The Equivalence of Fermat's Least Time with a Hypothetical Velocity

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In this note we try to show directly that Fermat's minimum time principle is equivalent to a hypothetical velocity approach where this velocity acts as a hypotenuse to the actual angled directions of light. We consider in particular problems for which there are two rays of light with possibly different speeds (at angles) and a medium with a surface of length L along the x axis. This approach applies to a fixed mirror, a mirror moving at constant v in the y direction and Snell's law. A goal is to find the point x on the medium surface at which the first ray hits. The distance traveled by the first ray is cta and that by the second ray ctb where t is time. For one to have a minimum of T=ta+tb: dT/dx = 0 = dta/dx + dtb/dx. We argue that dta/dx is of the form of a reciprocal velocity 1/H and that dtb/dx=(-1)1/H. The -1 for the tb case follows from the appearance of d/dx (L-x) = -1 whereas d/dx x = 1.
H is a hypothetical velocity and so one may solve such problems in terms of this H instead of minimizing time which requires taking a derivative. In this note, we try to show that such a velocity approach is equivalent to a minimized time one.

The General Problem

Consider problems of the form of two rays of light linked to a medium of fixed length L along the x axis. The first ray hits the medium (e.g. a mirror or an actual different medium e.g. water) at an angle AA with the normal. The second ray makes an angle with the normal of BB and travels such that its projection on the x axis ranges from x to L, i.e. a length of L-x. The second ray may be a reflected one or in the case of Snell's law, a refracted one. The first ray has a speed of c1 and the second, c2.

Fermat's principle states that time must be minimized i.e.

T=ta+tb = minimum or dT/dx=0 = dta/dx + dtb/dx ((1))

dta/dx and dtb/dx are in the form of inverse velocities 1/H and -1/H. The distances traveled by the two rays are: cta and ctb so:

Cta sin(AA) = x and ctb sin(BB) = L-x ((2))

One may arrange the problem so that the y distance for the two rays is the same i.e.

C1c 1tata = yy + xx and c2c2 tbtb = yy + (L-x)(L-x) ((3)) (Pythagoreas' theorem)

Taking derivatives with respect to x yields:

C1c1 ta dta/dx = y dy/dx + x ((4a)) and c2c2 tb dtb/dx = y dyd/x - (L-x) ((4b))

Dividing by c1ta in ((4a)) and c2tb in ((4b)) yields:

c1dta/dx = (y/c1ta) dy/dx + sin(AA) ((5a)) and c1dtb/dx = (y/c2tb) dy/dx - sin(BB) ((5b))

We use c1 and c2 because light travels at different speeds in different media and we are considering the refraction problem as well as reflection.
Let us examine y in more detail. For a stationary mirror or refraction problem, y is simply a constant so dy/dx =0. In such a case:

(dy/dx=0) c1dta/dx = sin(AA) and c2dtb/dx = -sin(BB) but dta/dx = -dtb/dx and c1=c/n1 and c2=c/n2 where n is the index of refraction.

Thus: sin(AA) / sin(BB) = n2/ n1 ((6)) which is Snell’s law.

For light reflecting from a fixed mirror n2=n1 and sin(AA) = sin(BB) i.e. the angle of reflection equals the angle of incidence.

In the case of a moving mirror (constant velocity in the downwards y direction), then dy/dx = dy/dta dta/dx instead of 0. This result holds for both rays, but dta/dx = -dtb/dx. In such a case ((5a)) and ((5b)) lead to (with c1=c2=1)

dta/dx (1- y/ta dy/dta) = sin(AA) ((7a)) and dtb/dx (1+y/tb dy/dta) = -sin(BB) ((7b))

If dta/dx=1/H and dtb/dx = 1/H then:

H sin(AA) = (1- y/ta dy/dta) ((8a)) and H sin(BB) = (1+y/tb dy/dta) ((8b))

Thus H acts like a hypotenuse velocity and the quantities in () like a kind of relative velocity. Examining these specifically for the moving mirror gives:

y= d + vta where d is a fixed number and v is the velocity of the mirror downward in the y direction.

(1-y/ta dy/dta) = c - cos(AA)v ((9a)) and (1+y/tb dy/dta) = c+ cos(BB) v ((9b))

vcos(AA) and vcos(BB) are the projections of the mirror velocity along the ray directions so one has indeed relative velocities in ((9a)) and ((9b)). Thus for the moving mirror problem:

v(relative b) / sin(AA) = v(relative b)/ sin(BB) ((10))

This result represents the hypothetical hypotenuse approach, but is equivalent to the minimization of time which is Fermat’s approach.

Conclusion

In conclusion we argue that Fermat's principle of least time applied to a problem of two rays hitting a horizontal surface of length L such that the first ray hits at x for the minimum time solution and the second ray's projected length on the x axis is L-x is equivalent to a hypothetical velocity approach where one creates relative velocities for the two rays and considers them as Hcos(90-AA) = v(relative a) and H cos(90-BB) = v(relative b). Such an approach holds for a fixed mirror, moving mirror (in the y direction) and Snell's law.

References

1. Gjurchinovski, A. Einstein's Mirror and Fermat's Principle of least time (Am. J. of Phys 72 (10) Oct. 2004)
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