth. This light is then energy trapped at a fixed x point in the moving frame, in other words localized or rest energy. From the point of view of the lab frame, the light moves in a triangle, but the momentum also follows this triangle. If both momentum and length perpendicular to the motion of the frame remain unchanged then (x,t) and (p,E) transform in the same manner for light because E=pc. Furthermore the Lorentz transformation obtained seems to hold to particles with rest mass because the light in the moving frame bouncing in the y direction is rest energy i.e. rest mass cc. In fact, the result for a moving mirror (i.e. reflected angle in terms of incident) given in (3) can be obtained from a transformation of p,E without considering x or t. Thus Fermat's principle together with the assumption that light has the same speed in all frames (an assumption used in (2)) and the idea that momentum and length components perpendicular to the direction of motion do not change in different frames, allow one to obtain the Lorentz transformation which applies to both (x,t) and (p,E) not only for light, but also for particles with rest mass. The final result is consistent with Fermat's principle applied directly to the lab frame because following (1) light follows a geodesic which means least time.

Lorentz Transformation from Light Following a Geodesic

In (1) light is said to follow a geodesic as a fundamental principle. In other words it moves in a straight line (extremum of time) regardless of the frame. Thus one may apply Fermat's principle to different frames and obtain consistent results. One, however, needs to know how variables such as x and t transform in the different frames. At this point, we consider only light and the variables x and t.

In (2) light reflects from a mirror of length L along the x axis and moving with a velocity v in the negative y direction. The angle of incidence and reflection are different. No information about initial momentum magnitude or final momentum magnitude is given because the author applies Fermat's principle of minimum time assuming that c, the speed of light, is the same for the incident and reflected rays. No knowledge of Lorentz transformations is needed as stressed in (2).

If one were to go into a frame moving with the mirror, then it would be stationary and one would apply Fermat's minimum time to this scenario obtaining the usual angle of incidence equals angle of reflection. From Newtonian mechanics one also has conservation of momentum parallel to the mirror. Furthermore in the lab frame one also has conservation of momentum parallel to the mirror. In other words:

(A) The momentum component parallel to the mirror surface (perpendicular to motion) does not change regardless of the frame.

At this point we make two assumptions:

(B) Light has the same speed in all frames (Einstein's 1905 assumption)
(C) Length perpendicular to motion of the frame is unchanged as seen from the moving frame and the stationary lab one.

(A) and (C) are of the same form which is an interesting result because it leads to (x,t) and (p,E) transforming in the same manner. Thus using Fermat's principle in two frames and trying to find transformation properties between the two leads to an extension from (x,t) to (x,t) and (p,E). It will be seen that it leads to a second extension, namely from light to particles with rest mass.

To find the Lorentz transformation which links x',t' in the moving frame to x, t in the lab frame for light, one may use Einstein's 1905 arguments and consider light moving in the y direction as seen in a frame moving with constant velocity v in the x direction. The light bounces back and forth in the y direction and so is localized energy i.e. equivalent to rest mass we argue. As seen in the lab, however, it makes a triangle with the hypotenuse represented by c and the adjacent by v. If L is the distance of the opposite which is the same as seen from both frames then:

L=ct' and cctt = LL + vv tt (lab) or t' = t sqrt(1-vv/cc) ((1))

This shows the matrix element which transforms t in the Lorentz matrix except that one considers t as the final frame and t' as the initial because the rest energy is in the (') frame. Consider next a momentum triangle as seen from the lab. It has the same angle as the c velocity triangle described above with py being the same in both frames. Using E=pc and E'=p'c and px = p cos(theta) = v/c (as seen in the lab):

E'=p'c and E = c sqrt(ppvv/cc + E'E'/cc) which yields E= E'/ sqrt(1-vv/cc) ((2))

We argued above that E’ is rest mass cc (of the photon bouncing up and down in the moving frame. This is seen as rest mass (cc) / sqrt(1-vv/cc) from the lab which considers the rest mass to be moving. This is a result for particles with rest mass, but shows that sqrt(1-vv/cc) is a factor associated with not only time, but E. To find the full Lorentz matrix one may actually consider a particle with rest mass. From the above arguments we consider the quantities seen in the moving frame as rest quantities and those in the lab as quantities subject to motion.

In a rest frame a particle has energy mo cc and is at x=0 at time to. Seen from a moving frame it has velocity x/t = v. Thus 1/sqrt(1-vv/cc) is the factor changing time to so v/sqrt(1-vv/cc) changes x. Thus one may suggest the matrix:

| 1/sqrt(1-vv/cc) -v/sqrt(1-vv/cc) | ((3))
| -v/sqrt(1-vv/cc)   1/sqrt(1-vv/cc)|

This applies not only to (x,t), but to (p,E) and not just to light, but to particles with rest mass as well. Thus applying Fermat’s principle in two frames yields much more information than simply using it in the lab frame in which the mirror is moving. One may now obtain information about not only the reflected angle in terms of the incident, but also about p’,E’ and p,E. In particular, one may apply ((3)) to momentum and energy to find a relationship between reflected and incident rays i.e. for c=1

(p’ incident perpendicular to the mirror, E’) = Matrix ((3)) (pcos(theta), E) where theta is the incident angle relative to the normal. Let G= 1/sqt(1-vv/cc).

E’ = Gv(-v cos(theta) + 1)   and E’ cos(theta’) = Gw cos(theta) -GvE so

cos(theta’ i.e. in the moving frame) = (cos(theta)-v) / (1-cos(theta)) ((4))

In the moving frame the reflected angle has the same magnitude as the incident so:

(Elab cos(beta, Elab) = ((3)) with v changed to -v acting on (-E’cos(theta’), E’) so

Beta is the reflected angle in the lab.

Elab cos(beta)= -E’ G cos(theta’) + Gv E’ and Elab= -GvE’cos(theta’) +GE’ yielding

cos(beta=reflected angle in lab) = { [-cos(theta)+v]/(1-vcos(theta) + v} / {
-v[cos(theta-v)/[1-vcos(theta)+1] ((5))

This essentially matches the value listed in (2) and is consistent with the result obtained from Fermat’s principle applied in the lab frame (as done in (2)).

Conclusion

In conclusion, (1) states that Fermat's principle is equivalent to stating that light travels along geodesics. Thus one may apply this principle in various frames moving at constant velocity because one does not know one is in a moving frame according to Einstein. In particular one may use Fermat's principle to find the reflected angle in terms of the incident for light bouncing off a mirror moving at constant v in the negative y direction. Equivalently one may use Fermat's principle in the moving frame. In such a frame one has the usual angle of incidence equals the angle of reflection. This result follows from Fermat's principle, but is also linked to conservation of momentum along the mirror surface which also holds in the lab frame. Thus we suggest that there is an important link between (x,t) and (p,E) for light. In other words, both x and p perpendicular to motion do not change. This suggests that both (x,t) and (p,E) transform in the same way.
One may assume that the speed of light in all frames is c (a standard assumption) and that lengths perpendicular to motion do not change as seen in moving or rest frames. We consider the problem of light bouncing in the y direction in a frame moving at constant v in the x direction. This represents localized energy or rest mass energy so the transformation not only holds for light, but for particles with rest mass. From the point of view of the lab one sees the light moving in a triangle with c as the hypotenuse and v the adjacent. This same triangle, however, applies to momentum with py being unchanged in different frames. From this one may show that E and t transform in the same way. Using the idea of a particle with rest mass at x=0 at time to (in the case, the rest frame is actually the moving one because that is the one with the localized light bouncing in the y direction) one may argue that as seen from a moving frame, the particle moves with velocity v so x/t=v. Thus x must transform as v t. T or E transform as 1/sqrt(1-vv/cc) so the other element of a 2x2 Lorentz matrix is v/sqrt(1-vv/cc). If the matrix is the same as its transpose then one has the Lorentz transformation for both (x,t) and (p,E). This may be used to solve the moving mirror problem as Einstein already did (3). The result is compatible with Fermat's principle applied to the lab frame (no Lorentz transformations) as shown in (2).

References

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