What are Foundations of Geometry and Algebra?

F.W. Lawvere, University of Buffalo

From observation of, and participation in, the ongoing actual practice of Mathematics, Decisive Abstract General Relations (DAGRs) can be extracted; when they are made explicit, these DAGRs become a guide to further rational practice of mathematics. The worry that these DAGRs may turn out to be as numerous as the specific mathematical facts themselves is overcome by viewing the ensemble of DAGRs as a 'Foundation', expressed as a single algebraic system whose current description can be finitely-presented. The category of categories (as a cartesian closed category with an object of small discrete categories) aims to serve as such a Foundation. One basic DAGR is the contrast between space and quantity, and especially the relation between the two that is expressed by the role of spaces as domains of variation for intensively and extensively variable quantity; in that way, the foundational aspects of cohesive space and variable quantity inherently includes also the conceptual basis for analysis, both for functional analysis and for the transformation from continuous cohesion to combinatorial semi-discreteness via abstract homotopy theory. Function spaces embody a pervasive DAGR.

The year 1960 was a turning point. Kan, Isbell, Grothendieck and Yoneda had further developed the Eilenberg-Mac Lane Theory of Naturality. Their work implicitly pointed towards such a Foundation as a foreseeable goal. Although the work of those four great mathematicians was still unknown to me, I had independently traversed a sufficient fragment of a similar path to encourage me to become a student of Professor Eilenberg. As I slowly became aware of the importance of those earlier developments, I attempted to participate in the realization of a Foundation in the sense described above, first through concentration on the particular docrine known as Universal Algebra, making explicit the fibered category whose base consists of abstract generals (called theories) and whose fibers are concrete generals (known as algebraic categories). The term 'Functorial Semantics' simply refers to the fact that in such a fibered category, any interpretation $T' \to T$ of theories induces a map in the opposite direction between the two categories of concrete meanings; this is a direct generalization of the previously observed cases of linear algebra, where the abstract generals are rings and the fibers consist of modules, and of group theory where the dialectic between abstract groups and their actions had long been fundamental in practice. This kind of fibration is special, because the objects T in the base are themselves categories, as I had noticed after first rediscovering the notion of clone, but then rejecting the latter on the basis of the principle that, to compare two things, one must first make sure that they are in the same category; when the two are (a) a theory and (b) a background category in which it is to be interpreted, comparisons being models., the category of categories with products serves. Left adjoints to the re-interpretation functors between fibers exist in this particular doctrine of general concepts, unifying a large number of classical and new constructions of algebra. Isbell conjugacy can provide a first approximation to the general space vs quantity pseudo-duality, because recent developments (KIGY) had shown that also spaces themselves are determined by categories (of figures and incidence relations inside them).

My 1963 thesis clearly explains that presentations (having a signature consisting of names for generators and another signature consisting of names for equational axioms) constitute one important source of theories. This syntactical left adjoint directly generalizes the presentations known from elimination theory in linear algebra and from word problems in group theory. No one would confuse rings and groups themselves with their various syntactical presentations, but previous foundations of algebra had under-emphasized the existence of another important method for constructing examples, namely the Algebraic Structure functor. Being a left adjoint , it can be calculated as a colimit over finite graphs. Fundamental examples, like cohomology operations as studied by the heroes of the 50's, show that typically an abstract general (such as an isometry group) arises by naturality; to find a syntactical presentation for it may then be an important question. This extraction, by naturality from a particular family of cases, provides much finer invariants, and as a process bears a profound resemblance to the basic extraction of abstract generals from experience.