The Distribution Of Prime Numbers And The Continued Fraction

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Abstract. In this paper, we discovered a new sequence of all odd prime numbers in the conjecture 1, wich can be calculated in two different ways that give the same result, the first way is using the greatest common divisor (gcd), the second way consists of using the denominator of the continued fraction defined by

$$\frac{B_m(n)}{A_m(n)} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5}{\ddots \, (n-1) - \cfrac{n}{n-(n+m)}}}}}$$

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Introduction

A continued fraction is an expression of the form

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{\ddots}}}$$

Other notation

$$a_0 + \frac{b_0}{a_1 +}\,\frac{b_1}{a_2 +}\,\frac{b_2}{a_3 +}\cdots$$

Where a_i and b_i are real numbers.

The distribution of prime numbers has been analyzed for a formula helpful in generating the prime numbers or testing if the given numbers is prime. In this paper, we present some known formulas.

Mill's showed that there exists a real number $A > 1$ such that $f(n) = \left[\, A^{3^n} \,\right]$ is a prime number for any integers n, approximately A=1.306377883863,.. (see A051021). The first few values

$f(n) =$ {2, 11, 1361, 2521008887, 16022236204009818131831320183,..}, (see A051254)

Euler's quadratic polynomial $n^2 + n + 41$ is prime for all n between 0 and 39, however, it is not prime for all integers.

The Rowland sequence of prime numbers composed entirely of 1's and primes, the sequence defined by the recurrence relation

$$r(n) = r(n-1) + \gcd\,(n\,, r(n-1));\;\; r(1) = 7$$

The sequence of differences $r(n+1) - r(n)$

1, 1, 1, 5, 3, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 23, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,1, 1, 1, 47, 3, 1, 5, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 101, 3, 1, 1, 7, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 13, 1, 1, 1, 1, 1, 1, 1, 1, 1,.. (see A132199).

For more details and formulas see [1] and [2]. In this paper, we present an interesting sequence which plays the same role as Rowland's sequence composed by a prime number or 1. Moreover, our sequence gives all prime numbers in order.

In this section, we present our sequence of prime numbers defined in the conjecture as follows.

Conjecture 1. The continued fraction

$$\frac{b(n-2)+b(n-3)}{2n-1} = \cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{4-\cfrac{5}{\ddots\, (n-1)-\cfrac{n}{n+1}}}}};n \geq 2$$

Where $b(n)$ satisfy the recursive formula as follow

$$b(n) = (n+2)\big(b(n-1) - b(n-2)\big)$$

With the starting conditions $b(-1) = 0$ and $b(0) = 1$

The following expression finds all odd prime numbers

$$a(n) = \frac{2n-1}{gcd\big(2n-1, b(n-2)+b(n-3)\big)};n \geq 2$$

Where gcd(x, y) denotes the greatest common divisor of x and y.

The values of $a(n)$

3, 5, 7, 3, 11, 13, 1, 17, 19, 1, 23, 1, 1, 29, 31, 1, 1, 37, 1, 41, 43, 1, 47, 1, 1, 53, 1, 1, 59, 61, 1, 1, 67, 1, 71, 73, 1, 1, 79, 1, 83, 1, 1, 89, 1, 1, 1, 97, 1, 101, 103, 1, 107, 109, 1, 113, 1, 1, 1, 1, 1, 1, 127, 1, 131, 1, 1, 137, 139, 1, 1, 1, 1, 149, 151, 1, 1, 157, 1, 1, 163, 1, 167…

Every term of this sequence is either a prime number or 1.

Conjecture 2. The continued fraction

$$\frac{b(n)}{n^2-n-1} = \cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{4-\cfrac{5}{\ddots\, (n-1)-\cfrac{n}{n-(n+1)}}}}}$$

Where $b(n)$ satisfy the recursive formula as follow

$$b(n) = (n-1)b(n-1) - nb(n-2)$$

With the starting conditions $b(1) = b(2) = -1$.

The following expression finds all odd prime numbers which ends with a 1 or 9 (except for 5)

$$a(n) = \frac{n^2 - n - 1}{gcd(b(n),\ n^2 - n - 1)} \ \ ;\ for\ n \geq 2$$

The values of $a(n)$

1, 5, 11, 19, 29, 41, 11, 71, 89, 109, 131, 31, 181, 19, 239, 271, 61, 31, 379, 419, 461, 101, 29, 599, 59, 701, 151, 811, 79, 929, 991, 211, 59, 41, 1259, 1, 281, 1481, 1559, 149, 1721, 1, 61, 1979, 2069, 2161, 1, 2351, 79, 2549, 241, 1, 2861, 2969, 3079, 3191,…(see A356247)

We conjectured that :

$*$ Every term of this sequence is either a prime number or 1.

$*$ Except for 5, the primes all appear exactly twice, such that

$$a(n) = a(a(n) - n + 1)$$

Consequently, let us consider the values of n and m such that we get:

$$a(n) = a(m) = n + m - 1$$

And

$$a(n) = a(m) = \gcd(n^2 - n - 1, m^2 - m - 1)$$

Conjecture 3. The continued fraction

$$\frac{2b(n-3) + nb(n-4)}{n^2 - 2} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5}{\ddots\, (n-1) - \cfrac{n}{n - (n+2)}}}}}$$

Where $b(n)$ satisfy the recursive formula as follow

$$b(n) = (n+2)\big(b(n-1) - b(n-2)\big)$$

With the starting conditions $b(-1) = 0$ and $b(0) = 1$.

The expression of the sequence $a(n)$ of prime numbers as follow

$$a(n) = \frac{n^2 - 2}{gcd\big(n^2 - 2,\ 2b(n-3) + nb(n-4)\big)} \ \ ;\ for\ n \geq 2$$

The values of $a(n)$.

7, 7, 23, 17, 47, 31, 79, 7, 17, 71, 167, 97, 223, 127, 41, 23, 359, 199, 439, 241, 31, 41, 89, 337, 727, 1, 839, 449, 137, 73, 1087, 577, 1223, 647, 1367, 103, 1, 47, 73, 881, 1, 967, 1, 151, 2207, 1151, 2399, 1249, 113, 193, 401, 1, 3023, 1567, 191, 41, 71…

the sequence $a(n)$ takes only 1’s and primes.

Generalisation

The unreduced sequence can be calculated by using the denominator of the continued fraction as follow

$$\frac{B_m(n)}{A_m(n)} = \cfrac{1}{2 - \cfrac{3}{3 - \cfrac{4}{4 - \cfrac{5}{\ddots \, (n-1) - \cfrac{n}{n-(n+m)}}}}}$$

Where $B_m(n)$ is the numerator related with the recursive formula as follow

$$b(n) = (n+2)\big(b(n-1) - b(n-2)\big), with\ b(-1) = 0\ and\ b(0) = 1$$

The denominator $A_m(n)$ is a polynomials in term n for various values of m, we can obtain the sequence of the unreduced denominator of the continued fraction as follow

$$a_m(n) = \frac{A_m(n)}{gcd\big(A_m(n), B_m(n)\big)}$$

For $m = -(n+1)$, we obtain the sequence in the conjecture 1, we can see that the sequence finds all prime numbers in order (except for n=5), also for $m = 1$, we find the sequence in the conjecture 2 which give all prime number which ends with a 1 or 9. Moreover, for $m = 2$, we find the sequence in the conjecture 3 that contains only 1's and the primes.

References

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